EQUIVARIANT GROMOV-WITTEN INVARIANTS

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The objective of this paper is to describe some construction and applications of the equivariant counterpart to the Gromov-Witten (GW) theory, i.e. intersection theory on spaces of (pseudo-) holomorphic curves in (almost-) Kahler manifolds.

Given a Killing action of a compact Lie group G on a compact Kahler manifold X, the equivariant GW-theory provides, as we will show in Section 3, the equivariant cohomology space $H^*_G(X)$ with a *Frobenius structure* (see [2]). We discuss applications of the equivariant theory to the computation ([7],[11]) of quantum cohomology algebras of flag manifolds (Section 5), to the simultaneous diagonalization of the quantum cup-product operators (Sections 7,8), to the S^1-equivariant Floer homology theory on the loop space LX (see Section 6 and [10],[9]) and to a "quantum" version of the Serre duality theorem (Section 12).

In Sections 9 — 11 we combine the general theory developed in Sections 1 — 6 with the fixed point localization technique [3] in order to prove the mirror conjecture (in the form suggested in [10]) for projective complete intersections.

By the mirror conjecture one usually means some intriguing relations (discovered by physicists) between symplectic and complex geometry on a compact Kahler Calabi–Yau n-fold and respectively complex and symplectic geometry on another Calabi-Yau n-fold called the mirror partner of the former one. The remakable application [16] of the mirror conjecture to enumeration of rational curves on Calabi–Yau 3-folds (1991, see the *theorem* below) raised a number of new mathematical problems — challenging maturity tests for modern methods of symplectic topology.

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On the other hand, in 1993 I suggested that the relation between symplectic and complex geometry predicted by the mirror conjecture can be extended from the class of Calabi–Yau manifolds to more general compact symplectic manifolds *if one admits non-constant holomorphic functions on suitable non-compact Calabi–Yau manifolds in the role of the mirror partners*. According to this generalized form of the mirror conjecture Gromov–Witten invariants of a symplectic manifold can be reinterpreted in terms of oscillating integrals over the mirror partner and saddle-point asymptotics of these integrals near critical points of the holomorphic function.

We refer to [10, 9] for a detailed discussion of the generalized mirror conjecture supported there by the examples of complex projective spaces and general toric symplectic manifolds. In this paper we prove the conjecture (see Corollary 11.10, Corollary 10.8, Corollary 9.2 and the remark following it) for complete intersections in $\mathbb{C}P^n$ given by r equations of degrees $(l_1, ..., l_r)$ with $l_1 + ... + l_r \leq n + 1$, that is for Fano ($<$) and Calaby–Yau ($=$) projective complete intersections.

In particular we explain in Section 11 how to pass the following maturity test:

Theorem. *Consider the Picard-Fuchs differential equation*

$$(\frac{d}{dt})^4 I = 5e^t(5\frac{d}{dt}+1)(5\frac{d}{dt}+2)(5\frac{d}{dt}+3)(5\frac{d}{dt}+4)I$$

satisfied by the periods

$$I(t) = \int_{\gamma_t^3} \frac{du_0 \wedge ... \wedge du_4}{d(u_0 + ... + u_4) \wedge d(u_0 ... u_4)}$$

of the non-vanishing holomorphic 3*-forms on the Calabi-Yau* 3*-folds* Y_t *with Hodge numbers* $h^{2,1} = 1,\ h^{1,1} = 101$ *given by the affine equations* $Y_t : u_0 + ... + u_4 = 1, u_0 ... u_4 = e^t$.

Pick the basis $I_0, ..., I_3$ *of solutions to this differential equation determined by*

$$I_0(t) + I_1(t)P + I_2(t)P^2 + I_3(t)P^3 = \sum_{d=0}^{\infty} e^{(P+d)t} \frac{\Pi_{m=1}^{5d}(5P+m)}{\Pi_{m=1}^{d}(P+m)^5} \quad (\ mod\ P^4).$$

Introduce the new variable $T(t) = I_1(t)/I_0(t)$.

Then

$$\frac{I_0}{I_0} + \frac{I_1}{I_0}(t(T))P + \frac{I_2}{I_0}(t(T))P^2 + \frac{I_3}{I_0}(t(T))P^3 \ =$$

$$e^{PT}+\frac{P^2}{5}\sum_{d=1}^{\infty}n_d d^3\sum_{k=1}^{\infty}\frac{e^{(P+kd)T}}{(P+kd)^2}\quad (\ mod\ P^4),$$

where the components of the RHS form the basis of solutions to the differential equation

$$(\frac{d}{dT})^2\frac{1}{K(e^T)}(\frac{d}{dT})^2\ J\ =0\quad with\quad K(q)=5+\sum_{d=1}^{\infty}n_d d^3\frac{q^d}{1-q^d},$$

and n_d is the virtual number of degree d rational curves in $\mathbb{C}P^4$ situated on a generic degree 5 hypersurface X, a Calabi-Yau 3-fold with Hodge numbers $h^{2,1}=101,\ h^{1,1}=1$.

An analogous result holds for any non-singular Calabi-Yau 3-dimensional projective complete intersection X.

The virtual numbers of rational curves on a Calabi-Yau 3-fold X are defined in several equivalent ways in the quantum cohomology theory [1] and are equal to the algebraic numbers of such curves on X provided with a generic almost Kahler structure. It is known that for generic quintic hypersurfaces $X\subset\mathbb{C}P^4$ the virtual number n_d coincides with the number of the degree d rational curves in $\mathbb{C}P^4$ situated in X at least for $d\leq 9$. The number $n_1=2875$ of straight lines on a generic quintic 3-fold has been known since the last century, $n_2=609250$ and $n_3=317206375$ were found (see [16]) several years ago, while $n_4=242467530000$ was predicted in [16] and confirmed in [3] (as an illustration of a method that allows in principle to find each n_d). The simultaneous description of all the numbers n_d given in the theorem was conjectured in [16] on the basis of physical ideas of mirror symmetry between the Calabi-Yau manifolds X and Y whose Hodge diamonds happened to be mirror-symmetric to one another.

As far as we know, our Theorem and its generalization to Calabi-Yau projective complete intersections given in Section 11 provide the first examples of Calabi-Yau manifolds for which predictions of the mirror symmetry are verified for rational curves of all degrees.

The results of Sections 9 – 11 can be immediately carried over to complete intersections in products of projective spaces. The method can be also applied to complete intersections in general toric varieties where however some generalization of our algebraic formalism and some refinement in foundations of the equivariant Gromov – Witten theory would be necessary.

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[1]see for instance [20, 19] and also [15, 14] where the problem of counting multiple covers is resolved.

to M. Kontsevich who taught me his approach to Gromov – Witten theory. The proof of the theorem formulated above has grown out of our joint attempt in Spring 95 to prove it using the method [3] of summation over trees. The influence of our discussions on other results of this paper is also significant.

1 Moduli spaces of stable maps

It was M. Gromov [8] who first suggested to construct (and constructed some) topological invariants of a symplectic manifold X as bordism classes of spaces of pseudo-holomorphic curves in X. Recently M. Kontsevich [3] suggested the concept of *stable maps* which gives rise to an adequate compactification of these spaces. We recall here some basic facts from [3] about these compactifications.

Let (C, p) be a compact connected complex curve with only double singular points and with n ordered non-singular *marked points* $(p_1, ..., p_n)$. Two holomorphic maps $(C, p) \to X$, $(C', p') \to X$ to an almost-Kahler manifold X are called *equivalent* if they can be identified by a holomorphic isomorphism $(C, p) \to (C', p')$. A holomorphic map $(C, p) \to X$ is called *stable* if it does not have infinitesimal automorphisms (or, equivalently, if its automorphism group is finite). In other words, a map is unstable if either it is constant on a genus 0 irreducible component of C with < 3 *special* (= marked or singular) points or if C is a torus, carries no marked points and the map is constant.

According to Gromov's compactness theorem [8], any sequence of holomorphic maps $C \to X$ of a nonsingular compact curve C has a subsequence Hausdorff-convergent to a holomorphic map $\hat{C} \to X$ of (may be reducible) curve $\hat{C}$ of the same genus g and representing the same total homology class $d \in H_2(X, \mathbb{Z})$. A refinement of this theorem from [3] says that equivalence classes of stable maps $C \to X$ with given g, n, d form a single compact Hausdorff space — the moduli space of stable maps — which we denote $X_{g,n,d}$. Here $g = \dim H^1(C, \mathcal{O}) = 1 - \chi(C \backslash C^{\text{sing}})/2$.

In the case $X = pt$ the moduli spaces coincide with Deligne-Mumford compactifications $\bar{\mathcal{M}}_{g,n}$ of moduli spaces of genus g Riemannian surfaces with n marked points. They are compact nonsingular orbifolds (i.e. local quotients of nonsingular manifolds by finite groups) and thus bear the rational fundamental cycle which allows one to build up an intersection theory. In general, the moduli spaces $X_{g,n,d}$ are singular and may have "wrong" dimension, and the idea of the program started in [4, 3] is to provide $X_{g,n,d}$ with virtual fundamental cycles insensitive to perturbations of the almost-Kahler structure on X. In some nice cases however the spaces $X_{0,n,d}$ are already nonsingular orbifolds of the "right" dimension.

Beginning with this point we will use only genus zero stable maps and use the notation $X_{n,d}$ for the moduli spaces $X_{0,n,d}$.

A compact complex manifold is called *convex* if it is a homogeneous space of its Lie algebra of holomorphic vector fields.

Theorem 1.1 ([3, 6]) *If X is convex then all non-empty moduli spaces $X_{n,d}$ of genus 0 stable maps are compact nonsingular complex orbifolds of "right" dimension $\langle c_1(T_x), d\rangle + \dim_{\mathbb{C}} X + n - 3$.*

Additionally, there are canonical morphisms $X_{n,d} \to X_{n-1,d},\ X_{n,d} \to \bar{\mathcal{M}}_{0,n},\ X_{n,d} \to X^n$ between the moduli spaces $X_{n,d}$ called *forgetful, contraction* and *evaluation* (and defined by forgetting one of the marked points, forgetting the map and evaluating the map at marked points respectively). We refer to [3, 6] for details of their construction.

In the rest of this paper we will stick to convex manifolds; we comment however on which results are expected to hold in greater generality. A number of recent preprints by B. Behrend – B. Fantechi, J.Li – G. Tian, T. Fukaya – K. Ono shows that Kontsevich's "virtual fundamental cycle" program is being realized successfully and leaves no doubts that these generalizations are correct. Still some verifications are necessary in order make them precise theorems.

2 Equivariant correlators

The Gromov-Witten theory borrows from the quantum field theory the name *(quantum) correlators* for numerical topological characteristics of the moduli spaces $X_{n,d}$ (characteristic numbers) and borrows from the bordism theory the construction of such correlators as integrals of suitable wedge-products of various universal cohomology classes (characteristic classes of the GW theory) over the fundamental cycle.

We list here some such characteristic classes.

1. Pull-backs of cohomology classes from X^n by the evaluation maps $e_1 \times \cdots \times e_n : X_{n,d} \to X^n$ at the marked points.

2. Any polynomial of the first Chern classes $c^{(1)}, \ldots, c^{(n)}$ of the line bundles over $X_{n,d}$ consisting of tangent lines to the mapped curves at the marked points. One defines these line bundles (by identifying the Cartesian product of the forgetful and evaluation maps $X_{n+1,d} \to X_{n,d} \times X$ with the *universal stable map* over $X_{n,d}$) as normal line bundles to the n embeddings $X_{n,d} \to X_{n+1,d}$ defined by the n marked points of the universal stable map. We will call these line bundles *the universal tangent lines* at the marked points.

3. Pull-backs of cohomology classes of the Deligne - Mumford spaces by contraction maps $\pi : X_{n,d} \to \bar{\mathcal{M}}_{0,n}$. We will make use of the classes $A_I := A_{i_1,\ldots,i_k}$ Poincare-dual to fundamental cycles of fibers of forgetful maps $\bar{\mathcal{M}}_{0,n} \to \bar{\mathcal{M}}_{0,k}$.

We define the *GW-invariant*

$$A_I\langle \phi_1, ..., \phi_n\rangle_{n,d} := \int_{X_{n,d}} \pi^* A_I \wedge e_1^* \phi_1 \wedge ... \wedge e_n^* \phi_n.$$

It has the following meaning in enumerative geometry: it counts the number of pairs "a degree-d holomorphic map $\mathbb{C}P^1 \to X$ with given k points mapped to given k cycles, a configuration of $n-k$ marked points mapped to the $n-k$ given cycles".

Suppose now that the convex manifold X is provided with a hamiltonian Killing action of a compact Lie group G. Then G act also on the moduli spaces of stable maps. The evaluation, forgetful and contraction maps are G-equivariant, and one can define correlators $A_I\langle \phi_1, \ldots \phi_n\rangle_{n,d}$ of *equivariant* cohomology classes of X.

The equivariant cohomology $H^*_G(M)$ of a G-space M is defined as the ordinary cohomology $H^*(M_G)$ of the homotopic quotient $M_G = EG \times_G M$ — the total space of the M-bundle $p : M_G \to BG$ associated with the universal principal G-bundle $EG \to BG$. The algebra $H^*(BG) = H^*_G(pt)$ of characteristic classes of principal G-bundles plays the role of the coefficient ring of the equivariant theory (so that $H^*_G(M)$ is a $H^*_G(\mathrm{pt})$-module). If M is a compact manifold with smooth G-action, the push-forward $p_* : H^*_G(M) \to H^*_G(\mathrm{pt})$ ("fiberwise integration") provides the equivariant cohomology of M with intersection theory with values in $H^*_G(\mathrm{pt})$. In the case of hamiltonian actions the corresponding intersection pairing $\langle \cdot, \cdot \rangle$ is non-degenerate over $H^*_G(pt)$.

We introduce the *equivariant GW-invariants*, $A_I(\langle\phi_1,\ldots,\phi_n\rangle_{n,d}$, with values in $H^*(BG)$, where $\phi_1,\ldots,\phi_n\in H^*_G(X)$. Values of such invariants on fundamental cycles of maps $B\to BG$ are accountable for enumeration of rational holomorphic curves in families of complex manifolds with the fiber X associated with the principal G-bundles over a finite-dimensional manifold B.

3 The WDVV equation

One of the main structural results about Gromov-Witten invariants — the composition rule [20],[19] — expresses all genus-0 correlators via the 3-given-marked-point ones, (we denote them $\langle\phi_1,...,\phi_n\rangle_{n,d}$ since the corresponding $A_I=1$) satisfying additionally the so-called *Witten-Dijkgraaf-Verlinde-Verlinde* equation. We will see here that the same result holds true for equivariant Gromov-Witten invariants (at least in the convex case).

Following [21], introduce the *potential*

$$F=\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{d}q^d\ \langle t,\ldots,t\rangle_{n,d}\ . \tag{1}$$

It is a formal function on the vector (super -) space $H^*_G(X)$ with values in the coefficient ring $\Lambda=H^*_G(pt,\mathbb{C}[[q]])$. Here $\mathbb{C}[[q]]$ stands for some completion of the group algebra $\mathbb{C}[H_2(X,\mathbb{Z})]$ so that the symbol $q^d=q_1^{d_1}\ldots q_k^{d_k}$ represents the class $(d_1,\ldots,d_k)$ in the lattice $\mathbb{Z}^k=H_2(X,\mathbb{Z})$ of 2-cycles. Fundamental classes of holomorphic curves in X have non-negative coordinates with respect to a basis of Kahler forms so that the formal power series algebra $\mathbb{C}[[q]]$ can be taken on the role of the completion. Strictly speaking, the formula 1 defines F up to a quadratic polynomial of t since the spaces $X_{n,0}$ are defined only for $n\geq 3$.

Denote ∇ the gradient operator with respect to the equivariant intersection pairing $\langle\ ,\ \rangle$ on $H^*_G(X)$. The WDVV equation is an identity between third directional derivatives of F. It says that

$$\langle\nabla F_{\alpha,\beta},\nabla F_{\gamma,\delta}\rangle \tag{2}$$

is totally symmetric (up to usual signs) with respect to permutations of the four directions $\alpha,\beta,\gamma,\delta\in H^*_G(X)$.

Theorem 3.1 *The WDVV equation holds for convex X.*

Notice that

$$\langle \nabla \int_X a \wedge t, \nabla \int_X b \wedge t \rangle = \langle a, b \rangle \tag{3}$$

has geometrical meaning of integration $\int_{\Delta \subset X \times X} a \otimes b$ over the diagonal in $X \times X$.

In order to prove the non-equivariant version of the WDVV equation one interprets the 4-point correlators $A_{1234}\langle \alpha, \beta, \gamma, \delta \rangle_{4,d}$ which are totally symmetric in $\alpha, \beta, \gamma, \delta$ as integrals over the fibers Γ_λ of the contraction map $\pi : X_{4,d} \to \bar{\mathcal{M}}_4 = \mathbb{C}P^1$ and specializes the cross-ratio λ to $0, 1$ or ∞. Stable maps corresponding to generic points of, say, Γ_0 are glued from a pair of maps $f_1 : (\mathbb{C}P^1, p_1, p_2, a_1) \to X$, $f_2 : (\mathbb{C}P^1, p_3, p_4, a_2) \to X$ of degrees $d_1 + d_2 = d$ with three marked points each, satisfying the diagonal condition $f_1(a_1) = f_2(a_2)$. One can treat such a pair as a point in $X_{3,d_1} \times X_{3,d_2}$ situated on the inverse image Γ_{d_1,d_2} of the diagonal $\Delta \subset X \times X$ under the evaluation map $e_3 \times e_3$. The *glueing map* $\sqcup_{d_1+d_2=d} \Gamma_{d_1,d_2} \to \Gamma$ is an isomorphism at generic points and therefore it identifies the analytic fundamental cycles. This means that

$$A_{1234}\langle \alpha, \beta, \gamma, \delta \rangle_{4,d} = \sum_{d_1+d_2=d} \langle \nabla \langle \alpha, \beta, t \rangle_{3,d_1}, \nabla \langle \gamma, \delta, t \rangle_{3,d_2} \rangle \ .$$

The above argument applies to the correlators $A_{1234}\langle \alpha, \beta, \gamma, \delta, t, \ldots, t \rangle_{n+4,d}$ with additional marked points and gives rise to

$$\langle \nabla F_{\alpha,\beta}, \nabla F_{\gamma,\delta} \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_d q^d A_{1234} \langle \alpha, \beta, \gamma, \delta, t \ldots t \rangle_{4+k,d} \tag{4}$$

which is totally symmetric in $\alpha, \beta, \gamma, \delta$.

Convexity of X is used here only in order to make sure that the moduli spaces have fundamental cycles and that the diagonal in $X \times X$ consists of regular values of the evaluation map $e_3 \times e_3$.

In order to justify the above argument in the equivariant situation, it is convenient to reduce the problem to the case of tori actions (using maximal torus of G) and use the De Rham version of equivariant cohomology theory.

For a torus $G = (S^1)^r$ acting on a manifold M the equivariant De Rham complex [1] consists of G-invariant differential forms on M with coefficients in $\mathbb{C}[u_1, \ldots, u_r] = H_G^*(pt)$, provided with the coboundary operator $d_G = d + \sum_{s=1}^{r} u_s i_s$ where i_s are the operators of contraction by the vector fields generating the action. Applying the ordinary Stokes formula to G-invariant forms and G-*invariant* chains we obtain well-defined functionals $H_G^*(M) \to \mathbb{C}[u]$ of *integration over invariant cycles.* The identity 4 follows now from the obvious G-invariance of the analytic varieties Γ_λ, Γ, Γ_{d_1,d_2}.

A similar argument proves a composition rule that reduces computation of all equivariant correlators $A_I\langle ... \rangle$ to that of $\langle ... \rangle$.

4 Convex vector bundles

The following construction was designed by M. Kontsevich in order to extend the domain of applications of WDVV theory to complete intersections in convex Kahler manifolds.

Let $V \to X$ be a *convex* bundle, that is, a holomorphic vector bundle spanned by its holomorphic sections. For stable $f : (C, p) \to X$ (of degree d, genus 0, with n marked points), the spaces $H^0(C, f^*V)$ form a holomorphic vector bundle $V_{n,d}$ over the moduli space $X_{n,d}$. If f is glued from f_1 and f_2 as in the proof of (4), then $H^0(C, f^*V) = \ker(H^0(C_1, f_1^*V) \oplus H^0(C_2, f_2^*V) \stackrel{e_1-e_2}{\longrightarrow} e_1^*V = e_2^*V)$ where $e_i : H^0(C_i, f_i^*V) \to e_i^*V$ is defined by evaluation of sections at the marked point a_i.

This allows one to construct a solution F to the WDVV equation starting with a convex G-equivariant bundle V and any invertible G-equivariant multiplicative characteristic class E (the total Chern class would be a good example).

Redefine

$$\langle a, b \rangle \ := \ \int_X a \wedge b \wedge E(V) \ ,$$

$$\langle t, \ldots, t \rangle_{n,d} \ := \ \int_{X_{n,d}} e_1^* t \wedge \ldots e_n^* t \wedge E(V_{n,d}) \ ,$$

$$F(t) \ = \ \sum_{n=0}^{\infty} \frac{1}{k!} \sum_d q^d \langle t, \ldots, t \rangle_{n,d} \ .$$

Then $\langle \nabla F_{\alpha,\beta}, \nabla F_{\gamma\delta} \rangle$ *is totally symmetric in* $\alpha, \beta, \gamma, \delta$.

This construction bears a limit procedure from the total Chern class to the (equivariant) Euler class, and the limit of F corresponds to the GW-theory on the submanifold $X' \subset X$ defined by an (equivariant) holomorphic section s of the bundle V. Namely, the section s

induces a holomorphic section $s_{n,d}$ of $V_{n,d}$, and the (equivariant) Euler class $Euler\ (V_{n,d})$ becomes represented by some cycle $[X'_{n,d}]$ situated in the zero locus $X'_{n,d} := s^{-1}_{n,d}(0)$ of the induced section. The variety $X'_{n,d}$ consists of stable maps to X', the Euler cycle $[X'_{n,d}]$ plays the role of the virtual fundamental cycle in $X'_{n,d}$, and the correlators

$$\langle t, ..., t \rangle_{n,d} := \int_{X_{n,d}} e_1^* t \wedge ... \wedge e_n^* t\ Euler\ (V_{n,d}) = \int_{[X'_{n,d}]} e_1^* t \wedge ... \wedge e_n^* t$$

are correlators of GW-theory on X' between the classes t which come from the ambient space X.

According to [28] one can consider the GW-theory with these correlators as the GW-theory on the *super-manifold* with the structural sheaf to be the sheaf of exterior forms on the dual bundle V^*.

Another solution of the $WDVV$-equation can be obtained from the bundles $V'_{d,k} := H^1(C, f^*V^*)$: one should put $\langle a, b\rangle := \int_X a \wedge b \wedge E^{-1}(V^*)$, $\langle t, \ldots, t\rangle_{n,d} = \int_{X_{n,d}} e_1^* t \wedge \cdots \wedge e_n^* t \wedge E(V'_{n,d})$ for $d \neq 0$ and $\langle t, \ldots, t\rangle_{n,0} = \int_{X_{n,0}} e_1^* t \wedge \cdots \wedge e_n^* t \wedge E^{-1}(V^*)$.

In Section 12 we will prove some duality theorem for the two solutions of the WDVV-equations in the case when $X = \mathbb{C}P^n$ and V is the sum of positive line bundles. Choosing the (equivariant) Euler class on the role of $E(V^*)$ one comes to the GW-theory on the *non-compact* total space of the bundle V^*. Using slight modifications of the above correlators and the constructions of the next Section one can also define quantum versions of both the ordinary and compactly supported cohomology algebras of this manifold. We leave the details of this construction to the reader.

5 Quantum cohomology

One interprets the WDVV equation as the associativity identity for the *quantum cup-product* on $H_G^*(X)$ defined by

$$\langle \alpha * \beta, \gamma \rangle = F_{\alpha,\beta,\gamma}\ .$$

It is a deformation of the ordinary cup-product (with t and q in the role of parameters) in the category of (skew)-commutative algebras *with unity*:

$$\langle \alpha * 1, \gamma \rangle = \langle \alpha, \gamma \rangle\ . \tag{5}$$

Indeed, the push-forward by the forgetful map $\pi : X_{n,d} \to X_{n-1,d}$ (with $n \geq 3$) sends $1 \in H^*_G(X_{n,d})$ to 0 unless $d = 0$ and $k = 3$ in which case $X_{n,d} = X$ and $X_{n-1,d}$ is not defined.

The structure usually referred in the literature as the *quantum cohomology algebra* corresponds to the restriction of the deformation $*_{t,q}$ to $t = 0$. As it is shown in [4], in many cases the function F can be recovered on the basis of WDVV-equation from the structural constants $F_{\alpha,\beta,\gamma}|_{t=0}(q)$ of the quantum cohomology algebra due to the following symmetry of the potential F. Let $u \in H^2_G(X)$ and $(u_1, ..., u_k)$ be its coordinates with respect to the basis of the lattice $(\mathbb{Z}^k)^* = H^2(X) = H^2_G(X)/H^2_G(pt)$ (so that $u_i \in H^*_G(pt)$). Then

$$\partial_u \, F_{\alpha,\beta,\gamma} = \sum_{i=1}^{k} u_i q_i \partial F_{\alpha,\beta,\gamma}/\partial q_i \quad \forall \alpha, \beta, \gamma \in H^*_G(X) \; . \tag{6}$$

The identity (6) follows from the obvious push-forward formula $\pi_* u = \sum d_i u_i$.

The symmetry (6) can be interpreted in the way that the quantum deformation of the cup-product restricted to $t = 0$ is equivalent to the deformation with $q = 1$ and t restricted to the 2-nd cohomology of X (in the equivariant setting it is better however to keep both parameters in place — see Sections $7, 8$).

In this paper, we will use the term *quantum cup-product* for the entire (q, t)-deformation and reserve the name *quantum cohomology algebra* for the restriction of the quantum cup-product to $t = 0$.

I have heard some complaints about such terminology because it allows many authors to compute quantum cohomology algebras without even mentioning the deformation in t-directions. There are some indications however that (despite the equivalence (6)) the q-deformation has a somewhat different nature than the t-deformation. The loop space approach [9] and our computations in Sections 9 – 11 seem to emphasize this distinction.

Quantum cohomology algebras of the classical flag manifolds have been computed in [7], [11] on the basis of several conjectures about properties of U_n-*equivariant* quantum cohomology (see also [5] where a slightly different formalism was applied). The answer (in terms of generators and relations [2]) for complete flag manifolds U_n/T^n is strikingly related to

[2] Several weeks after the first version of this paper had been completed some new results arrived: S. Fomin, S. Gelfand, A. Postnikov [25] found structural constants of the quantum cohomology algebra of the flag manifold with respect to the basis of Schubert polynomials, and B. Kim [13] proved the relation of quantum cohomology algebras of generalized flag manifolds G/T with the Toda lattice (of the Langlands-dual group G^*).

conservation laws of Toda lattices. The conjectures named in [7] the *product, induction* and *restriction* properties and describing behavior of equivariant quantum cohomology under some natural constructions, were motivated by interpretation of the quantum cohomology in terms of Floer theory on the loop space LX. Although a construction of the equivariant counterpart of the Floer - Morse theory on LX remains an open problem, the three conjectured axioms can be justified within the Gromov-Witten theory. This was done by B.Kim [12]. The induction and restriction properties follow directly from definitions given in this paper and hold for the entire quantum deformation (not only at $t = 0$), while the "product" axiom (that the $G_1 \times G_2$-equivariant quantum cohomology algebra of $X_1 \times X_2$ is the tensor product of the G_i-equivariant quantum cohomology algebras of the factors X_i) has been verified in [12] for convex manifolds.

Behavior of the quantum cup-product at $t \neq 0$ under the Cartesian product operation on the target manifolds is much more complicated than the operation of the tensor product.

We complete this section with a discussion of some remarkable relation between quantum cohomology algebras of manifolds $F(n_0, ..., n_k)$ of partial flags $\mathbb{C}^{n_0} \subset \mathbb{C}^{n_0+n_1} ... \subset \mathbb{C}^{n_0+...+n_k} = \mathbb{C}^n$ (equivariant with respect to the action of U_n on $\mathbb{C}^n$) and the action-angle coordinates of the Toda lattice — an integrable system with the Hamilton function $p_1^2/2+...+p_n^2/2-\exp(t_1-t_2) - ... - \exp(t_{n-1} - t_n)$ (with respect to the symplectic structure $dp_1 \wedge dt_1 + ... + dp_n \wedge dt_n$). The equivariant quantum cohomology algebras of these manifolds were computed in [11, 12]. The answer can be formulated as follows.

Consider the chain fraction

$$\frac{P(x)}{Q(x)} := P_0(x) + \frac{q_1}{P_1(x) + \frac{q_2}{P_2(x) + \frac{...}{... + \frac{q_k}{P_k(x)}}}}$$

where $P_0, ..., P_k$ are *monic* polynomials of some positive degrees (which we denote $n_0, ..., n_k$) and $q_1, ..., q_k$ are some non-zero constants. Given $P_0, ..., P_k$ and $q_1, ..., q_k$, the chain fraction uniquely determines the two monic polynomials P, Q of degrees n, $n - n_0$. Let the coefficient of the polynomial $P_i = x^{n_i} + c_1^{(i)} x^{n_i - 1} + ... + c_{n_i}^{(i)}$ denote the Chern classes of the tautological n_i-dimensional vector bundle over $F(n_0, ..., n_k)$, $(q_1, ..., q_k)$ denote the parameters of the quantum deformation in the quantum cohomology algebra of $F(n_0, ..., n_k)$ (see [11]). Then the relation $P = x^n + c_1 x^{n-1} + ... + c_n$ expresses a basis of relations between the generators $(c_j^{(i)})$ of the quantum cohomology algebra of the partial flag manifold and the Chern classes

$(c_1, ..., c_n)$ of the tautological vector bundle over BU_n (so that $\mathbb{C}[c_1, ..., c_n]$ plays the role of the coefficient ring of the U_n-equivariant cohomology theory).

Notice that in the "classical" equivariant cohomology algebra the same relation holds with $P = P_0 ... P_k$. This indicates that a quantum generalization of the multiplicative property of the total Chern class should involve chain fractions.

Consider now the reduced rational function Q/P with monic Q. For a generic P it can be written as the sum of simple fractions

$$\frac{a_1}{x - x_1} + ... + \frac{a_n}{x - x_n}, \quad \sum a_i = 1.$$

Introduce the following n commuting flows with time variables $\tau_1, ..., \tau_n$:

$$x_i \mapsto x_i, \; a_i \mapsto \frac{a_i e^{x_i \tau_i}}{a_1 e^{x_1 \tau_1} + ... + a_n e^{x_n \tau_n}}.$$

This dynamics preserves the hyperplane $a_1 + ... + a_n = 1$ corresponding to monic polynomials Q. For generic Q the transformation of the sum Q/P of simple fractions to the chain fraction

$$P/Q = x - p_1 + \frac{q_1}{x - p_2 + \frac{q_2}{... + \frac{...}{... + \frac{q_{n-1}}{x - p_n}}}}$$

defines n commuting flows on the space with coordinates $(p_1, ..., p_n, q_1, ..., q_{n-1})$ (this chain fraction corresponds to the complete flag manifold $F(1, ..., 1)$). We put $q_i = \exp(t_i - t_{i+1})$. It is easy to check that the dynamics of the Toda system (in the (p, t)-space) coincides with the diagonal flow with $\tau_1 = ... = \tau_n = \tau$.

I am thankful to N. Reshetikhin who pointed to me the references [27, 26] where these facts about Toda lattices are described.

Despite of several recent papers (see for instance [24]), the actual relation of quantum cohomology with Toda dynamics as well as the interrelations between quantum cohomology algebras of partial flag manifolds (whose spectra fit nicely as various strata in the space of polynomials Q) yet to be understood.

6 Floer theory and D-modules

Structural constants $\langle \alpha * \beta, \gamma \rangle$ of the quantum cup-product are derivatives $\partial_\beta F_{\alpha,\gamma}$ of the same function. This allows to interpret the WDVV-equation as integrability condition of some connections $\nabla_\hbar$ on the tangent bundle T_H of the space $H = H^*(X, \mathbb{C})$. Namely, put $t = \sum t_\alpha p_\alpha$ where $p_1 = 1, p_2, \ldots, p_N$ is a basis in H and define

$$\nabla_\hbar = \hbar d - \sum (p_\alpha *) dt_\alpha \wedge : \Omega^0(T_H) \to \Omega^1(T_H)$$

where $p_\alpha *$ are operators of quantum multiplication by p_α. Then $\nabla_\hbar \circ \nabla_\hbar = 0$ *for each value of the parameter* $\hbar$. Notice that the integrability condition that reads "the system of differential equations $\hbar \partial_\alpha I = p_\alpha * I$ has solutions $I \in \Omega^0(TH)$" is actually obtained as a somewhat combinatorial statement (the WDVV-equation) about coefficients of the series F.

In [9], [10] we attempted to improve this unsatisfactory explanation of the integrability property by describing a direct geometrical meaning of the solutions I in terms of S^1-equivariant Floer theory on the loop space LX. Briefly, the universal covering $\widetilde{LX}$ carries the action of the covering transformation lattice $\pi_2(X)$ with generators $q_1, \ldots, q_k$ and the S^1-action by rotation of loops which preserves natural symplectic forms $\omega_1, \ldots, \omega_k$ on LX and thus defines corresponding Hamiltonians $H_1, \ldots, H_k$ on $\widetilde{LX}$ (the action functionals). The Duistermaat–Heckman forms $\omega_i + \hbar H_i$ (here $\hbar$ is the generator of $H^*_{S^1}(pt)$) are equivariantly closed, and operators p_i of exterior multiplication by these forms have the following Heisenberg commutation relations with the covering transformations:

$$p_i q_j - q_j p_i = \hbar q_j \delta_{ij}.$$

Conjecturally, this provides S^1-equivariant Floer cohomology of $\widetilde{LX}$ with a $\mathcal{D}$-module structure which is equivalent to the above system of differential equations (restricted to $t = 0$, $q \neq 0$) and reduces to the quantum cohomology algebra in the quasi-classical limit $\hbar = 0$ (see [9, 7]).

In this section we describe solutions to $\nabla_\hbar I = 0$ by imitating the S^1-equivariant Floer theory (which is still to be constructed) within the framework of Gromov–Witten theory. This construction turns out to be crucial in our proof in Section 11 of the mirror conjecture for Calabi-Yau projective complete intersections.

One may think of the graph of an algebraic loop $\mathbb{C}P^1 \backslash \{0, \infty\} \to X$ of degree d as of a stable map $\mathbb{C}P^1 \to X \times \mathbb{C}P^1$ of bidegree $(d, 1)$. Denote $L_d(X)$ a moduli space of genus zero stable maps to $X \times \mathbb{C}P^1$ of bidegree $(d, 1)$ (we do not specify the number of marked points in this notation). Our starting point consists in interpretation of $L_d(X)$ of such as a degree-d

approximation to $\widetilde{LX}$ and application of equivariant Gromov–Witten theory to the action of S^1 on the second factor $\mathbb{C}P^1$ with the fixed points $\{0, \infty\}$.

In the theorem below we assume X to be convex. It is natural to expect however that the theorem holds true whenever the non-equivariant Gromov–Witten theory works for X since the S^1-action is non-trivial only on the factor $\mathbb{C}P^1$ which is convex on its own.

Let $\langle\ ,\ \rangle$ be the Poincare pairing on $H = H^*(X, \mathbb{C})$. The equivariant cohomology algebra $H^*_{S^1}(X \times \mathbb{C}P^1)$ is isomorphic to $H \otimes_{\mathbb{C}} \mathbb{C}[p, \hbar]/(p(p-\hbar))$ with the S^1-equivariant pairing

$$(\varphi, \psi) = \frac{1}{2\pi i} \oint \frac{\langle \varphi, \psi \rangle dp}{p(p-\hbar)}.$$

Localization in $\hbar$ allows to introduce coordinates $\varphi = tp/\hbar + \tau(\hbar - p)/\hbar$, $\tau, t \in H$, diagonalizing the equivariant pairing:

$$((\tau, t), (\tau', t')) = \frac{\langle t, t' \rangle - \langle \tau, \tau' \rangle}{\hbar}.$$

The potential $\mathcal{F}(t, \tau, \hbar, q, q_0)$ satisfying the equivariant WDVV-equation for $X \times \mathbb{C}P^1$ expands as

$$\mathcal{F} = \mathcal{F}^{(0)} + q_0 \mathcal{F}^{(1)} + q_0^2 \mathcal{F}^{(2)} \dots$$

according to contributions of stable maps of degree $0, 1, 2, \dots$ with respect to the second factor. Denote $F = F(t, q)$ the potential (1) of the GW-theory for X.

Theorem 6.1 (a) $\mathcal{F}^{(0)} = (F(t, q) - F(\tau, q))/\hbar$.

(b) *The matrix* $(\Phi_{\alpha\beta}) := (\partial^2 \mathcal{F}^{(1)}/\partial \tau_\alpha \partial t_\beta)$ *is a fundamental solution of* $\nabla_{\pm\hbar} I = 0$*:*

$$-\hbar \frac{\partial}{\partial \tau_\gamma} \Phi = \hat{p}_\gamma(t) \Phi\ ,$$

$$\hbar \frac{\partial}{\partial t_\gamma} \Phi^* = \hat{p}_\gamma(\tau) \Phi^*\ ,$$

where $\hat{p}_\gamma = (p_\alpha^\beta)_\gamma$, $\gamma = 1, \dots, N$, *are matrices of quantum multiplication by* p_γ, *and* Φ^* *is transposed to* Φ.

Proof. Moduli spaces of bidegree-$(d, 0)$ stable maps to $X \times \mathbb{C}P^1$ coincide with $X_{n,d} \times \mathbb{C}P^1$. This implies (a) and shows that the WDVV-equation for $\mathcal{F}$ *modulo* q_0 follows from the

WDVV-equation for F. Part (b) follows now directly from the WDVV-equation for $\mathcal{F}$ *modulo* q_0^2 and from

$$\hbar\frac{\partial}{\partial t_1}\Phi_{\alpha\beta} = \Phi_{\alpha\beta} = -\hbar\frac{\partial}{\partial \tau_1}\Phi_{\alpha\beta}$$

due to (5) and (6). Here $\partial/\partial t_1, \partial/\partial\tau_1$ are derivatives in the direction $1 \in H^*(X)$ of the identity components of t and τ respectively.

The following corollary is obtained by expressing equivariant correlators $\Phi_{\alpha\beta}$ via localization of equivariant cohomology classes of moduli spaces $L_d(X)$ to fixed points of the S^1 action.

Define the matrix $\psi = (\phi_{\alpha\beta}(t,q,\hbar))$ by

$$\psi_{\alpha\beta} = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_d q^d\langle\frac{p_\beta}{\hbar+c}, t, \ldots, t, p_\alpha\rangle_{n+2,d} \tag{7}$$

where $c := c^{(1)}$ is the first Chern class of the line bundle over $X_{k,d}$ introduced in Section 2 as "the universal tangent line at the first marked point", and $\langle\frac{p_\beta}{\hbar+c}, p_\alpha\rangle_{2,0} := \langle p_\alpha, p_\beta\rangle$.

Corollary 6.2 *$\hbar\partial\psi/\partial t_\gamma = \hat{p}_\gamma(t)\psi$, i.e., the matrix ψ is (another) fundamental solution of $\nabla_\hbar I = 0$.*

Proof. A fixed point in $L_d(X)$ is represented by a stable map $C_0 \cup \mathbb{C}P^1 \cup C_\infty \to X \times \mathbb{C}P^1$ where $\varphi_i : C_i \to X \times \{i\}$ are stable maps of degrees $d_0 + d_\infty = d$ connected by a "constant loop" $\mathbb{C}P^1 \xrightarrow{\cong} \{x\} \times \mathbb{C}P^1$ (notice that $d_i = 0$ corresponds to empty C_i.) Thus components of $L_d(X)^{S^1}$ can be identified with submanifolds in $X_{d_0,k_0+1} \times X_{d_\infty,k_\infty+1}$ defined by the diagonal constraint $e_1(\varphi_0) = e_1(\varphi_\infty)$, with $\hbar^2(\hbar + c(0))(\hbar - c(\infty))$ to be the equivariant Euler class of the normal bundle. This gives rise to

$$\hbar^2\Phi_{\alpha\beta} = \sum_{\varepsilon,\varepsilon'}\psi_{\alpha\varepsilon}(\tau,-\hbar)\ \eta^{\varepsilon\varepsilon'}\psi_{\varepsilon'\beta}(t,h) \tag{8}$$

where $\sum\eta^{\varepsilon\varepsilon'}p_\varepsilon \otimes p_{\varepsilon'}$ is the coordinate expression of the diagonal cohomology class of $X \subset X \times X$. Now the differential equations of Theorem 6.1 for Φ imply the differential equation for ψ.

We give here several reformulations which will be convenient for computation of quantum cohomology algebras in Sections 9 – 11.

Consider the specialization of the connection $\nabla_\hbar$ to the parameter subspace corresponding to the deformation of the quantum cup-product along the 2-nd cohomology (this is accomplished by putting first $t=0$ and then replacing q^d by $\exp \sum d_i t_i$ where $(t_1, ..., t_k)$ are coordinates on $H^2(X)$ with respect to the basis $p^{(1)}, ..., p^{(k)} \in H^2(X)$. In this new setting put

$$s_{\alpha,\beta} := \sum_d e^{dt} \langle p_\beta \frac{e^{pt/\hbar}}{\hbar + c}, p_\alpha \rangle_{2,d}$$

where $pt := \sum p^{(i)} t_i$ and $dt = \sum d_i t_i$.

Corollary 6.3. *The matrix $(s_{\alpha,\beta}(t))$ is a fundamental solution to*

$$\nabla_\hbar \; s = 0 : \; \hbar \frac{\partial}{\partial t_i} \; s = \hat{p}^{(i)} \; s.$$

Proof. One should combine Corollary 6.2 with iterative applications of the following symmetries generalizing (5), (6):

$$\langle f(c), ..., 1 \rangle_{n+1,d} = \langle \frac{f(0) - f(c)}{c}, ... \rangle_{n,d},$$

$$\langle f(c), ..., p^{(i)} \rangle_{n+1,d} = d_i \langle f(c), ... \rangle_{n,d} + \langle p^{(i)} \frac{f(0) - f(c)}{c}, ... \rangle_{n,d}.$$

Here f is a function of one variable with values in $H^*(X)$.

The symmetries are easily verified on the basis of the following geometrical properties of universal tangent lines:

(i) Consider the push-forward along the map $\pi : X_{n+1,d} \to X_{n,d}$ (forgetting the last marked point). It is easy to see that the difference $\pi^*(c) - c$ between the Chern class of the universal tangent line at the 1-st marked point and the pull-back of its counterpart from $X_{n,d}$ is represented by the fundamental cycle of the section $i : X_{n,d} \to X_{n+1,d}$ defined by the first marked point.

(ii) $i^*(c) = c$.

In particular $\pi_*(1/(\hbar + c)) = 1/[\hbar(\hbar + c)]$.

Corollary 6.4. *Consider the functions*

$$s_\beta := \sum_d e^{dt} \langle p_\beta \frac{e^{pt/\hbar}}{\hbar + c}, 1 \rangle_{2,d} \; .$$

Let $P(\hbar\partial/\partial t, \exp t, \hbar)$ be a differential operator annihilating simultaneously all the functions s_β. Then the relation $P(p^{(1)}, ..., p^{(k)}, q_1, ..., q_k, 0) = 0$ holds in the quantum cohomology algebra of X (we assume here that P depends only on non-negative powers of $\hbar$).

Proof. The functions s_β form the first row in the fundamental solution matrix $S = (s_{\alpha,\beta})$. Application of the differential operator P to the matrix S is equal to the matrix product $(P_0 + \hbar P_1 + ... + \hbar^M P_M)S$ where P_0 is the matrix of the quantum multiplication by the symbol $P(p, \exp t, 0)$. Our hypotheses mean that the first row in this product vanishes. Since the fundamental solution matrix S is non-degenerate, this implies that the first row in P_0 vanishes too. In other words, $\langle P(p, q, 0), p_\beta \rangle = 0$ for all β and thus $P(p, q, 0) = 0$ in the quantum cohomology algebra.

All results of this section extend literally to the equivariant setting and/or to the generalization to convex vector bundles described in Section 4. We will apply them in this extended form in Sections 9–11.

Remarks. 1) The universal formula (7) for solutions of $\nabla_\hbar I = 0$ was perhaps discovered independently by several authors. I first learned this formula from R. Dijkgraaf. It can also be found in [2] in the *axiomatic* context of conformal topological field theory. One can prove it directly from a recursion relation (in the spirit of WDVV-equation) for so-called *gravitational descendents* — correlators involving the first Chern classes of the universal tangent lines (or, in a slightly different manner, by describing explicitly the divisor in $X_{n,d}$ representing c). Our approach provides an interpretation of (7) in terms of fixed point localization in equivariant cohomology.

2) One can generalize our theorem to bundles over $\mathbb{C}P^1$ with the fiber X. This seems to indicate that a straightforward "open-string" approach to S^1-equivariant Floer theory on $\widetilde{LX}$ would be more powerful and flexible than the approximation by Gromov–Witten theory on $X \times \mathbb{C}P^1$ described above.

3) Although the theorem provides a geometrical interpretation of solutions to $\nabla_\hbar I = 0$, it does not eliminate the combinatorial nature of the integrability condition. Indeed, the theorem is deduced from an equivariant WDVV-equation which in its turn can be interpreted as an integrability condition. Of course one can explain it using the $S^1 \times S^1$-equivariant WDVV-equation on $(X \times \mathbb{C}P^1) \times \mathbb{C}P^1$, etc. It would be interesting to find out whether this process converges.

7 Frobenius structures

In [2], B. Dubrovin studied geometrical structures defined by solutions of WDVV-equations on the parameter space and reduced classification of generic solutions to the classification of trajectories of some Euler-like non-autonomous Hamiltonian systems on so_N^*. We show here how this approach to equivariant Gromov–Witten theory yields analogous Hamiltonian systems on the affine Lie coalgebras $\widehat{so}_N^*$.

The quantum cup-product on $H = H_G^*(X)$ considered as an N-dimensional vector space over the field of fractions K of the algebra $H_G^*(pt)$ defines a formal *Frobenius structure* on H. The structure consists of the following ingredients.

1. A symmetric K-bilinear inner product $\langle\ ,\ \rangle$,

2. a (formal) function $F : H \to K$ whose third directional derivatives $\langle a * b, c\rangle := F_{a,b,c}$ provide tangent spaces T_tH with the Frobenius algebra structure (i.e. associative commutative multiplication $*$ satisfying $\langle a * b, c\rangle = \langle a, b * c\rangle$).

3. The constant vector field $\mathbb{1}$ of unities of the algebras $(T_tH, *)$ whose flow preserves the multiplication $*$ (i.e. $L_{\mathbb{1}}(*) = 0$).

4. Grading: In the non-equivariant case axiomatically studied by B. Dubrovin it can be described by the *Euler* vector field E, such that the tensor fields $\mathbb{1}$, $*$ and $\langle\ ,\ \rangle$ are homogeneous (i.e. are eigen-vectors of the Lie derivative L_E) of degrees $-1, 1$ and D respectively (where $D = \dim_{\mathbb{C}} X$ in the models arising from the GW-theory). In the equivariant GW-theory this grading axiom should be slightly modified since the grading of the structural ring $H_G^*(pt)$ is non-trivial and thus the natural Euler operator L_E is $\mathbb{C}$-linear but not K-linear [3]

The fact that the multiplication $*$ is defined on tangent vectors to H means that the algebra $(\Omega^0(T_H), *)$ can be naturally considered as the algebra $K[L]$ of regular functions on some subvariety $L \subset T^*H$ in the cotangent bundle. A point $t \in H$ is called *semi-simple* if

[3]One may also think of the $H_G^*(pt)$-module $H_G^*(X)$ as of the module of sections of a vector bundle over the spectrum of $H_G^*(pt)$. The fibers of the bundle then carry Frobenius structures satisfying the axioms 1 – 3 while the Euler vector field is not tangent to the fibers.

the algebra $(T_tH, *)$ is semi-simple, that is if $L \cap T_t^*H$ consists of N linearly independent points.

Flatness of the connection (defined on T_H)

$$\nabla_\hbar = \hbar d - \sum_\alpha p_\alpha * dt_\alpha \tag{9}$$

implies [7] that L is a Lagrangian submanifold in T^*H near a semisimple t. Following [2], introduce local *canonical coordinates* $(u_1, \ldots, u_N)$ such that the sections $(du_1, \ldots, du_N)$ of T^*H are the N branches of L near t, and transform the connections $\nabla_\hbar$ to these local coordinates and to a (suitably normalized) basis $f_1, \ldots, f_N$ of vector field on H diagonalizing the $*$-product.

The result of this transformation can be described as follows.

(a) The basis $\{f_i\}$ can be normalized in a way that in the transformed form

$$\nabla_\hbar = \hbar d - \hbar A^1 \wedge - D^1 \wedge \tag{10}$$

of the connection $\nabla_\hbar$ with $D^1 = \mathrm{diag}(du_1, \ldots, du_N)$, and $A_{ij} = V_{ij}(u)d(u_i - u_j)/(u_i - u_j)$ for all $i \neq j$, we will have additionally $A_{ii} = 0\ \forall i$.

(b) The vector field $\mathbb{1}$ in the canonical coordinates assumes the form $\sum_k \partial_k$ where $\partial_k := \partial/\partial u_k$ are the canonical idempotents of the $*$-product:

$$\partial_i * \partial_j = \delta_{ij}\partial_j. \tag{11}$$

(c) The (remaining part of the) integrability condition $\nabla_\hbar^2 = 0$ reads $d(A^1) = A^1 \wedge A^1$ or

$$\partial_i \phi_\alpha^j = \phi_\alpha^i V_{ij}/(u_i - u_j),\ i \neq j, \tag{12}$$

where (ϕ_α^j) is the transition matrix, $\partial/\partial t_\alpha = \sum_i \phi_\alpha^i f_i$; it can be reformulated as compatibility of the PDE system (12) for (ϕ_α^j) completed by

$$\sum_k \partial_k \phi_\alpha^j = 0. \tag{13}$$

(d) The Frobenius property $\langle a * b, c\rangle = \langle a, b * c\rangle$ of the $*$-product shows that the diagonalizing basis $\{f_i\}$ is orthogonal, that its normalization by $\langle f_i, f_j\rangle = \delta_{ij}$ obeys $A_{ii} = 0$ and, additionally, implies anti-symmetricity $A_{ij} = -A_{ji}$, or

$$V_{ij} = -V_{ji}. \tag{14}$$

The presence of the grading axiom (4) of Frobenius structures over $K = \mathbb{C}$ allows B.Dubrovin to describe anti-symmetric matrices $V = (V_{ij}) \in so_N^*$ satisfying the integrability conditions (12) and (13) in *quasi-homogeneous* canonical coordinates (i.e. $L_E u_i = u_i$ so that $E = \sum u_k \partial_k$) as trajectories of N commuting non-autonomous Hamiltonian systems (see [2]):

$$\partial_i V = \{H_i, V\}$$

where the Poisson-commuting non-autonomous quadratic Hamiltonians H_i on so_N^* are given by

$$H_i = \frac{1}{2} \sum_{j \neq i} \frac{V_{ij} V_{ji}}{u_i - u_j}.$$

Consider now the following model modification of the grading axiom: $K = \mathbb{C}[[\lambda^{\pm 1}]]$, $\deg \lambda = 1$. In quasi-homogeneous canonical coordinates $(u_1, \ldots, u_N, \lambda)$ the Euler vector field takes then on

$$L_E = \sum_k u_k \partial_k + \lambda \partial_\lambda. \tag{15}$$

Introduce the connection operator

$$\mathbb{V} = \lambda \partial_\lambda - V \in \widehat{so}_N^*$$

and the qudratic Hamiltonians on the Poisson manifold $\widehat{so}_N^*$

$$\mathcal{H}_i(\mathbb{V}) = \oint H_i(V) \frac{d\lambda}{\lambda}. \tag{16}$$

Proposition 7.1 *The Hamiltonians $\mathcal{H}_1, \ldots, \mathcal{H}_N$ are in involution. The operator $\mathbb{V}$ of a Frobenius manifold over K satisfies the non-autonomous system of Hamiltonian equations*

$$\partial_i \mathbb{V} = \{\mathcal{H}_i, \mathbb{V}\}, \; i = 1, \ldots, N. \tag{17}$$

The columns $\phi_\alpha = (\phi_\alpha^i)$ of the transition matrix are eigen-functions of the connection operator $\mathbb{V}$:

$$\mathbb{V} \phi_\alpha := (\lambda \partial_\lambda - V) \phi_\alpha = \left(\frac{n}{2} - \deg t_\alpha + 1 \right) \phi_\alpha. \tag{18}$$

Proof. It can be obtained by a straightforward calculation quite analogous to that in [2].

In our real life the model equations (15–18) describe the structure of Frobenius manifolds *over each semi-simple orbit of the grading Euler field in the ground parameter space. This parameter space is the spectrum of the coefficient algebra* $H^*_G(pt, \mathbb{C}) \otimes \mathbb{C}[q_1^{\pm 1}, \ldots, q_k^{\pm 1}]$ (its field of fractions can be taken on the role of the ground field K). An orbit of the Euler vector field in this parameter space is semi-simple if the corresponding $\mathbb{C}$-Frobenius algebras are semi-simple.

The equations (15–18) over semi-simple Euler orbits should be complemented by the additional symmetries (6).

In the next section we will show how the canonical coordinates of the axiomatic theory of Frobenius structures emerge from localization formulas in equivariant Gromov–Witten theory.

8 Fixed point localization

We consider here the case of a circle T^1 acting by Killing transformations on a compact Kahler manifold X with *isolated* fixed points only. The case of tori actions with isolated fixed points requires only slight modification of notations which we leave to the reader. Our results are rigorous for convex X (which includes homogeneous Kahher spaces of compact Lie group and their maximal tori) while applications to general toric manifolds (which are typically not convex) yet to be justified.

It is the Borel localization theorem that reduces computations in torus-equivariant cohomology to computations near fixed points. Let $\{p_\alpha\}, \alpha = 1, ..., N$, be the fixed points of the action. We will denote with the same symbols p_α the equivariant cohomology class of X which restricts to $1 \in H^*_T(p_\alpha)$ at p_α and to 0 at all the other fixed points. These classes are well-defined over the field of fractions $\mathbb{C}(\lambda)$ of the coefficient ring $H^*_T(pt) = \mathbb{C}[\lambda]$ and form the basis of canonical idempotents in the semi-simple algebra $H^*_T(X, \mathbb{C}(\lambda)$. The equivariant Poincare pairing reduces to $\langle p_\alpha, p_\beta \rangle = \delta_{\alpha,\beta}/e_\beta$ where $e_\alpha \in \mathbb{C}[\lambda]$ is the equivariant Euler class of the normal "bundle" $T_{p_\alpha}X \to p_\alpha$ to the fixed point.

The results described below apply to the setting of Section 4 of a manifold X provided with a convex vector bundle in which case e_α's should be modified accordingly.

The same localization theorem reduces computation of GW-invariants to that near the fixed point set (orbifold) in the moduli spaces $X_{n,d}$. A fixed point in the moduli space is represented by a stable map to X of a (typically reducible) curve C such that each component of C is mapped to (the closure of) an orbit of the complexified action $T_{\mathbb{C}} : X$. Any such an orbit is either one of the fixed points p_α or isomorphic to $(\mathbb{C}-0)$ connecting two distinct fixed points p_α and p_β corresponding to 0 and ∞. Respectively, there are two types of components of C:

(i) Each component of C which carries 3 or more special points must be mapped to one of the fixed points p_α.

(ii) All other components are multiple covers $z \mapsto z^d$ of the non-constant orbits, and their special points may correspond only to $z = 0$ or ∞.

The *combinatorial structure* of such a stable map can be described by a tree whose edges correspond to *chains* of components of type (ii) and should be labeled by the total degree of this chain as a curve in X, and vertices correspond to the ends of the chains. The ends may carry 0 or 1 marked point, or correspond to a (tree of) type-(i) components with 1 or more marked points and should be labeled by the indices of these marked points and by the target point p_α.

The fixed stable maps with different combinatorial structure belong to different connected components of the fixed point orbifold in $X_{n,d}$.

The results below are based on the observation that a stable map with the first $k \geq 3$ marked points in a *given* generic configuration (i.e. with the given generic value of the contraction map $X_{n.d} \to \bar{\mathcal{M}}_{0,k}$) must have in an irreducible component C_0 in the underlying curve C which contains this given configuration of k *special* points, (so that the corresponding first k marked points are located on the branches outgoing these special points of C_0). The cause is hidden in the definition of the contraction map (see [3, 6]).

We will call the component C_0 *special*.

The observation applied to a fixed stable map of the circle action allows to subdivide all fixed point components in $X_{3+n,d}$ into N *types* p_i according to the fixed points p_i where the special component is mapped to. We introduce the superscript notation $(...)^i$ for the contribution (via Borel's localization formulas) of type-p_i components into various equivariant correlators. For example,

$$F^i_{\alpha\beta\gamma} = \sum_n \frac{1}{n!} \sum_d q^d \langle p_\alpha, p_\beta, p_\gamma, t, ..., t\rangle^i_{3+n,d}$$

where $t = \sum_{\alpha=1}^N t_\alpha p_\alpha$ is the general class in $H^*_T(X, \mathbb{C}(\lambda))$, so that $F_{\alpha\beta\gamma} = \sum_i F^i_{\alpha\beta\gamma}$.

We introduce also the notations

- $\Psi^i_{\alpha\beta}$ — for contributions to $e_i F^i_{\alpha\beta i}$ of those fixed points which have the third marked point situated directly on the special component C_0 (it is convenient here to introduce the normalizing factor $e_i \in H^*_T(pt)$, the Euler class of the normal "bundle" to the fixed point p_i in X);

- $\Psi^i_\alpha := \Psi^i_{\alpha 1\!\!1} = \sum_\beta \Psi^i_{\alpha\beta}$;

- D^i_α — for contributions to $e_i F_{\alpha ii}$ of those fixed points which have the second and third marked points situated directly on the special component;

- Δ^i — for contributions to $e_i F_{iii}$ of those fixed points which have the first three marked points situated directly on the special component;

- $u_i = t_i +$ contributions to $e_i F_{ii}$ of all those fixed point components in $X_{2+n,d}$ for which the first two marked points belong to the same vertex of the tree describing the combinatorial structure.

The correlators u_i can be also interpreted as contributions to the *genus*-1 equivariant correlators

$$\sum_n \frac{1}{n!} \sum_d q^d (t, ..., t)_{n,d}$$

with *given* complex structure of the elliptic curve of those T-invariant classes which map the (only) genus-1 component of the curve C to the fixed point p_i.

Theorem 8.1. *(a) The functions $u_1(t), ..., u_N(t)$ are the canonical coordinates of the Frobenius structure on $H^*_G(X, \mathbb{C}(\lambda))$.*

(b) The functions $D^i_\alpha(t)$ are eigen-values of the quantum multiplication by p_α: $du_i = \sum_\alpha D^i_\alpha dt_\alpha$.

(c) The transition matrix (Ψ^i_α) provides simultaneous diagonalization of the quantum cup product: $F^i_{\alpha\beta\gamma} = \Psi^i_\alpha D^i_\beta \Psi^i_\gamma$ and obeys the following orthogonality relations:

$$\sum_i \Psi^i_\alpha \Psi^i_\beta = \delta_{\alpha\beta}/e_\beta, \sum_\alpha \Psi^i_\alpha \Psi^j_\alpha = \delta_{ij} \ .$$

(d) The Euclidean structure on the cotangent bundle of the Frobenius manifold (defined by the equivariant intersection pairing in $H^*_T(X)$) in the canonical coordinates u_i takes on $\langle du_i, du_j\rangle = (\Delta^i)^2 \delta_{ij} e_j$ and additionally

$$(\Delta^i)^{-1} = \sum_\alpha \Psi^i_\alpha, \ \ \Psi^i_\alpha = \frac{D^i_\alpha}{\Delta^i}, \ \ \Psi^i_{\alpha\beta} = \frac{D^i_\alpha D^i_\beta}{\Delta^i} \ .$$

Proof. We first apply the localization formula

$$A_{1234}\langle ... \rangle_{n,d} = \sum_i A_{1234}\langle ... \rangle^i_{n,d}$$

to the 4-point equivariant correlators with the fixed cross-ratio z of the 4 marked points and only after this specialize the cross-ratio to 0,1 or ∞. This gives rise to the *local* WDVV-identities

$$\Psi^i_{\alpha\beta}\Psi^i_{\gamma\delta} \textit{ is totally symmetric in } \ \alpha, \beta, \gamma, \delta$$

which is independent of the global WDVV-equation. When combined with the global identities

$$A_{1234}\langle \mathbb{1}, p_\alpha, p_\beta, p_\gamma\rangle_{n,d} = \langle p_\alpha, p_\beta, p_\gamma\rangle_{n,d}$$

they yield the orthogonality relation $\sum_i \Psi^i_\alpha \Psi^i_\beta = \delta_{\alpha\beta}/e_\beta$ and localization formulas

$$F_{\alpha\beta\gamma} = \sum_i \Psi^i_{\alpha\beta}\Psi^i_\gamma$$

for the structural constants of the quantum cup-product.

A similar argument with $>$ 4-point correlators $A_{12345...}\langle ... \rangle^i$ proves the diagonalization

$$\langle p_\alpha * p_\beta, p_\gamma\rangle = \sum_i \Psi^i_\alpha D^i_\beta \Psi^i_\gamma / e_i \ ,$$

$$\langle p_\alpha * p_\beta * p_\gamma, p_\delta\rangle = \sum_i \Psi^i_\alpha D^i_\beta D^i_\gamma \Psi^i_\delta / e_i$$

and the identities

$$\Psi^i_{\alpha\beta} = \Psi^i_\alpha D^i_\beta,\ (\Delta^i)^{-1} = \sum_\alpha \Psi^i_\alpha\ .$$

Finally, the identity $du_i = \sum_\alpha D^i_\alpha dt_\alpha$ follows directly from the definition of u_i and implies that $u_1, ..., u_N$ are the canonical coordinates of the Frobenius structure.

9 Projective complete intersections

We are going to describe explicitly solutions of the differential equations arising from quantum cohomology of projective complete intersections. Lex X be such a non-singular complete intersection in $Y := \mathbb{C}P^n$ given by r equations of the degrees $(l_1, ..., l_r)$. If $l_1 + ... + l_r = n+1$ then X is a Calabi-Yau manifold and its quantum cohomology is described by the mirror conjecture. In this and the next sections we study respectively the cases $l_1 + ... + l_r < n$ and $l_1 + ... + l_r = n$ when the 1-st Chern class of X is still positive. In the case $l_1 + ... + l_r > n+1$ (which from the point of view of enumerative geometry can be considered as "less interesting" for rational curves generically occur only in finitely many degrees) the "mirror symmetry" problem of hypergeometric interpretation of quantum cohomology differential equations remains open.

Let E_d be the Euler class of the vector bundle over the moduli space $Y_{2,d}$ of genus 0 degree d stable maps $\phi : (C, x_0, x_1) \to \mathbb{C}P^n$ with two marked points, with the fiber $H^0(C, \phi^* H^{l_1} \oplus ... \oplus \phi^* H^{l_r})$ where H^l is the l-th tensor power of the hyperplane line bundle over $\mathbb{C}P^n$.

Consider the class

$$S_d(\hbar) := \frac{1}{\hbar + c_1^{(0)}}\ E_d\ \in\ H^*(Y_{2,d})$$

where $c_1^{(0)}$ is the 1-st Chern class of the "universal tangent line at the marked point x_0 ", and e_0, e_1 are the evaluation maps. Due to the factor E_d this class represents the push forward along $X_{2,d} \to Y_{2,d}$ of the class $1/(\hbar + c_1^{(0)})\ \in H^*(X_{2,d})$ (by the very construction of $X_{2,d}$ in Section 4).

In the cohomology algebra $\mathbb{C}[P]/(P^{n+1})$ of $\mathbb{C}P^n$, consider the class

$$S(t, \hbar) := e^{Pt/\hbar} \sum_{d=0}^{\infty} e^{dt} (e_0)_* (S_d(\hbar))$$

where $(e_0)_*$ represents the push-forward along the evaluation map (and for $d = 0$, when $Y_{2,d}$ is not defined, we take $Euler\ (\oplus_j H^{\otimes l_j})$ on the role of $(e_0)_* S_0$).

Considered as a function of t, S is a curve in $H^*(\mathbb{C}P^n)$ whose components are solutions of the differential equation we are concerned about. Indeed, according to Section 6 a similar sum represents the solutions of the quantum cohomology differential equation for X, and S is just the push-forward of that sum from $H^*(X)$ to $H^*(Y)$. (Strictly speaking S carries information only about correlators between those classes which come from the ambient projective space; also if X is a surface $\operatorname{rk} H_2(X)$ can be greater than 1 and S mixes information about the curves of different degrees in X when they have the same degree in Y.)

Theorem 9.1. *Suppose that* $l_1 + ... + l_r < n$. *Then*

$$S = e^{Pt/\hbar} \sum_{d=0}^{\infty} e^{dt} \frac{\Pi_{m=0}^{dl_1}(l_1 P + m\hbar)...\Pi_{m=0}^{dl_r}(l_r P + m\hbar)}{\Pi_{m=1}^{d}(P + m\hbar)^{n+1}}.$$

The formula coincides with those in [9, 10] (found by analysis of toric compactifications of spaces of maps $\mathbb{C}P^1 \to \mathbb{C}P^n$) for solutions of differential equations in S^1-equivariant cohomology of the loop space.

Corollary 9.2. (see [9, 10]) *The components* $s := \langle P^i, S\rangle, i = 0, ..., n - r$, *of* S *form a basis of solutions to the linear differential equation*

$$(\hbar \frac{d}{dt})^{n+1-r}\ s\ =\ e^t \Pi_{j=1}^{r}\ l_j\ \Pi_{m=1}^{l_j - 1}\ \hbar(l_j \frac{d}{dt} + m)\ s\ .$$

This implies (combine [10] with [18]) that the solutions have an integral representation of the form

$$\int_{\gamma^{n-r} \subset X'_t} e^{(u_0 + ... + u_n)/\hbar}\ \frac{du_0 \wedge ... \wedge du_n}{dF_0 \wedge dF_1 \wedge ... \wedge dF_r}$$

where

$$F_0 = u_0 ... u_n,\ F_1 = u_1 + ... u_{l_1},\ F_2 = u_{l_1+1} + ... + u_{l_1+l_2}, ...,\ F_r = u_{l_1+...+l_{r-1}+1} + ... + u_{l_1+...+u_{l_r}}$$

and the "mirror manifolds" X'_t are described by the equations

$$X'_t = \{(u_o, ..., u_n) \mid F_0(u) = e^t,\ F_1(u) = 1, ...,\ F_r(u) = 1\}.$$

This proves for X the mirror conjecture in the form suggested in [10].

Corollary 9.3. *If* $\dim_{\mathbb{C}} X \neq 2$ *the cohomology class* p *of hyperplane section satisfies in the quantum cohomology of* X *the relation*

$$p^{n+1-r} = l_1^{l_1}...l_r^{l_r} q p^{l_1+...+l_r-r}.$$

When X is a surface the same relation holds true in the quotient of the quantum cohomology algebra which takes in account only degrees of curves in the ambient $\mathbb{C}P^n$ (we leave to figure out a precise description of this quotient to the reader; quadrics $\mathbb{C}P^1 \times \mathbb{C}P^1$ in $\mathbb{C}P^3$ provide a good example: $(p_1+p_2)^3 = 4q(p_1+p_2) \mod p_1^2 = q = p_2^2$.)

This corollary is consistent with the result of A. Beauville [22] describing quantum cohomology of complete intersections with $\sum l_j \leq n+1-\sum(l_j-1)$ and with results of M. Jinzenji [23] on quantum cohomology of projective hypersurfaces ($r=1$) with $l_1 < n$.

Corollary 9.4. *The number of degree* d *holomorphic maps* $\mathbb{C}P^1 \to X^{n-r} \subset \mathbb{C}P^n$*, which send* 0 *and* ∞ *to two given cycles and send* $n+1-r$ *given points in* $\mathbb{C}P^1$ *to* $n+1-r$ *given generic hyperplane sections, is equal to* $l_1^{l_1}...l_r^{l_r}$ *times the number of degree* $d-1$ *maps which send* 0 *and* ∞ *to the same cycles and* $l_1+...+l_r-r$ *given points — to* $l_1+...+l_r-r$ *given hyperplane sections.*

This is the enumerative meaning of Corollary 9.3; of course in this formulation numerous general position reservations are assumed.

Control examples. 1. $l_1 = ... = l_r = 1$: The above formulas for quantum cohomology and for solutions of the differential equations in the case of a hyperplane section give rise to the same formulas with $n := n-1$.

2. $n=5, r=1, l=2$: X is the Plucker embedding of the grassmanian $Gr_{4,2}$. Its quantum cohomology algebra is described by the relations $c_1^3 = 2c_1c_2$, $c_2^2 - c_2c_1^2 + q = 0$ between the Chern classes of the tautological plane bundle. For the 1-st Chern class $p = -c_1$ of the determinant line bundle we deduce the relation $p^5 = 4pq$ prescribed by Corollary 9.3.

We will deduce Theorem 9.1 from its equivariant generalization. Consider the space $\mathbb{C}^{n+1}$ provided with the standard action of the $(n+1)$-dimensional torus T. The equivariant cohomology algebra of $\mathbb{C}^{n+1}$ coincides with the algebra of characteristic classes $H^*(BT^{n+1}) = \mathbb{C}[\lambda_0, ..., \lambda_n]$. The equivariant cohomology algebra of the projective space $(\mathbb{C}^{n+1}-0)/\mathbb{C}^\times$

in these notations is identified with $\mathbb{C}[p,\lambda]/((p-\lambda_0)...(p-\lambda_n))$ and the push-forward $H_T^*(\mathbb{C}P^n)\to H_T^*(pt)$ is given by the residue formula

$$f(p,\lambda)\mapsto \frac{1}{2\pi i}\oint \frac{f(p,\lambda)dp}{(p-\lambda_0)...(p-\lambda_n)}\ .$$

Here $-p$ can be considered as the equivariant 1-st Chern class of the Hopf line bundle provided with the natural lifting of the torus action. We will use $\phi_i := \Pi_{j\neq i}(p-\lambda_j),\ i=0,...,n$, as a basis in $H_T^*(\mathbb{C}P^n)$.

Consider the T-equivariant vector bundle $\oplus_{j=1}^r H^{\otimes l_j}$ and provide it with the fiberwise action of the additional r-dimensional torus T'. The equivariant Euler class of this bundle is equal to $(l_1p-\lambda'_1)...(l_rp-\lambda'_r)$ where $\mathbb{C}[\lambda'] = H^*(BT')$.

Introduce the equivariant counterpart S' of the class S in the $T\times T'$-equivariant cohomology of $\mathbb{C}P^n$. This means that we use the equivariant class p instead of P and replace the Euler classes E_d and $c_1^{(0)}$ by their equivariant partners.

Theorem 9.5. Let $l_1+...+l_r<n$. Then

$$S' = e^{pt/\hbar}\sum_{d=0}^{\infty} e^{dt}\frac{\Pi_0^{dl_1}(l_1p-\lambda'_1+m\hbar)...\Pi_0^{dl_r}(l_rp-\lambda'_r+m\hbar)}{\Pi_1^d(p-\lambda_0+m\hbar)...\Pi_1^d(p-\lambda_n+m\hbar)}\ .$$

Theorem 9.1 follows from Theorem 9.5 by putting $\lambda=0,\lambda'=0$ which corresponds to passing from equivariant to non-equivariant cohomology.

The vector-function S' satisfies the differential equation

$$\Pi_{i=0}^r(\hbar\frac{d}{dt}-\lambda_i)\ S' = e^t\Pi_{m=1}^{l_1}(l_1\hbar\frac{d}{dt}-\lambda'_1+m\hbar)...\Pi_{m=1}^{l_r}(l_r\hbar\frac{d}{dt}-\lambda'_r+m\hbar)\ S'.$$

We intend to prove Theorem 9.5 by means of localization of S' to the fixed point set of the torus T action on the moduli spaces $Y_{2,d}$. As it is shown in [3], all correlators of the equivariant theory on $\mathbb{C}P^n$ are computable at least in principle, and in practice the computation reduces to a recursive procedure which can be understood as a summation over trees and can be also formulated as a non-linear fixed point (or critical point) problem. We will see below that in the case of correlators $\langle\phi_i, S'\rangle$ certain reasons of a somewhat geometrical character cause numerous cancellations between trees so that the recursive procedure reduces to a "summation over chains" and respectively to a *linear* recurrence equation. The formula of Theorem 9.5 is simply the solution to this equation.

In the proof of Theorem 9.5 below we write down all formulas for $r = 1$ (it serves the case when X is a hypersurface in $\mathbb{C}P^n$ of degree $l < n$). The proof for $r > 1$ differs only by longer product formulas.

Let us abbreviate $c_1^{(0)}$ as c, denote E'_d the equivariant Euler class of the vector bundle over $Y_{2,d}$ whose fiber over the point $\psi : (C, x_0, x_1) \to Y = \mathbb{C}P^n$ consists of holomorphic sections of the bundle $\psi^*(H^l)$ *vanishing at* x_0, and introduce the following equivariant correlator:

$$Z_i := \sum_{d=0}^{\infty} q^d \int_{Y_{2,d}} e_0^*(\phi_i) \frac{1}{\hbar + c} E'_d.$$

We have

$$\langle \phi_i, S' \rangle = e^{\lambda_i t/\hbar} (l\lambda_i - \lambda')(Z_i|_{q=e^t})$$

Proposition 9.6.

$$Z_i = 1 + \sum_{d>0} (\frac{q}{\hbar^{n+1-l}})^d \int_{Y_{2,d}} \frac{(-c)^{(n+1-l)d-1}}{1 + c/\hbar} \; E'_d \; e_0^*(\phi_i) \; .$$

Proof. We have just dropped first several terms in the geometrical series $1/(\hbar + c)$ since their degree added with the degrees of other factors in the integral over $Y_{2,d}$ is still less than the dimension of $Y_{2,d}$. It is important here that all the equivariant classes involved including ϕ_i are defined in the equivariant cohomology over $\mathbb{C}[\lambda, \lambda']$ without any localization.

It is a half of the geometrical argument mentioned above. The other half comes from the description of the fixed point set in $Y_{2,d}$ given in [11].

Consider a fixed point of the torus T action on $Y_{2,d}$. It is represented by a holomorphic map of a possibly reducible curve with complicated combinatorial structure and with two marked points on some components. Each component carrying 3 or more special points is mapped to one of the $n+1$ fixed points of T on $\mathbb{C}P^n$, and the other components are mapped (with some multiplicity) onto the lines joining the fixed points and connect the point-mapped components in a tree-like manner.

In the Borel localization formula for $\int e_0^*(\phi_i)...$ the fixed point will have zero contribution unless the marked point x_0 is mapped to the i-th fixed point in $\mathbb{C}P^n$ (since ϕ_i has zero localizations at all other fixed points.

Consider a fixed point curve C whose marked point x_0 is indeed mapped to the i-th fixed point in $\mathbb{C}P^n$. There are two options

(i) the marked point x_0 is situated on an irreducible component of C mapped with some degree d' onto the line joining the i-th fixed point with the j-th fixed point in $\mathbb{C}P^n$ with $i \neq j$;

(ii) the marked point x_0 is situated on a component of C mapped to the i-th fixed point and carrying two or more other special points.

Consider first the option (ii) and the contribution of such a connected component of the fixed point set in $Y_{2,d}$ to the Borel localization formula for $\int c^{(n+1-l)d-1}...$. The connected component itself is the (product of the) Deligne-Mumford configuration space of, say, $s+1$ special points: the marked point x_0, may be the marked point x_1, and respectively $s-1$ or s endpoints of other components of C mapped onto the lines outgoing the i-th fixed point in $\mathbb{C}P^n$.

Lemma 9.7. *The type (ii) fixed point component in $Y_{2,d}$ has zero contribution to the Borel localization formula for $\int_{Y_{2,d}} c^{(n+1-l)d-1}...$*

Proof. Restriction of the class c from $Y_{2,d}$ to the type (ii) fixed point component coincides with the 1-st Chern class of the line bundle on the Deligne-Mumford factor $\bar{\mathcal{M}}_{0,s+1}$ of the component defined as "the universal tangent line as the marked point x_0" and is thus nilpotent in the cohomology of the component. Since the number of straight lines in a curve of degree d does not exceed d we find that the dimension $s-2$ of the factor $\bar{\mathcal{M}}_{0,s+1}$ is less than d which in its turn does not exceed $(n+1-l)d-1$ for $d>0$ (because we assumed that $n+1-l \geq 2$).

Consider now the option (i). The irreducible component C' of the curve $C = C' \cup C''$ carrying the marked point x_0 is mapped with the multiplicity $d' \leq d$ onto the line joining i-th fixed point in $\mathbb{C}P^n$ with the j-th one *while the remaining part $C'' \to \mathbb{C}P^n$ of the map represents a fixed point in $Y_{2,d-d'}$*. Moreover, the normal space to the fixed point component at the type (i) point (the equivariant Euler class of the normal bundle occurs in the denominator of the Borel localization formula) is the sum of (a) such a space N'' for $C'' \to \mathbb{C}P^n$, (b) the space N' of holomorphic vector fields along the map $C' \to \mathbb{C}P^n$ vanishing at the fixed point j factorized by infinitesimal reparametrizations of C', (c) the tensor product L of the tangent lines to C' and C'' at their intersection point. Since the space V of holomorphic sections of H^l restricted to C (and vanishing at x_0) admits a similar decomposition $V' \oplus V''$, we arrive to the following linear recursion relation for Z_i.

Proposition 9.8. *Put* $z_i(Q,\hbar) := Z_i(\hbar^{(n+1-l)}Q,\hbar)$. *Then*

$$z_i(Q,\hbar) = 1 + \sum_{j\neq i}\sum_{d'>0} Q^{d'} Coeff\ {}_i^j(d')\ z_j(Q,(\lambda_j-\lambda_i)/d')$$

where

$$Coeff\ {}_i^j(d') = \frac{[(\lambda_j-\lambda_i)/d']^{(n+1-l)d'-1}}{1+(\lambda_i-\lambda_j)/d'\hbar}\frac{Euler\ (V')}{Euler\ (N')}\ \phi_i|_{p=\lambda_i}\ .$$

Proof. Here $(\lambda_i-\lambda_j)/d'$ is the localization of c, and the key point is that the equivariant Chern class of the line bundle L over $Y_{2,d-d'}$ is what we would denote $\hbar + c$ *for the moduli space* $Y_{2,d-d'}$ but with $\hbar = (\lambda_j-\lambda_i)/d'$. This is how the recursion for the correlators z_i becomes possible. The rest is straightforward.

Remark. Our reduction to the linear recursion relation can be interpreted in the following more geometrical way: contributions of all non-isolated fixed points cancel out with some explicit part of the contribution from *isolated* fixed points; the latter are represented by chains of multiple covers of straight lines connecting the two marked points.

Let us write down explicitly the factor $Coeff\ {}_i^j(d)$ from Proposition 9.8 (compare with [3]). $Coeff\ {}_i^j(d) =$

$$\frac{\Pi_{m=1}^{ld}(l\lambda_i-\lambda'+m(\lambda_j-\lambda_i)/d)[(\lambda_j-\lambda_i)/d]^{(n+1-l)d-1}}{d(1+(\lambda_i-\lambda_j)/\hbar d)\Pi_{\alpha=0}^{n}\ {}_{m=1}^{d}\ {}_{(\alpha,m)\neq(j,d)}(\lambda_i-\lambda_\alpha+m(\lambda_j-\lambda_i)/d)} =$$

(here the product in the numerator is $Euler(V')$, the denominator — it is essentially $Euler(N')$ where however the cancellation with $\phi_i|_{p=\lambda_i}$ is taken care of — has been computed using the exact sequence $0\to\mathbb{C}\to\mathbb{C}^{n+1}\otimes H\to T_Y\to 0$ of vector bundles over $Y=\mathbb{C}P^n$, and the "hard-to-explain" extra-factor d is due to the orbifold structure of the moduli spaces (the d-multiple map of C' onto the (ij)-line in $\mathbb{C}P^n$ has a discrete symmetry of order d)

$$= \frac{1}{[(\lambda_i-\lambda_j)/\hbar + d]}\frac{\Pi_{m=1}^{ld}(l\frac{(\lambda_i-\lambda')d}{\lambda_j-\lambda_i}+m)}{\Pi_{\alpha=0}^{n}\ {}_{m=1}^{d}\ {}_{(\alpha,m)\neq(j,d)}(\frac{(\lambda_i-\lambda_\alpha)d}{\lambda_j-\lambda_i}+m)}\ .$$

Now it is easy to check

Proposition 9.9. *The correlators* $z_i(Q,1/\omega)$ *are power series* $\sum C_i(d)Q^d$ *in* Q *with coefficients* $C_i(d)$ *which are* reduced *rational functions of* ω *with poles of the order* ≤ 1 *at*

$\omega = d'/(\lambda_j - \lambda_i)$ with $d' = 1, ..., d$. The correlators z_i are uniquely determined by these properties, the recursion relations of Proposition 9.8 *and the initial condition $C_i(0) = 1$.*

The proof of Theorem 9.5 is completed by the following

Proposition 9.10. *The series*

$$z_i = \sum_{d=0}^{\infty} Q^d \frac{\Pi_{m=1}^{ld}((l\lambda_i - \lambda')\omega + m)}{d!\Pi_{\alpha\neq i}\Pi_{m=1}^{d}((\lambda_i - \lambda_\alpha)\omega + m)}$$

satisfy all the conditions of Proposition 9.9.

Proof. The recursion relation is deduced by the decomposition of the rational functions of ω into the sum of simple fractions (or, equivalently, from the Lagrange interpolation formula for each numerator through its values at the roots of the corresponding denominator).

10 Complete intersections with $l_1 + ... + l_r = n$

Let $X \subset Y = \mathbb{C}P^n$ be a non-singular complete intersection given by equations of degrees $(l_1, ..., l_r)$ with $l_1 + ... + l_r = n$. There are only two points where our proof of Theorem 9.1 would fail for such X. One of them is the Lagrange interpolation formula in the proof of Proposition 9.10. Namely, the rational functions of ω there are not reduced — the degree dl of the numerator *is equal* to the degree dn of the corresponding denominator. The other one is Lemma 9.7. Namely, we have the following lemma instead.

Lemma 10.1. *The type (ii) fixed point component in $Y_{2,d}$ makes zero contribution via Borel localization formulas to $\int_{Y_{2,d}} c^{d-1}...$* unless *it consists of maps $(C' \cup C'', x_0, x_1) \to Y$ where C' is mapped to a fixed point in $\mathbb{C}P^n$ and carries both marked points, and C'' is a disjoint union of d irreducible components (intersecting C' at d special points) mapped (each with multiplicity* 1*) onto straight lines outgoing the fixed point. All type (ii) components make zero contribution to $\int_{Y_{2,d}} c^d...$.*

Let us modify the results of Section 9 accordingly. As we will see, the LHS in Theorem 9.5 is now only *proportional* to the RHS, and we will compute the proportionality coefficient (a series in q) directly.

Proposition 10.2. *Put $z_i(Q, \hbar) := Z_i(\hbar Q, \hbar)$. Then*

$$z_i(Q, \hbar) = 1 + \sum_{d>0} Q^d Coeff\ _i(d) + \sum_{j\neq i}\sum_{d'>0} Q^{d'} Coeff\ _i^j(d')\ z_j(Q, (\lambda_j - \lambda_i)/d')$$

where $Coeff\ _i(d)$ *is equal to the contribution of type (ii) fixed point components to* $\int_{Y_{2,d}}(-c)^{d-1}E'_d\ e_0^*(\phi_i)$*, and*

$$Coeff\ _i^j(d) = \frac{1}{[(\lambda_i-\lambda_j)/\hbar+d]}\ \frac{\Pi_{a=1}^r\Pi_{m=1}^{dl_a}(\frac{(l_a\lambda_i-\lambda'_a)d}{\lambda_j-\lambda_i}+m)}{\Pi_{\alpha=0}^n\ {}_{m=1}^{d}\ {}_{(\alpha,m)\neq(j,d)}(\frac{(\lambda_i-\lambda_\alpha)d}{\lambda_j-\lambda_i}+m)}\ .$$

Corollary 10.3. *The correlators* $z_i(Q,1/\omega)$ *are power series* $\sum_d C_i(d)Q^d$ *with coefficients*

$$C_i(d) = P_d(\omega,\lambda,\lambda')/\Pi_\alpha\Pi_{m=1}^d((\lambda_i-\lambda_\alpha)\omega+m)$$

where $P_d = P_d^0\omega^{nd}+...$ *is a polynomial in* ω *of degree* nd*. The correlators* z_i *are uniquely determined by these properties, the recursion relations of Proposition* 10.2 *and the initial conditions*

$$\sum_d Coeff\ _i(d)Q^d = \sum_d Q^d\frac{P_d^0}{d!\Pi_{\alpha\neq i}(\lambda_i-\lambda_a)^d}\ .$$

Proposition 10.4. *The series*

$$z'_i = \sum_{d=0}^{\infty} Q^d\frac{\Pi_{a=1}^r\Pi_{m=1}^{l_ad}((l_a\lambda_i-\lambda'_a)\omega+m)}{d!\Pi_{\alpha\neq i}\Pi_{m=1}^d((\lambda_i-\lambda_\alpha)\omega+m)}$$

satisfy the requirements of Corollary 10.3 *with the initial condition*

$$\sum_d Q^d\frac{\Pi_{a=1}^r(l_a\lambda_i-\lambda'_a)^{l_ad}}{d!\Pi_{\alpha\neq i}(\lambda_i-\lambda_\alpha)^d} \ = \ \exp\{Q\frac{\Pi_a(l_a\lambda_i-\lambda'_a)^{l_a}}{\Pi_{\alpha\neq i}(\lambda_i-\lambda_\alpha)}\}.$$

Now let us compute $Coeff\ _i(d)$ using the description of type (ii) fixed point components given in Lemma 10.1.

Proposition 10.5. *Contribution of the type (ii) fixed point components to* $\sum_d Q^d\int_{Y_{2,d}}(-c)^{d-1}E'_d\ \phi_i$ *is*

$$\exp\{Q\frac{\Pi_a(l_a\lambda_i-\lambda'_a)^{l_a}}{\Pi_{\alpha\neq i}(\lambda_i-\lambda_\alpha)}\}\ \exp\{-Q\ l_1!...l_r!\}\ .$$

Proof. Each fixed point component described in Lemma 10.1 is isomorphic to the Deligne - Mumford configuration space $\bar{\mathcal{M}}_{0,d+2}$. Our computation is based on the following known formula (see for instance [3]) for correlators between Chern classes of universal tangent lines at the marked points:

$$\int_{\bar{\mathcal{M}}_{0,k}} \frac{1}{(w_1+c_1^{(1)})...(w_k+c_1^{(k)})} = \frac{(1/w_1+...+1/w_k)^{k-3}}{w_1...w_k} \ .$$

Consider the type (ii) fixed point component specified by the following combinatorial structure of stable maps: d degree 1 irreducible components join the i-th fixed point with the fixed points with indices $j_1,...,j_d$. Using the above formula and describing explicitly the normal bundle to this component in $Y_{2,d}$ and localization of the Euler class E'_d we arrive to the following expression for the contribution of this component to $\int_{Y_{2,d}}(-c)^{d-1}\phi_i E'_d$:

$$\Pi_{s=1}^d \frac{\Pi_{a=1}^r \Pi_{m=1}^{l_a}(l_a\lambda_i-\lambda'_a+m(\lambda_{j_s}-\lambda_i))}{(\lambda_i-\lambda_{j_s})\Pi_{\alpha\neq j_s,i}(\lambda_{j_s}-\lambda_\alpha)} \ .$$

Summation over all type (ii) components in all $Y_{2,d}$ with weights Q^d gives rise to

$$\exp\{-Q\sum_{j\neq i}\frac{\Pi_a\Pi_{m=1}^{l_a}(l_a\lambda_i-\lambda'_a+m(p-\lambda_i))}{\Pi_{\alpha\neq j}(p-\lambda_\alpha)} \ |_{p=\lambda_j}\} \ .$$

The exponent can be understood as a sum of residues at $p\neq\lambda_i,\infty$ and is thus opposite to the sum

$$l_1!...l_r! - \frac{\Pi_a(l_a\lambda_i-\lambda'_a)^{l_a}}{\Pi_{\alpha\neq i}(\lambda_i-\lambda_\alpha)}$$

of residues at ∞ and λ_i.

Corollary 10.6. $z_i(Q,1/\omega)=z'_i(Q,\omega)\ \exp(-l_1!...l_r!Q)$.

Proof. Multiplication by a function of Q does not destroy the recursion relation of Proposition 10.2 but changes the initial condition.

We have proved the following

Theorem 10.7. *Suppose* $l_1+...+l_r=n$. *Then*

$$S' = e^{(pt-l_1!...l_r!e^t)/\hbar}\sum_{d=0}^{\infty} e^{dt}\ \frac{\Pi_0^{dl_1}(l_1p-\lambda'_1+m\hbar)...\Pi_0^{dl_r}(l_rp-\lambda'_r+m\hbar)}{\Pi_1^d(p-\lambda_0+m\hbar)...\Pi_i^d(p-\lambda_n+m\hbar)} \ .$$

$$S = S'|_{\lambda=0,\lambda'=0} = e^{(Pt-l_1!...l_r!e^t)/\hbar}\frac{\Pi_{j=1}^r\Pi_{m=0}^{dl_j}(l_jP+m\hbar)}{\Pi_{m=1}^d(P+m\hbar)^{n+1}} \ \ (\text{mod } P^{n+1}) \ .$$

Corollary 10.8. *Let* $D = \hbar d/dt + l_1!...l_r!e^t$. *Then*

$$D^{n+1-r}S = l_1...l_r e^t \Pi_{j=1}^r (l_jD + \hbar)...(l_jD + (l_j - 1)\hbar)\ S.$$

Corollary 10.9. *In the quantum cohomology algebra of* X *the class* p *of hyperplane sections satisfies the following relation (with the same reservation in the case* $\dim X \leq 2$ *as in Corollary* 9.3*):*

$$(p + l_1!...l_r!q)^{n+1-r} = l_1^{l_1}...l_r^{l_r} q(p + l_1!...l_r!q)^{n-r}\ .$$

Control examples. [4] 1. $X = pt$ in $\mathbb{C}P^1$ ($n = 1, r = 1, l = 1$). The above relation takes on $p + q = q$, or $p = 0$. Since $P^2 = 0$, we also find from Theorem 10.7 that $S = P\exp(-e^t)\sum_d e^{dt}/d! = P$, or $\langle 1, S\rangle = 1$ as it should be for the solution of the differential equation $\hbar d/dt\ s = 0$ that arises from quantum cohomology of the point.

2. $X = \mathbb{C}P^1$ embedded as a quadric into $\mathbb{C}P^2$ ($n = 2, r = 1, l = 2$). We get $(p + 2q)^2 = 4q(p + 2q)$, or $p^2 = 4q^2$. Taking into account that p is twice the generator in $H^2(\mathbb{C}P^1)$ and the line in $\mathbb{C}P^1$ has the degree 2 in $\mathbb{C}P^2$ we conclude that this is the correct relation in the quantum cohomology of $\mathbb{C}P^1$. This example was the most confusing for the author: predictions of the loop space analysis [9] appeared totally unreliable because they gave a wrong answer for the quadric in $\mathbb{C}P^2$. As we see now, the loop space approach gives correct results if $l_1 + ... + l_r < n$ and require "minor" modification (by the factor $\exp(-l_1!...l_r!q/\hbar)$) in the boundary cases $l_1 + ...l_r = n$; the quadric on the plane happens to be one of such cases.

3. $n = 3, r = 1, l = 3$. We have $(p + 6q)^3 = 27q(p + 6q)^2$, or $p^3 = 9qp^2 + 6^3q^2p + 27 \cdot 28q^3$. In particular, $\langle p * p, p\rangle = 9q\langle p, p\rangle + 6^3q^2\langle p, 1\rangle + 27 \cdot 28q^3\langle 1, 1\rangle = 27q + 0 + 0$ which indicates that there should exist 27 discrete lines on a generic cubical surface in $\mathbb{C}P^3$.

11 Calabi-Yau projective complete intersections

Let $L_d(Y)$ denote, as in Section 6, the moduli space of stable maps $\psi : \mathbb{C}P^1 \to \mathbb{C}P^n \times \mathbb{C}P^1$ of bidegree $(d, 1)$ with 2 marked points mapped to $\mathbb{C}P^n \times \{0\}$ and $\mathbb{C}P^n \times \{\infty\}$ respectively. Let $\mathcal{E}_d$ denote the equivariant Euler class of the vector bundle over $L_d(Y)$ with the fiber

[4] I am thankful to A. Collino [30] who pointed to me that in the hypersurface case the statement of Corollary 10.9 was conjectured independently by several authors on the basis of numerical data.

$H^0(\mathbb{C}P^1, \psi^*(V))$ where V is the bundle on $\mathbb{C}P^n \times \mathbb{C}P^1$ induced from our convex bundle $\oplus_a H^{l_a}$ by the projection to the first factor.

Consider the equivariant correlator

$$\Phi = \int_Y \, Euler^{-1}(V) \; S'(t,\hbar) \; S'(\tau,-\hbar) \; =$$

$$= \sum_{d,d'} e^{dt} e^{d'\tau} \sum_i \frac{\Pi_a(l_a\lambda_i - \lambda'_a)}{\Pi_{j\neq i}(\lambda_i - \lambda_j)} \int_{Y_{2,d}} E'_d \frac{e^{pt/\hbar} e_0^*(\phi_i)}{\hbar + c} \; \int_{Y_{2,d'}} E'_{d'} \frac{e^{-p\tau/\hbar} e_0^*(\phi_i)}{-\hbar + c}.$$

In the case $l_1 + ... + l_r < n$ it is easy to check using the explicit formula for S' from Theorem 9.5 that

$$\Phi = \frac{1}{2\pi i} \oint e^{p(t-\tau)/\hbar} [\sum_d e^{d\tau} \frac{\Pi_a \Pi_{m=0}^{l_a d}(l_a p - \lambda'_a - m\hbar)}{\Pi_{j=0}^n \Pi_{m=0}^d (p - \lambda_j - m\hbar)}] \; dp.$$

This is an equivariant version of a formula found in [9] in the context of loop spaces and toric compactifications of spaces of rational maps. Namely, consider the projective space L'_d of $(n+1)$-tuples of polynomials in one variable of degree $\leq d$ each, up to a scalar factor (notice that L'_d has the same dimension $d(n+1)+n$ as L_d). It inherits the component-wise action of the torus T^{n+1} and the action of S^1 by the rotation of the variable ("rotation of loops"). Integration over the equivariant fundamental cycle in L'_d is given by the residue formula

$$f(p,\lambda,\hbar) \mapsto \frac{1}{2\pi i} \oint \frac{f dp}{\Pi_{j=0}^n \Pi_{m=0}^d (p - \lambda_j - m\hbar)}.$$

Consider the equivariant vector bundle over L'_d such that substitution of the $(n+1)$ polynomials into r (invariant) homogeneous equations in $\mathbb{C}P^n$ of degrees $l_1, ..., l_r$ produces a section of this bundle. The equivariant Euler class of the bundle is

$$\mathcal{E}'_d = \Pi_{a=1}^r \Pi_{m=0}^{l_a d} (l_a p - \lambda'_a - m\hbar).$$

The formula for Φ indicates that there should exist a close relation between the spaces L_d and L'_d. This relation is described in the following lemma whose proof will be given in the end of this Section.

The Main Lemma. *There exists a natural $S^1 \times T^{n+1}$-equivariant map $\mu : L_d \to L'_d$. Denote $-p$ the equivariant 1-st Chern class of the Hopf bundle over L'_d induced by μ to L_d. Then*

$$\Phi(t,\tau) = \sum_d e^{d\tau} \int_{L_d} e^{p(t-\tau)/\hbar} \mathcal{E}_d.$$

Define $\Phi'(q,z,\hbar) := \Phi|_{t=\tau+z\hbar, q=e^{\tau}}$. (By the way the limit of the series Φ' at $\hbar = 0$ has the topological meaning of what is called in [7] the *generating volume function*, and the meaning of this limit procedure in terms of differential equations satisfied by Φ is the *adiabatic approximation*.)

Corollary 11.1. $\Phi'(q,z) := \sum_d q^d \int_{L_d} e^{pz} \mathcal{E}_d =$

$$= \frac{1}{2\pi i} \oint e^{pz} \sum_d \frac{q^d \; E_d(p,\lambda,\lambda',\hbar)}{\Pi_{j=0}^n \Pi_{m=0}^d (p-\lambda_j - m\hbar)} dp$$

where $E_d = \mu_*(\mathcal{E}_d)$ is a polynomial *(of degree $< (n+1)d$) of all its variables.*

Proof. The integrals $E^{(k)} = \int_{L_d} p^k \mathcal{E}_d$, $k = 0, ..., \dim L'_d$, which determine the push-forward $\mu_*(\mathcal{E})$ are polynomials in $(\lambda, \lambda', \hbar)$. The matrix $\int_{L'_d} p^{i+\dim L'_d - j}$ is triangular with all eigenvalues equal to 1. This means that there exists a unique polynomial in p with coefficients *polynomial in* $(\lambda,\lambda',\hbar)$ which represents the push-forward with any given polynomials $E^{(k)}(\lambda,\lambda',\hbar)$.

The last argument also proves

Proposition 11.2. *Suppose that a series*

$$s = \sum_d q^d \frac{P_d(p,\lambda,\lambda',\hbar)}{\Pi_j \Pi_{m=0}^d (p-\lambda_j - m\hbar)}$$

with coefficients P_d which are polynomials of p of degree $\leq \dim L_d$ has the property that for every $k = 0,1,2,...$ the q-series $\oint sp^k dp$ has polynomial coefficients in $(\lambda,\lambda',\hbar)$. Then the coefficients of all P_d are polynomials of $(\lambda,\lambda',\hbar)$, and vice versa.

The coefficient $E_d(p,\lambda,\lambda',\hbar)$ in the series Φ' has the total degree $(l_1 + ... + l_r)d + r$ according to the dimension of the vector bundle whose Euler class it represents. Consider the following operations with the series Φ:

(i) multiplication by a series of e^t and / or e^{τ};

(ii) simultaneous change of variables $t \mapsto t + f(e^t), \tau \mapsto \tau + f(e^{\tau})$.

(iii) multiplication by $\exp[C(f(e^t) - f(e^{\tau}))/\hbar]$ (here the factor C should be a linear function of (λ,λ') in order to obey homogeneity).

Proposition 11.3. *The property of the series Φ to generate polynomial coefficients $E_d(p, \lambda, \lambda', \hbar)$ is invariant with respect to the operations (i),(ii),(iii).*

Proof. The polynomiality property of coefficients in Φ' is equivalent, due to Proposition 11.2, to the fact that for all k the q-series $(\partial/\partial z)^k|_{z=0}\Phi'$ has polynomial coefficients.

Multiplication by a series of q does not change this property, which proves the invariance with respect to multiplication by functions of e^τ.

The roles of t and τ can be interchanged by the substitutions $p \mapsto p + \hbar d, \hbar \mapsto -\hbar$ in each summand of Φ. This proves the invariance with respect to multiplication by functions of e^t.

The operation (ii) transforms Φ' to

$$\frac{1}{2\pi i}\sum_d q^d e^{df(q)} \oint \exp\{p\frac{z\hbar + f(qe^{z\hbar}) - f(q)}{\hbar}\} \frac{E_d(p)}{\Pi_j\Pi_m(p - \lambda_j - m\hbar)} dp.$$

Since the exponent is in fact divisible by $\hbar$, the derivatives in z at $z = 0$ still have polynomial coefficients. This proves the invariance with respect to (ii). The case of the operation (iii) is analogous.

We are going to use the above polynomiality and invariance properties of the correlator Φ in order to describe quantum cohomology of *Calabi-Yau* complete intersections in $\mathbb{C}P^n$ (in which case $l_1 + ... + l_r = n + 1$). We will use this polynomiality in conjunction with recursion relations based on the fixed point analysis of Sections $9, 10$. The result can be roughly formulated in the following way: the hypergeometric functions of Theorem 9.5 in the case $l_1 + ... + l_r = n + 1$ can be transformed to the correlators S' by the operations (i),(ii),(iii). Notice that in the Calabi – Yau case all our formulas are homogeneous with the grading $\deg q = 0, \deg p = \deg \hbar = \deg \lambda = \deg \lambda' = 1, \deg z = -1$. In particular the transformations (i)–(iii) also preserve the degrees of the numerators E_d in Φ'. In the "positive" case $l_1 + ... + l_r \leq n$ where $\deg q = n + 1 - \sum l_a > 0$ the transformations (i)–(iii) in fact increase degrees of the numerators E_d and are "not allowed". The only exception is the operation (iii) with $f(q) = \text{const}\, q$ in the case $l_1 + ... + l_r = n$ when $\deg q = 1$. The right value $-l_1!...l_r!$ of the constant can be found by counting contributions of curves of degree 1. In Section 10 we have found this answer by a straightforward computation involving curves of all degrees.

Consider now the correlator Φ (defined in The Main Lemma) in the Calabi–Yau case $l_1 + ... + l_r = n+1$. Localization to S^1-fixed points in L_d (as in Section 6) expresses Φ via the correlators Z_i (defined in Section 9) as follows:

$$\Phi = \sum_i \frac{\Pi_a(l_a\lambda_i - \lambda'_a)}{\Pi_{j\neq i}(\lambda_i - \lambda_j)} \; e^{\lambda_i(t-\tau)/\hbar} \; Z_i(e^t, \hbar) \; Z_i(e^\tau, -\hbar).$$

Proposition 11.4. (1) *The coefficients of the power series* $Z_i(q,\hbar) = \sum_d q^d C_i(d)$ *are rational functions*

$$C_i(d) = \frac{P_d^{(i)}}{d!\hbar^d \Pi_{j\neq i}\Pi_{m=1}^d (\lambda_i - \lambda_j + m\hbar)}$$

where $P_d^{(i)}$ *is a polynomial in* $(\hbar, \lambda, \lambda')$ *of degree* $(n+1)d$.

(2) *The polynomial coefficients* $E_D(p)$ *in* Φ' *are determined by their values*

$$E_D(\lambda_i + d\hbar) = \Pi_a(l_a\lambda_i - \lambda'_a) \; P_d^{(i)}(\hbar) P_{D-d}^{(i)}(-\hbar)$$

at $p = \lambda_i + d\hbar$, $i = 0, ..., n$, $d = 0, ..., D$.

(3) *The correlators* $z_i(Q,\hbar) := Z_i(Q\hbar, \hbar)$ *satisfy the recursion relation*

$$z_i(Q,\hbar) = 1 + \sum_{d>0} \frac{Q^d}{d!} R_{i,d} + \sum_{d>0}\sum_{j\neq i} Q^d \frac{Coeff\ _i^j(d)}{\lambda_i - \lambda_j + d\hbar} \; z_j(Q, \frac{(\lambda_j - \lambda_i)}{d})$$

where $R_{i,d} = R_{i,d}^{(0)}\hbar^d + R_{i,d}^{(1)}\hbar^{d-1} + ...$ *is a polynomial of* $(\hbar, \lambda, \lambda')$ *of degree* $\leq d$, *and*

$$Coeff\ _i^j(d) = \frac{\Pi_a \Pi_{m=1}^{l_a d}(l_a\lambda_i - \lambda'_a + m(\lambda_j - \lambda_i)/d)}{d!\Pi_{\alpha\neq i}\ {}_{m=1}^{d}\ {}_{(\alpha,m)\neq(j,d)}(\lambda_i - \lambda_\alpha + m(\lambda_j - \lambda_i)/d)} \; .$$

For any given $\{R_{i,d} | i = 0, ..., n, \ d = 1, 2, ...\}$ *these recursion relations have a unique solution* $\{z_i\}$.

Proof. (3) We have

$$Z_i = 1 + \sum_{d>0} q^d [\sum_{k=0}^{d-1} \hbar^{-k-1} \int_{Y_{2,d}} E'_d e_0^*(\phi_i)(-c)^k] + \sum_{d>0} q^d \hbar^{-d} \int_{Y_{2,d}} E'_d e_0^*(\phi_i) \frac{(-c)^d}{\hbar + c}$$

where the integrals of the last sum have zero contributions from the type (ii) fixed point components (Lemmas 9.7, 10.1). Thus these integrals have a recursive expression identical to those of Sections 9 and 10. The terms of the double sum constitute the initial condition $\{R_{i,d}\}$. The recursion relations have the form of the decomposition of rational functions of $\hbar$ (coefficients at powers of $Q = q/\hbar$) into the sum of simple fractions in the case when degrees of numerators exceed degrees of denominators. This proves existence and uniqueness of solutions to the recursion relations.

(1) follows directly from the form and topological meaning of the recursion relations.

(2) follows from the definition of Φ in terms of Z_i.

Introduce now the class $\mathcal{P}$ of solutions to the recursion relation 11.4(3) which give rise (via 11.4(2)) to *polynomial* coefficients E_d in the corresponding Φ'.

Proposition 11.5. *A solution from $\mathcal{P}$ is uniquely determined by the first two coefficients $R_{i,d}^{(0)}, R_{i,d}^{(1)}, i = 0, ..., n,\ 0 < d < \infty$, of its initial condition (that is by the first two terms in the expansion $Z_i = Z_i^{(0)} + Z_i^{(1)}/\hbar + ...$ as power series in $1/\hbar$.*

Proof. Perturbation theory: suppose that two solutions from the class $\mathcal{P}$ have the same initial condition up to the order $(d-1)$ inclusively. Then (2) shows that corresponding E_k for these solutions coincide for $k < d$ and the variation $\delta E_d(p)$ vanishes at $p = \lambda_i + k\hbar$ for $0 < k < d$. This means that the polynomial δE_d is divisible by $\Pi_j \Pi_{m=1}^{d-1}(p - \lambda_j - m\hbar)$. On the other hand (1) and (2) imply that the variation $\delta R_{i,d}$ of the initial condition satisfies

$$\delta R_{i,d}(\hbar)\ \Pi_a(l_a\lambda_i - \lambda'_a)\Pi_{j\neq i}\Pi_{m=1}^{d}(\lambda_i - \lambda_j + m\hbar) = \delta E_d|_{p=\lambda_i+\hbar d}$$

(since $R_{i,0} = 1$) and thus $\delta R_{i,d}$ is divisible by $\hbar^{d-1}$. Since $\delta R_{i,d}$ is a degree d polynomial, it leaves only the possibility

$$\delta R_{i,d} = \delta R_{i,d}^{(0)}\hbar^d + \delta R_{i,d}^{(1)}\hbar^{d-1}.$$

Thus if two class $\mathcal{P}$ solutions coincide in orders $\hbar^0, \hbar^{-1}$ then $\delta R_{i,d} = 0$, and thus the very solutions coincide.

Proposition 11.6. *The class $\mathcal{P}$ is invariant with respect to the following operations:*

(a) simultaneous multiplication $Z_i \mapsto f(q)Z_i$ by a power series of q with $f(0) = 1$;

(b) changes $Z_i(q,\hbar) \mapsto e^{\lambda_i f(q)/\hbar} Z_i(qe^{f(q)}, \hbar)$ with $f(0) = 0$;

(c) multiplication $Z_i \mapsto \exp(Cf(q)/\hbar)Z_i$ where C is a linear function of (λ, λ') and $f(0) = 0$.

Proof. The operations (a),(b),(c) give rise to the operations of type (i)-(iii) for corresponding polynomials E_d. Thus it suffices to show that the operations (a),(b),(c) transform a solution $\{z_i\}$ of the recursion relations to another solution.

Recall that the recursion relation 11.4(3) expresses z_i as formal power series in Q with coefficients (at Q^D) being rational functions of $\hbar$ decomposed into simple fractions with the poles $\hbar = (\lambda_j - \lambda_i)/d,\ d \leq D$, plus the polynomial parts $R_{i,D}(\hbar)$ of degrees $\leq D$.

Consider the recursion coefficient $Q^d\ Coeff\ _i^j(d)$ responsible for the simple fraction with the denominator $(\lambda_i - \lambda_j + d\hbar)$. Application of the operations (a), (b), (c) to the left and right hand sides of the recursion relation causes respectively the following modifications in this coefficient:

$$Q^d \mapsto f(Q\hbar)Q^d/f(Q(\lambda_j - \lambda_i)/d),$$

$$Q^d \mapsto Q^d \exp\{\frac{\lambda_i f(Q\hbar)}{\hbar} + df(Q\hbar) - \frac{\lambda_j f(Q(\lambda_j - \lambda_i)/d)}{(\lambda_j - \lambda_i)/d}\},$$

$$Q^d \mapsto Q^d \exp\{C\frac{f(Q\hbar)}{\hbar} - C\frac{f(Q(\lambda_j - \lambda_i)/d}{(\lambda_i - \lambda_j)/d}\}.$$

In the case of the change (b), additionally, the argument Q in z_j on the RHS of the recursion relation gets an extra-factor $\exp[f(Q\hbar) - f(Q(\lambda_j - \lambda_i)/d)]$.

All the modifying factors written above actually take on 1 at $\hbar = (\lambda_j - \lambda_i)/d$. This means that the recursion coefficient responsible for the simple fraction with the pole at $\hbar = (\lambda_j - \lambda_i)/d$ does not change and that the operations (a), (b), (c) modify only the polynomial initial conditions $R_{i,D}(\hbar)$.

Under our assumptions about f (that $f(0) = 1$ in (a) and $f(0) = 0$ in (b), (c)) the modifying factors depend on $\hbar$ only in the combination $Q\hbar$. This implies that the degrees of the new initial conditions $R_{i,D}(\hbar)$ still do not exceed D.

Let us consider now the hypergeometric series

$$Z_i^* = \sum_{d=0}^{\infty} q^d\ \frac{\Pi_{a=1}^r \Pi_{m=1}^{l_a d}(l_a\lambda_i - \lambda'_a + m\hbar)}{\Pi_{\alpha=0}^n \Pi_{m=1}^d(\lambda_i - \lambda_\alpha + m\hbar)}$$

where $l_1 + ... + l_r = n + 1$.

It is straightforward to see that $\{Z_i^*\}$ satisfy the recursion relations of Proposition 11.4(3) (see the proof of Proposition 9.10) and that the formulas of Proposition 11.4(2) generate corresponding

$$\Phi^* = \frac{1}{2\pi i}\oint\ e^{p(t-\tau)/\hbar}\sum_{d=0}^{\infty} e^{d\tau}\frac{\Pi_{a=1}^r \Pi_{m=0}^{l_a d}(l_a p - \lambda'_a - m\hbar)}{\Pi_{i=0}^n \Pi_{m=0}^d(p - \lambda_i - m\hbar)}dp$$

with polynomial numerators. Thus $\{Z_i^*\}$ is a solution from the class $\mathcal{P}$.

Computation of the first two terms in the initial condition gives

$$Z_i^{*\ (0)} = f(q) = \sum_{d=0}^{\infty} \frac{(l_1 d)! ... (l_r d)!}{(d!)^{n+1}}\ q^d\ ,$$

$$Z_i^{*\ (1)} = \lambda_i \sum_a l_a [g_{l_a}(q) - g_1(q)] + (\sum_\alpha \lambda_\alpha) g_1(q) - \sum_a \lambda'_a g_{l_a}(q)$$

where

$$g_l = \sum_{d=1}^{\infty} q^d \frac{\Pi_a (l_a d)!}{(d!)^{n+1}}\ (\sum_{m=1}^{ld} \frac{1}{m})\ .$$

Let us compare these initial conditions with those for $\{Z_i\}$.

Proposition 11.7. $Z_i^{(0)} = 1,\ Z_i^{(1)} = 0$.

Proof. The first statement follows from the definition of Z_i while the second means that $\int_{Y_{2,d}} E'_d e_0^*(\phi_i) = 0$ for all $d > 0$. It is due to the fact that the class $E'_d e_0^*(\phi_i)$ is a pull-back from $Y_{1,d}$. (In fact we have just repeated an argument proving (5) from Section 5 and thus the proposition can be deduced from general properties of quantum cohomology.)

Combining the last three propositions we arrive to the following

Theorem 11.8. *The hypergeometric solution $\{Z_i^*(q,\hbar)\}$ coincides with the solution $\{Z_i(Q,\hbar)\}$ up to transformations (a),(b),(c). More precisely, perform the following operations with $\{Z_i\}$*

1) put

$$Q = q \exp\{\sum_a l_a [g_{l_a}(q) - g_1(q)]/f(q)\}\ ,$$

2) multiply $Z_i(Q(q),\hbar)$ by

$$\exp\{\frac{1}{f(q)\hbar}[\sum_a (l_a \lambda_i - \lambda'_a) g_{l_a}(q) - (\sum_\alpha (\lambda_i - \lambda_\alpha)) g_1(q)]\},$$

3) multiply all Z_i simultaneously by $f(q)$.

Then the resulting functions coincide with hypergeometric series $Z_i^*(q,\hbar)$.

Proof. The three steps correspond to consecutive applications of operations of type (b),(c) and (a) to $\{Z_i\}$ and transform the initial condition of Proposition 11.7 to that for $\{Z_i^*\}$. According to Propositions 11.5, 11.6 this transforms the whole solution $\{Z_i\}$ to $\{Z_i^*\}$.

Consider the solutions

$$s_i = \; e^{\lambda_i T/\hbar} \; Z_i(e^T, \hbar)$$

to the equivariant quantum cohomology differential equations.

Corollary 11.9. *The operations*

1) change $T = t + \sum_\alpha l_a [g_{l_a}(e^t) - g_1(e^t)]/f(e^t)$,

2) multiplication by

$$f(e^t)\exp\{[g_1(e^t)(\sum_\alpha l_\alpha) - \sum_a \lambda'_a g_{l_a}(e^t)]/(\hbar f(e^t))\}$$

transform $\{s_i\}$ *to the hypergeometric solutions*

$$s_i^* = e^{pt/\hbar} \sum_d e^{dt} \frac{\Pi_a \Pi_{m=1}^{l_a d}(l_a p - \lambda'_a + m\hbar)}{\Pi_\alpha \Pi_{m=1}^{d}(p - \lambda_\alpha + m\hbar)} \; |_{p=\lambda_i}$$

of the differential equation

$$\Pi_\alpha(\hbar\frac{d}{dt} - \lambda_\alpha)s^* = e^t \; \Pi_a \Pi_{m=1}^{l_a}(\hbar l_a \frac{d}{dt} - \lambda'_a + m\hbar) \; s^* \; .$$

For $\lambda' = 0, \; \lambda_0 + ... + \lambda_n = 0$ *the solutions* s_i^* *have the following integral representation:*

$$\int_{\Gamma^n \subset \{F_0(u) = e^t\}} \frac{u_0^{\lambda_0/\hbar} ... u_n^{\lambda_n/\hbar} \; du_0 \wedge ... \wedge du_n}{F_1(u) \; ... \; F_r(u) \; dF_0}$$

where

$$F_1 = (1 - u_1 - ... - u_{l_1}), \; F_2 = (1 - u_{l_1+1} - ... - u_{l_1+l_2}), \; ..., F_r = (1 - u_{l_1+...+l_{r-1}+1} - ... - u_{l_1+...+l_r})$$

and $F_0 = u_0 ... u_n$.

Corollary 11.10. *The hypergeometric class* $S^*(t, \hbar) \in H^*(\mathbb{C}P^n) = \mathbb{C}[P]/(P^{n+1})$,

$$S^* = e^{Pt/\hbar} \sum_d e^{dt} \frac{\Pi_a \Pi_{m=0}^{l_a d}(l_a P + m\hbar)}{\Pi_{m=1}^{d}(P + m\hbar)^{n+1}}$$

whose $n + 1 - r$ *non-zero components are solutions to the Picard-Fuchs equation*

$$(\frac{d}{dt})^{n+1-r} s^* = l_1 ... l_r e^t \Pi_a \Pi_{m=1}^{l_a - 1}(l_a \frac{d}{dt} + m) s^*$$

for the integrals

$$\int_{\gamma^{n-r}\subset X'_t} \frac{du_0 \wedge ... \wedge du_n}{dF_0 \wedge dF_1 \wedge ... \wedge dF_r} \ ,$$

(here $X'_t = \{(u_0, ..., u_n) | F_0(u) = e^t, F_1(u) = 0, ..., F_r(u) = 0\}$*) are obtained from the class* S *(describing the quantum cohomology* $\mathcal{D}$*-module for the Calabi-Yau complete intersection* $X^{n-r} \subset \mathbb{C}P^n$*),*

$$S = e^{PT/\hbar} \sum_d e^{dT} (e_0)_* (\frac{E_d}{\hbar + c_1^{(0)}}),$$

by the change

$$T = t + \sum_a l_a [g_{l_a}(e^t) - g_1(e^t)]/f(e^t)$$

followed by the multiplication by $f(e^t)$*.*

Proof. Corollary 11.9 shows that for $\lambda' = 0, \sum \lambda_\alpha = 0$ these change and multiplication transform the corresponding equivariant classes S' and $S'\,^*$ to one another. The class $-p$ in the formula for s_i^* in Corollary 11.9 is the equivariant Chern class of the Hopf line bundle over $\mathbb{C}P^n$. In the limit $\lambda = 0$ it becomes $-P$ while S' and $S'\,^*$ transform to their non-equivariant counterparts S and S^*.

Remarks. 1) Notice that the components S_0^* and S_1^* in

$$S^* = l_1 ... l_r [P^r S_0^*(t) + P^{r+1} S_1^*(t) + ... + P^n S_n^*(t)]$$

are exactly $f(e^t)$ and $tf(e^t) + \sum_a l_a [g_{l_a}(e^t) - g_1(e^t)]$ respectively. Thus the inverse transformation from S^* to S consists in division by S_0^* followed by the change $T = S_1^*(t)/S_0^*(t)$ in complete accordance with the recipe [16, 18, 9] based on the mirror conjecture.

2) According to [17] the $(n-r)$-dimensional manifolds X'_t admit a Calabi-Yau compactification to the family $\bar{X}'_t$ of *mirror manifolds* of the Calabi-Yau complete intersection $X^{n-r} \subset \mathbb{C}P^n$. The Picard-Fuchs differential equation from Corollary 11.10 describes variations of complex structures for $\bar{X}'$. This proves the mirror conjecture (described in detail in [18]) for projective Calabi-Yau complete intersections and confirms the enumerative predictions about rational curves and quantum cohomology algebras made there (and in some other papers) on the basis of the mirror conjecture.

3) The description [15] of the quantum cohomology algebra of a Calabi-Yau 3-fold in terms of the numbers n_d of rational curves of all degrees d (see for instance [9] for the description of the corresponding class S in these terms) has been rigorously justified in [14].

Combining these results with Corollary 11.10 we arrive to the theorem formulated in the introduction.

Proof of The Main Lemma.

In our construction of the map $\mu : L_d \to L'_d$ we will denote L_d the moduli space of stable maps $C \to \mathbb{C}P^n \times \mathbb{C}P^1$ of bidegree $(d, 1)$ with no marked points (it also has dimension $d(n+1)+n$). The construction works for any given number of marked points but produces a map which is the composition of μ with the forgetful map. In this form it applies to the submanifold of stable maps with two marked points confined over 0 and ∞ in $\mathbb{C}P^1$ (this submanifold is what we denoted L_d in the formulation of The Main Lemma).

Let $\psi : C \to \mathbb{C}P^n \times \mathbb{C}P^1$ be a stable genus 0 map of bidegree $(d, 1)$. Then $C = C_0 \cup C_1 ... \cup C_r$ where C_0 is isomorphic to $\mathbb{C}P^1$ and $\psi|C_0$ maps C_0 onto the graph of a degree $d' \le d$ map $\mathbb{C}P^1 \to \mathbb{C}P^n$, and for $i = 1, ..., r$ the bidegree $(d_i, 0)$ map $\psi|C_i$ sends C_i into the slice $\mathbb{C}P^n \times \{p_i\}$ where $p_i \ne p_j$ and $d_1 + ... + d_r = d - d'$.

The map $\mu : L_d \to L'_d$ assigns to $[\psi]$ the $(n+1)$-tuple $(f_0 g : f_1 g : ... : f_n g)$ of polynomials (= binary forms) on $\mathbb{C}P^1$ where g is the polynomial of degree $d - d'$ with roots $(p_1, ..., p_r)$ of multiplicities $(d_1, ..., d_r)$ and the tuple $(f_0 : ... : f_n)$ of degree d' polynomials (with no common roots, including ∞) is the one that describes the map $\psi|C_0$.

In order to prove that the map μ is regular [5] let us give it more invariant description. The construction below can be generalized to any positive line bundle instead of $\mathcal{O}_{\mathbb{C}P^n}(1)$.

Denote $\hat{L}_d$ the moduli space of bidegree $(d, 1)$ stable maps with an extra-marked point and pull back to $\hat{L}_d$ the line bundle

$$H := Hom(\pi_1^* \mathcal{O}_{\mathbb{C}P^n}(1), \pi_2^* \mathcal{O}_{\mathbb{C}P^1}(d))$$

by the evaluation map $e : \hat{L}_d \to \mathbb{C}P^n \times \mathbb{C}P^1$ (where π_i are projections to the factors). Consider the push-forward sheaf $H^0 := R^0 \pi_* e^*(H)$ of the locally free sheaf $e^* H$ along the forgetful map $\pi : \hat{L}_d \to L_d$. To a small neighborhood $U \subset L_d$, it assigns the $\mathcal{O}_U$ -module $H^0(\pi^{-1}(U), e^* H)$ of sections of $e^* H$.

Claim. 1) *H^0 is a rank 1 locally free sheaf on L_d.*

2) *The fiber at $[\psi]$ of the corresponding line bundle can be identified with*

$$H^0(C_0, (\psi|C_0)^*(H) \otimes \mathcal{O}(-[p_1])^{\otimes d_1} ... \otimes \mathcal{O}(-[p_r])^{\otimes d_r}).$$

[5] I am thankful to M. Kontsevich who communicated to me another, more elementary proof of this statement.

3) The kernel of the natural map

$$h : H^0(C, \psi^* \pi_1^*(\mathcal{O}_{\mathbb{C}P^n}(1))) \to H^0(C, \psi^* \pi_2^*(\mathcal{O}_{\mathbb{C}P^1}(d))) = H^0(\mathbb{C}P^1, \mathcal{O}(d))$$

defined by a nonzero vector in this fiber consists of the sections vanishing identically on C_0.

Using this, we pick $n+1$ independent sections of $\mathcal{O}_{\mathbb{C}P^n}(1)$ (that is homogeneous coordinates on $\mathbb{C}P^n$), define corresponding sections of $e^*\pi_1^*\mathcal{O}_{\mathbb{C}P^n}(1)$ and apply the map h. By this we obtain a degree 1 map from the total space of the line bundle H^0 to the linear space $\mathbb{C}^{n+1} \otimes H^0(\mathbb{C}P^1, \mathcal{O}(d))$. Since the homogeneous coordinates on $\mathbb{C}P^n$ nowhere vanish simultaneously, we obtain a natural map

$$L_d \to L'_d = Proj(\mathbb{C}^{n+1} \otimes H^0(\mathbb{C}P^1, \mathcal{O}(d)))$$

which sends $[\psi]$ to $(f_0 g : ... : f_n g)$ and conclude that μ is regular.

The remaining statements of The Main Lemma are proved by looking at localizations of the equivariant class p at the $S^1 \times T^{n+1}$-fixed points in L'_d and L_d (in this paragraph we use the notation L_d for the same space as in the formulation of The Main Lemma). The fixed points in L'_d are represented by the vector-monomials $(0 : ... : 0 : x^{d'} : 0 : ... : 0)$ where p localizes to $\lambda_i + d'\hbar$. A fixed point in L_d is represented by ψ with $\psi(C_0) = (0 : ... : 0 : 1 : 0 : ... : 0)$, $r = 2$, $p_0 = 0$, $p_1 = \infty$ and the maps $\psi | C_k : C_k \to \mathbb{C}P^n$, $k = 1, 2$ representing T^{n+1}-fixed points respectively in $Y_{2,d'}$ and $Y_{2,d-d'}$ such that their (say) second marked points are mapped to the point $\psi(C_0)$. This implies that the class $\mu^*(p)$ localizes to $\lambda_i + d'\hbar$ at such a fixed point and thus the pull back of p to the fixed point set

$$\{[\psi] \in Y_{2,d'} \times Y_{2,d-d'} | (e_2 \times e_2)([\psi]) \in \Delta \subset Y \times Y$$

of the S^1-action on L_d coincides with the pull back through the common marked point of the T^{n+1}-equivariant class $p + d'\hbar$ on the diagonal $\Delta = \mathbb{C}P^n$. Now localizations of $\int_{L_d} e^{p(t-\tau)} \mathcal{E}_d$ to the fixed points of S^1-action identify the form of the correlator Φ given in The Main Lemma with the definition of Φ as the convolution of $S'(t, \hbar)$ and $S'(\tau, -\hbar)$.

In order to justify the *claim* we need to compute the space of global sections of the sheaf $e^*(H)$ over the formal neighborhood of the fiber $\pi^{-1}([\psi])$ of the forgetful map $\pi : \hat{L}_d \to L_d$. The fiber itself is isomorphic to the tree-like genus 0 curve C. Let (x_j, y_j), $j = 1, ..., N \geq r$ be some local parameters on irreducible components of C near the singular points such that $\varepsilon_j = x_j y_j$ are local coordinates on the *orbifold* L_d near $[\psi]$ (one should add some local coordinates ε' on the stratum $\varepsilon_1 = ...\varepsilon_N = 0$ of stable maps $C \to \mathbb{C}P^n$ in order to construct a complete local coordinate system on L_d). Such a description of local coordinates on L_d follows from the very construction of the moduli spaces of stable maps to convex manifolds; we refer the reader to [3, 6] for details.

A line bundle over the neighborhood of $C \subset \hat{L}_d$ can be specified by the set

$$u_j(x_j^{\pm 1}, \varepsilon), v_j(y_j^{\pm 1}, \varepsilon),\ j = 1, ..., N,$$

of non-vanishing functions describing transition maps between trivializations of the bundle inside and outside the neighborhoods (with local coordinates $(x_j, y_j, \varepsilon_1, ..., \hat{\varepsilon_j}, ..., \varepsilon_N, \varepsilon'))$ of the double points.

Let us consider first the following model case. Suppose that C consists of $r+1$ irreducible components $(C_0, C_1, ..., C_r)$ such that each C_j with $j > 0$ intersects C_0 at some point p_j. Let x_j be the local parameter on C_0 near p_j, and the line bundle (of the degree $-d_j \leq 0$ on C_j) be specified by $v_j = y_j^{-d_j}$.

In the neighborhood of p_j a section of such a bundle is given by a function $s(x_j, y_j, \hat{\varepsilon_j})$ satisfying

$$s = y_j^{-d_j} s_j(y_j^{-1}, \varepsilon)$$

where the function s_j represents the section in the trivialization over the neighborhood of $C_j - p_j$. Here $\hat{\varepsilon_j}$ means that ε_j is excluded from the set of coordinates ε (remember that $\varepsilon_j = x_j y_j$). This implies that $s_j = \varepsilon_j^{d_j} f_j(y_j^{-1} \varepsilon_j, \varepsilon)$ where f_j is some regular function. Thus this section in the neighborhood of $p \in C_0$ is given by a function $s(x_j, \varepsilon) = x_j^{d_j} f_j(x_j, \varepsilon)$ with zero of order d_j at $x_j = 0$, and the restriction of this section to the neighborhood of C_j is determined by s.

In other words, the $\mathbb{C}[[\varepsilon]]$-module of global sections in the formal neighborhood of C identifies with the module of global sections on C_0 for the line bundle given by the loops $x_j^{-d_j} u_j$ instead of u_j (this corresponds to the subtraction of the divisor $\sum d_j [p_j]$.

The more general situation where v_j is the product of $y_j^{-d_j}$ with an invertible function $w_j(y_j, x_j, \hat{\varepsilon_j})$ preserves the above conclusion with $w^{-1} s = x_j^{d_j} f_j(x_j, \varepsilon)$ instead of s.

Obviously, the above computation bears dependence on additional parameters.

Now we apply our model computation to the neighborhood of a general tree-like curve C *inductively* by decomposing the tree into simpler ones starting from the root component

C_0. We conclude that the $\mathbb{C}[[\varepsilon]]$-module of sections of the bundle $e^*(H)$ is identified with the module of sections of some line bundle over the product of C_0 with the polydisk with coordinates $(\varepsilon_1, ..., \varepsilon_r, ..., \varepsilon_N, \varepsilon')$, and that this line bundle is $e^*(H)$ for C_0 (given by the loops u_j in our current notations) twisted by the loops $x_j^{-d_j}$ in the punctured neighborhoods of the points $(p_1, ..., p_r)$, where $(d_1, ..., d_r)$ are the degrees of the maps $\psi|C_j : C_j \to \mathbb{C}P^n$ (in the notations of the *claim* so that $d_1 + ... + d_r = d - d'$).

This implies that the $\mathbb{C}[[\varepsilon]]$-module $\mathcal{H}^0$ of global sections can be identified with the module of those global sections of the degree $d - d'$ locally free sheaf $(\psi|C_0)^*(H) \otimes \mathbb{C}[[\varepsilon]]$ which have zeroes of order d_j at p_j for $j = 1, ..., r$. In particular

1) $\mathcal{H}^0$ is a free $\mathbb{C}[[\varepsilon]]$-module of rank 1,

2) $\mathcal{H}^0 \otimes_{\mathbb{C}[[\varepsilon]]} (\mathbb{C}[[\varepsilon]]/(\varepsilon))$ is the 1-dimensional space $H^0|_{[\psi]}$ described in the *claim*, and

3) non-zero vectors in $H^0|_[\psi]$ represent sections of $\psi^*(H)$ over C non-zero on C_0 (and thus their product with a non-zero on C_0 section of $\psi^*\pi_1^*(\mathcal{O}_{\mathbb{C}P^n}(1))$ can not vanish identically on C_0.)

Factorization by the discrete group $Aut(\psi)$ preserves $(1 - 3)$ with $\mathbb{C}[[\varepsilon]]$ replaced by $\mathbb{C}[[\varepsilon]]^{Aut(\psi)}$.

12 Quantum Serre duality

Results of Sections 9 – 11 on quantum cohomology algebras of projective complete intersections can be understood as a study of the recursion relations which arise from localization to fixed points of tori actions. In this Section we apply the same technique to the more general quantum cup-product structures defined by solutions of WDVV-equations. Comparing the solutions which correspond (see Section 4) to a convex vector bundle over $\mathbb{C}P^n$ and its dual we will arrive to a quantum analogue of the Serre duality theorem. The canonical coordinates of semisimple Frobenius manifolds that we discussed in Section 8 will play here a key role.

We begin with a known property of quantum correlators for $X = pt$. Consider the series

$$W(x,y) = \frac{1}{x+y} + \sum_{k=1}^{\infty} \frac{1}{k!} \langle \frac{1}{x+c}, f(c), ..., f(c), \frac{1}{y+c} \rangle_{k+2}$$

where $f(c) = f_0 + f_1 c + f_2 c^2 + ...$ is a given function of the 1-st Chern class of the universal tangent line on the Deligne - Mumford spaces $X_{k+2} := \bar{\mathcal{M}}_{0,k+2}$, and $\langle \ ... \ \rangle_{k+2}$ are defined by integration over these spaces.

Lemma 12.1.

$$W(x,y) = \frac{e^{U/x+U/y}}{x+y} \ ,$$

where U depends on f and does not depend on x and y.

Proof. This fact is well-known in the axiomatic theory [2] of Frobenius structures and their τ-functions. Consider the correlators

$$V(x) := \lim_{y\to\infty} yW(x,y), \ U := \lim_{x\to\infty} x(V(x)-1)$$

(they correspond to replacing $1/(y+c)$ or/and $1/(x+c)$ by 1 in the definition of W). The symmetries from the proof of Corollary 6.3 show that for the vector field $F = \partial/\partial f_0 - \sum f_{k+1}\partial/\partial f_k$

$$L_F U = 1, \ L_F V = V/x, \ L_F W = (\frac{1}{x}+\frac{1}{y})W$$

(the so called *string equations*). The degree argument shows that for $f_0 = 0$ we have $U = 0, V = 1, W = 1/(x+y)$. Thus $V(x) = \exp(U/x), \ W(x,y) = (\exp(U/x+U/y))/(x+y)$.

Notice that these identities are compatible with the WDVV-type equation $L_F W(x,y) L_F U = L_F V(x) L_F V(y)$.

This lemma can be deduces also from the explicit formula for correlators between universal tangent lines over Deligne-Mumford spaces which we exploited in the proof of Proposition 10.5. As we see from the above proof some convergence-providing assumptions about f are necessary in this lemma.

Consider now the following modification of the recursion relations of Sections 9 – 11:

$$W_i^j(x,y) = \frac{\delta_{ij}}{x+y} + \sum_{k\neq i}\sum_{d>0} \quad q^d e^{d(U_i-U_k)/(\lambda_i-\lambda_k)} \frac{C_i^k(d)}{(xd+\lambda_i-\lambda_k)} \ W_k^j((\lambda_k-\lambda_i)/d, y).$$

Given $U_0, ..., U_n$ and the coefficients $C_i^j(d)$, the recursion relation has a unique matrix solution (W_i^j) in formal power series of q.

Consider the torus-equivariant GW-theory on $Y = \mathbb{C}P^n$ provided with the r-dimensional convex vector bundle $V = \oplus H^{l_j}$. Introduce the correlator

$$Z_{ij} := \sum_{d=0}^{\infty} q^d \sum_{k=0}^{\infty} \frac{1}{k!} \int_{Y_{k+2,d}} \frac{\phi_i}{x + c^{(0)}}\ e_1^*(t)...e_k^*(t)\ \frac{\phi_j}{y + c^{(k+1)}}\ Euler_{k+2,d}$$

where $Euler_{k+2,d}$ is the equivariant Euler class of the vector bundle $H^0(C, e_{k+2}^* V)$ over $Y_{k+2,d}$, and $\int_Y \phi_i Euler(V) \phi_j/(x+y)$ is taken on the role of the ill-defined summand with $d = 0, n = 2$.

Introduce another correlator, Z_{ij}^*, replacing $E_{k+2,d}$ by the equivariant Euler class $E_{k+2,d}^*$ of the vector bundle $H^1(C, e_{k+2}^* V^*)$ for $d > 0$ and by $Euler^{-1}(V^*)$ for $d = 0$.

In both versions $t = \sum_{\alpha=0}^n t_\alpha \phi_\alpha / \Pi_{\beta \neq \alpha}(\lambda_\alpha - \lambda_\beta)$ denotes the general equivariant cohomology class of Y.

Proposition 12.2. 1)

$$Z_{ij} = \Pi_{a=1}^r (l_a \lambda_i - \lambda'_a)\ e^{u_i/x} W_i^j e^{u_j/y}\ \Pi_{\alpha \neq j}(\lambda_j - \lambda_\alpha)$$

where $u_0, ..., u_n$ are the homogeneous canonical coordinates at t ($u_\alpha = u_\alpha(t, q; \lambda, \lambda')$) of the Frobenius structure, and (W_i^j) is the solution to the recursion relation with $U_\alpha = u_\alpha$ and

$$C_i^j(d) = \frac{\Pi_{a=1}^r \Pi_{m=1}^{l_a d} (l_a \lambda_i - \lambda'_a + m(\lambda_j - \lambda_i)/d)}{\Pi_{\alpha=0}^n \Pi_{m=1\ (\alpha,m) \neq (j,d)}^d (\lambda_i - \lambda_\alpha + m(\lambda_j - \lambda_i)/d)}\ .$$

2)

$$Z_{ij}^* = e^{u_i^*/x} W_i^j e^{u_j^*/y} \Pi_{\alpha \neq j}(\lambda_j - \lambda_\alpha)\ /\ \Pi_{a=1}^r (\lambda'_a - l_a \lambda_j)$$

where $u_0^*, ..., u_n^*$ are the homogeneous canonical coordinates at t^* of the Frobenius structure, and (W_i^j) is the solution to the recursion relation with $U_\alpha = u_\alpha^*$ and

$$C_i^j(d) = \frac{\Pi_{a=1}^r \Pi_{m=1}^{l_a d} (-l_a \lambda_i + \lambda'_a - m(\lambda_j - \lambda_i)/d)}{\Pi_{\alpha=0}^n \Pi_{m=1\ (\alpha,m) \neq (j,d)}^d (\lambda_i - \lambda_\alpha + m(\lambda_j - \lambda_i)/d)}\ .$$

3) *In each case put*

$$V_i^j(x, U) = \lim_{y \to \infty} y W_i^j = \delta_{ij} + \hat{V}_i^j(U)/x + o(1/x).$$

Then

$$t_\alpha = u_\alpha + \sum_{j=0}^n \hat{V}_\alpha^j(u),\ \ t_\alpha^* = u_\alpha^* + \sum_{i=0}^n \hat{V}_i^\alpha(u^*).$$

Proof. The recursion for Z_{ij} and Z^*_{ij} is based on the same idea as in Sections 9 – 11. We use Borel's localization formula in order to reduce computation of the correlators to summation over all fixed point components in $Y_{k+2,d}$. The components are labeled by trees "walking" in the 1-skeleton of the n-simplex (the momentum polyhedron of the torus action on $\mathbb{C}P^n$). Each tree contains the chain of edges connecting the vertices i and j where the first and the last marked points are mapped to. We cut off the 1-st edge (connecting the vertex i with say vertex α). The rest of the chain contributes to the Borel localization formula by $Z_{\alpha j}((\lambda_\alpha - \lambda_i)/d, y)$ while the coefficient $C_i^j(d)$ takes in account the contribution of the edge $[i, \alpha]$ of degree d.

The subtlety hidden in this argument is due to the possibility that the first marked point can belong to a component of the curve which is mapped to the vertex i and carries k more special points giving birth to k branches of the tree (not containing the last marked point) and / or l extra marked points (carrying the cohomology class t). Cutting off the first edge we should take care of the weight obtained by integration over the factor $\bar{\mathcal{M}}_{k+l+2}$ of the fixed point set and by summation over all possibilities for the k branches.

However it is easy to see that this sum effectively reduces to the exponential series of Lemma 12.1 (with $f(c)$ unknown so far) and is thus equal to $\exp(U_i/x + U_i d/(\lambda_i - \lambda_\alpha))$. Moreover, compaing the description of the correlator U_i from the proof of Lemma 12.1 with the definition of "local" equivariant correlators u_i given in Section 8 and applying Theorem 8.1(a) we conclude that $U_i = u_i$ is the canonical coordinate of the Frobenius structure. In particular correlators U_i are well-defined as formal q-series.

In order to prove the relation between flat and canonical coordinates described in the part (3) of the Proposition, consider the correlators

$$Z_i^{(1)} = \frac{d}{d(1/x)}|_{1/x=0} \sum_j \lim_{y\to\infty} \frac{yZ_{ij}}{\Pi_{\alpha\neq j}(\lambda_j - \lambda_\alpha)}$$

$$= \sum_{k,d} q^d \langle \phi_i, t, ..., t, 1\rangle_{k+2,d}/k! = t_i\langle\phi_i, 1\rangle \ .$$

The recursion relation for Z_{ij} shows that $Z_i^{(1)} = (U_i + \sum_j \hat{V}_i^j)\langle\phi_i, 1\rangle$.

Similarly,

$$\frac{d}{d(1/y)}|_{1/y=0} \sum_i \lim_{x\to\infty} \frac{xZ^*_{ij}}{\Pi_{\alpha\neq i}(\lambda_i - \lambda_\alpha)} = t^*_j\langle 1, \phi_j\rangle = (U_j + \sum_i \hat{V}_i^j)\langle 1, \phi_j\rangle.$$

Remark. According to Section 6 the matrix $\lim_{y\to\infty} yZ_{ij}(\hbar, y)$ and its counterpart "with $*$" are essentially the fundamental solutions of the linear differential systems defined by the corresponding Frobenius structures on the equivariant cohomology space of $\mathbb{C}P^n$ provided with the convex vector bundle V (see Section 4). Proposition 12.2 identifies the fundamental solution *expressed in terms of canonical coordinates* with the solution of a linear recursion relation and additionally describes the non-linear transformation from canonical to flat coordinates. These equations are exactly identical to the critical point equations obtained by M. Kontsevich [3]. In particular the linear recursion relations can be also interpreted as critical point equations for a *quadratic* combinatorial "Lagrangian" with the "kinetic energy" (its terms should correspond to the edges of the momentum simplex) determined by the coefficients $(C_i^j(d))^{-1}$ and the "potential energy" (whose terms should correspond to the vertices) determined by the factors $\exp(U_i d_\alpha/(\lambda_i - \lambda_\alpha) + U_i d_\beta/(\lambda_i - \lambda_\beta))$. The important problem of finding the general solution to the linear recursion relation remains open as well as the role of these relations and of the quadratic combinatorial Lagrangian in the theory of isomonodromy deformations [2] accompanying the concept of canonical coordinates — unclear.

Corollary 12.3 (*Quantum Serre Duality).*

At the points t and t^* with the same canonical coordinates $U = (u_0, ..., u_n) = (u_0^*, ..., u_n^*)$ the correlators (Z_{ij}) and (Z_{ij}^*) satisfy the following relation:

$$Z_{ij}^*(U, x, y, q) = \frac{(-1)^r Z_{ij}(U, x, y, (-1)^{\sum l_a} q)}{\Pi_{a=1}^r (l_a \lambda_i - \lambda'_a) \Pi_{a=1}^r (l_a \lambda_j - \lambda'_a)} \ .$$

Remarks. This fact (which is proved by comparisson of the coefficients $C_i^j(d)$ of the two recursion relations) can be explained in the following way: for the curve C which is a chain of $\mathbb{C}P^1$'s with two marked points p_0 and p_∞ on the first and the last component, the Serre duality theorem provides a non-degenerate duality between $H^0(C, V \otimes \mathcal{O}[-p_0])$ and $H^1(C, V^* \otimes \mathcal{O}[p_\infty])$ since the canonical class $K + [p_0] + [p_\infty]$ on such a curve C is trivial.

The quantum Serre duality theorem shows that the Frobenius structures on $H_T^*(\mathbb{C}P^n)$ corresponding to V and V^* are equivalent, but the equivalence involves some transformation of the flat coordinates. The same equivalence statement holds true in the limit $\lambda = 0, \lambda' \neq 0$. It is natural to conjecture that such an equivalence of two Frobenius structures corresponding to V and V^* should be true for vector bundles over arbitrary Y. One of the two equivalent Frobenius structures coincides with the Frobenius structure of the *super-manifold* (in terminology of [28]) of dimension $(n|r)$ (here the fiber of the convex bundle V over $\mathbb{C}P^n$ is considered odd) and in the limit $\lambda' = 0$ degenerates to the Gromov-Witten theory on the codimension r complete intersections X defined by sections of V. The second one corresponds to an $n+r$ dimensional non-compact manifold — the total space of the bundle V^*. It would be interesting to study the relation between the two structures in greater detail.

Corollary 12.4. *The translations*
(a): $t_\alpha \mapsto t_\alpha + \tau$
(b): $t_\alpha \mapsto t_\alpha + \lambda_\alpha \tau$
on the Frobenius manifolds cause respectively the translations $\tilde{U}_\alpha = U_\alpha + \tau$ *and* $\tilde{U}_\alpha = U_\alpha + \lambda_\alpha \tau$ *of the canonical coordinates and the following transformations of the matrix* (Z_{ij})*:*

$$(a) \quad Z_{ij}(\tilde{U}, x, y, q) = e^{\tau/x + \tau/y} Z_{ij}(U, x, y, q) \ ,$$

$$(b) \quad Z_{ij}(\tilde{U}, x, y, q) = e^{\lambda_i \tau/x + \lambda_j \tau/y} Z_{ij}(U, x, y, q e^{\tau}) \ .$$

The same transformation formulas hold for the matrix (Z^*_{ij}).

Proof. The translation (a) corresponds to the vector field $\sum \partial/\partial t_\alpha = \sum \partial/\partial u_\alpha$ representing the unity of the quantum cup-product. The translation (b) corresponds to the symmetry generated by the vector field $q\partial/\partial q - \sum \lambda_\alpha \partial/\partial t_\alpha$. This justifies the effect of the translations on the canonical coordinates. The rest follows now directly from the form of the recursion relation.

Remarks. This corollary explains the origin of the invariance property with respect to the change of coordinates stated in Propositions 11.3, 11.6: it is a consequence of the symmetries (5), (6) from Section 5.

The proof of the mirror conjecture given in Sections 9 – 11 could be more straightforward and conceptual if we had at our disposal a well-developed theory of Frobenius structures, their flat and canonical coordinates, for the models of Landau-Ginzburg type (see [10]) more general than K. Saito's theory [29] of isolated critical points of holomorphic functions.

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