Fourth grade science: Sharp and vibrant

A parent's report on his son's school project

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The Big Question

For the greatest experimentalist of all times Michael Faraday, getting to the top of scientific Olympus took years of routine: reading books while binding them, helping his scientific boss Sir Humphry Davy with editing lecture notes, serving as his research assistant *and* servant, as well as a pet-sitter for his wife's dog. According to the 4th grade section of "Science Content Standards for California Public Schools" [1], nowadays young scientists take a shortcut:

> Scientific progress is made by asking meaningful questions and conducting careful investigations. As a basis for understanding this concept [...] students should develop their own questions and perform investigations.

So, one day my ten-year-old son comes from school: "Dad, we need to choose a Big Question for science project. This one looks interesting: Do plants grow faster when you yell at them?"

I am impressed.

Visionary ideas of math and science education in the US are a favorite laughingstock in some Second World countries. Our curriculum news that "substances have properties" has even reached central Moscow media and a presidential education committee of the Russian Federation. Comparing to such truisms, which are plentiful in the Standards, the reasoning, that the excess carbon dioxide we breathe out yelling can be used by plants to grow faster, sounds pleasingly refreshing. Yet, I try to kill the brilliant idea with arguments like these:

1. A rescuer administering CPR does not end up in jail for suffocating the victim.

2. You don't seem to have difficulty breathing when I yell at you, do you?

(The correct answer to the last question is "I do", but my son agrees that the plan to speed up plant growth can be dismissed without experimenting.) What I did *not* dare to tell him was:

3. Some plants in your hydroponic home garden have grown twice as big as some others though all of the same breed. Suppose that there is, on average, a (huge!) 5% improvement from yelling at the plants. To notice this, you would need to reduce the (say) 50% error of a single experiment to the level below 5% by repeating the experiment (not 10, but) over 100 times! This is because taking the average over n tries reduces the error by about $\sqrt{n}$ times.

My son is in the 4th grade, but his math is somewhere in grade 7, so he knows what $\sqrt{n}$ is. Still, how do I justify the above conclusion to him?

The standard explanation in probability theory is based on the Central Limit Theorem or on the Law of Large Numbers. The latter can be interpreted geometrically as an n-dimensional application of the Pythagorean Theorem. Namely, the deviations of n independent measurements from the mean value can be thought of as n pairwise perpendicular vectors of the same length (well, in the infinite dimensional space of random variables). Their sum, which looks like the diagonal in the cube of dimension n, is $\sqrt{n}$ times longer than the edge of the cube. Thus, the sum of n random errors, when divided by n, becomes $\sqrt{n}$ times smaller than the error of an individual measurement. "Understood?!"

OK, let's try something else. The phenomenon is related to diffusion, Brownian motion, heat conduction, statistics of random walks. Perhaps the simplest model is: A drunkard makes one step per second randomly to the left or to the right; how far will he be in an hour? One can start with a small number of steps $n = 0, 1, 2, 3, ...$ and calculate expected travel distances, discover a relation with Pascal's triangle, and may be even estimate π. Indeed, the expected *squared* distance is exactly n, but the expected *distance* squared is smaller, by a factor approaching $\pi/2$ when n is very large. Hmm, looks like a topic for another project!

— How about this: *What makes sounds into music?*

After some clarification we accept this as the Big Question, but there is one problem left, and I am supposed to settle it with the teacher.

While young Faradays of old days had to begin with studying their subject in depth, modern scientists are required to start with a *hypothesis*, and guess, even if out of the blue, an answer to the Big Question. I have a problem with this plan, not because it sounds nonsense (as it does), but because with my particular question, there are charming hypotheses to discover (I know — I've been there!), and if only I give them away upfront, the joy of discovery will be gone. So, I try to argue with the teacher (who knows a lot about musical intervals):

— Press down a guitar string at the 3rd fret to make it sound a *minor third* higher than the whole string. What portion of the whole string's length will it be?

— I have no idea!

— So, you don't have a *hypothesis*, but you can take your guitar and measure it, and you will be up to a surprise!

— Perhaps, so what?

— Therefore, by applying the scientific method we have to conclude that the hypothesis, that scientific investigations must always begin with hypothesizing, is incorrect.

— ???

It turns out that the scientific method doesn't apply to real life, and hypothesizing was always useful in scientific projects they made at the school of education, and her scientist husband finds out-of-the-blue guessing helpful, and science projects like this have been at school for years, and *they* are only little kids [so, they won't remember the nonsense they are taught], and even if it is true that math and science in US schools sucks, there is nothing we (me and her) can do about it. But I win; not because I am right (truth in our culture is a matter of opinion), but because this is *my* son (and our school is private). Hurrah, we are approved!

The Amusical Foot

The 1st (thinnest) string of a classical guitar is tuned to E4, the *mi* of the 4th octave of a properly tuned piano keyboard. The interval from E4 to E5 on the keyboard (an *octave*) is divided into 12 *half-steps*. Various musical intervals have the following traditional names:

3 half-steps: *minor third* (e.g. from E to G)
4 half-steps: *major third* (e.g. from G to B)
5 half-steps: *perfect fourth* (e.g. from B to E)
7 half-steps: *perfect fifth* (e.g. from E to B)
9 half-steps: *major sixth* (e.g. from G to E)

Our first aim is to find out what string lengths correspond to which musical intervals.

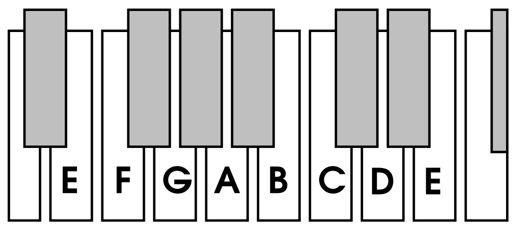

Figure 1

It is said that true alchemists analyze their stuff by mouth and meter it by tablespoon. We take the same bold approach and introduce a new unit of length, the *amusical foot*, equal to the lengths of my son's left shoe. Conveniently, our guitar's string is 3 amusical feet long. The distance from the 0th to the 12th fret is 1.5 amusical feet. It is truly the midpoint of the string, because both halves of it produce the same sound, E5.

A bit of theory helps. When a guitar string vibrates, it makes sound waves. These are compressions and decompressions of air, taking turns in time, which spread in space and reach our ear creating the sound. The frequency of vibration depends on the string's length. The truth is that, in principle, the whole string can vibrate as if two halves of it were moving separately (but in opposite directions, see Figure 2). The frequency of this vibration is twice the frequency of the vibration of the whole string. Likewise, the same string can vibrate as 3 thirds of the string, or 4 quarters, or 5 fifth, etc., with the frequency respectively 3,4,5, etc. times greater than the frequency of the string vibrating as a whole. In a real sound, all these vibrations: of the whole string, of its halves, thirds, etc., are mixed together.

All this is not too hard to *tell*, but to *explain* the phenomenon, one would need to solve a Partial Differential Equation, use Fourier series, or at least know some trigonometry.

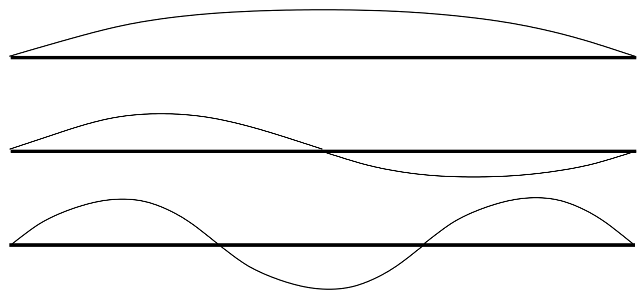

Figure 2

Fortunately, this pedagogical problem has a high-tech solution. To our microphone-equipped laptop, we dounload an open-source sound-editing package called *Audacity*. It includes a frequency analyzer. For example, Figure 3 shows the frequency spectrum of E4 pre-recorded from our piano keyboard.

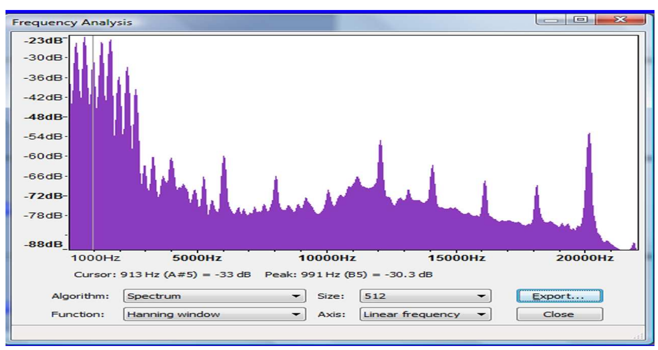

Figure 3

Seeing is believing. We see peaks at approximately integer multiples of 330 Hz. Likewise, the piano key E5 generates peaks at approximately integer multiples of 660 Hz. It is now the time for our first revelation. The key E5 produces no new frequencies compared to those of E4. This explains why E5 sounds “the same as E4 but only an octave higher.”

Our next goal is to measure the perfect fifth, and here the time comes to put forward scientific hypotheses. Looking at the guitar, my son makes an educated guess, and confirms it by measurement. The distance from the 0th to the 7th fret turns out to be exactly 1 amusical foot! This means that the *perfect fifth* interval corresponds to the shortening of the string by a factor of 2/3, and respectively to the 1.5 times increase in the frequency of vibration. Therefore, when a perfect fifth is played (e.g. the keys E4 and G4) the 3rd multiple of the main frequency of the lower note coincides with the 2nd multiple (i.e. with G5 in this example) of the higher note.

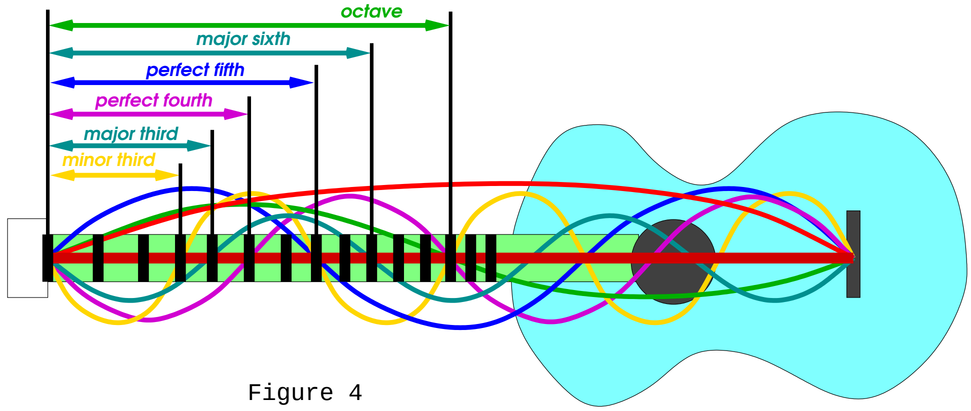

Figure 4

Our next surprise is that the 3rd fret *bisects* the amusical foot between frets 0 and 7. This means that a minor third corresponds to the shortening of the string by a factor of 5/6. Furthermore, the major third between frets 3 and 7 corresponds to the ratio of 2/3 to 5/6, i.e. 4/5; the perfect fourth between frets 7 and 12 corresponds to the ratio of 1/2 to 2/3, i.e. 3/4; the major sixth between frets 3 and 12 corresponds to the ratio of 1/2 to 5/6, i.e. 3/5. These theoretical predictions can be then confirmed by measuring in amusical feet the distances (Figure 4) from fret 0 to frets 4, 5, and 9 respectively.

It becomes more clear now "what makes sounds into music" at least in the sense of understanding what makes basic musical intervals and simple chords sound *consonant*. When main frequencies of participating sounds have relatively simple ratios, integer multiples of these frequencies, which are always present in the real sound, do not become congested but instead reinforce each other, apparently creating the impression of harmony. Were the ratios chosen randomly, their integer multiples would be all different, thus creating the impression of *dis*harmony.

Our theory is well-grounded, and we can now find ratios of string lengths corresponding to all (but one) of the 12 half-steps of an octave without any further measurement (feasibility of which, based on the precision provided by the amusical foot, would become questionable anyway). For example, the interval between A and C is a minor third, and thus the string length corresponding to C should be 5/6 of the length 3/4 corresponding to A, i.e. 5/8, as shown on Figure 5.

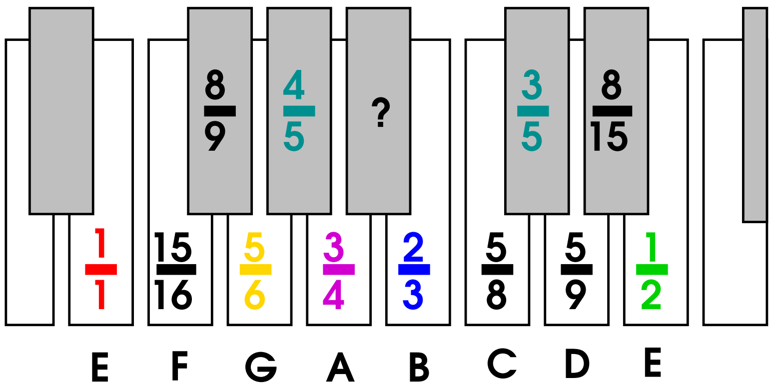

Figure 5

Such computations extended beyond one octave help us realize that all the fractions involved have numerators and denominators expressed as products of only three prime numbers: 2, 3, and 5. The significance of this observation can be illustrated with the example of the 7 half-step interval from F to C. The corresponding fractions 15/16 and 5/8 have the ratio 2/3 required for the perfect fifth. Harmony of sophisticated sequences of notes seems to depend on this amazing property of the musical scale to contain many intervals with ratios of the frequencies found in consonant musical intervals. For a series of fractions made of 2,3, and 5, it may not be so surprising that many of their ratios are equal to 2/3, 3/4, 5/6, 4/5, 3/5.

Let's summarize our accomplishments. We did make scientific progress by asking meaningful questions and conducting careful investigations, just as the "Science Content Standard" [1] purports. The questions we asked were developed not by fourth graders but by the ancient Greek philosopher Pythagoras (whose theory, however, made use of only two prime numbers: 2

and 3). We did not aim to understand any abstract nonsense *about* science: this would not qualify in our view for the title of a science project; our goal was to *do* science in order to explain the world. Namely, we applied some physics of string vibrations to understand the basics of musical harmony, and we encountered many of the common attributes of genuine scientific research: units, measurements, conjectures, bare-feet and high-tech devices, theoretical predictions, experimental verifications, and ... contradictions.

As if to exemplify Kuhn's theory of scientific revolutions [2], our initially successful approach to explaining consonance of sounds in music is facing serious difficulties. One is that some of the 2 half-step intervals on Figure 5 correspond to the ratio 8/9 while some others to 9/10. How can a musical tune be transposed into another key (say, from E-minor to D-minor) if various intervals containing the same number of half-steps are not equal to each other? Another problem: Suppose that intervals containing the same number of steps are *made* equal; then the question mark on Figure 5 must be $\sqrt{1/2}$, since each of the intervals: from E to A$^\sharp$ and from A$^\sharp$ to E, contain 6 half-steps. How can intervals be expressed by fractions if $\sqrt{2}$ is irrational? Apparently, using Kuhn's terminology, a *paradigm shift* is impending.

The Musical Root

If instead of amusical foot we introduced a more accurate measuring tool, we would have missed all our neat observations about the role of fractions for the musical scale. Now it is time to abandon the alchemist's attitude and determine where on the guitar's neck the frets are *really* placed.

We find that the total length of the string (to fret 0) on our guitar is 648 *mm*, and measure with the precision of 0.5 *mm* such lengths to frets 1, 2, ..., 11, and 12 respectively. Then, in order to compare to each other the 12 half-steps of one octave, we compute 12 ratios of our consecutive measurements: of the length to fret 1 to the length to fret 0, of the length to fret 2 to the length to fret 1, and so on. Using the virtual scientific calculator on out laptop we find that all the 12 ratios fall within the interval between 0.942 and 0.944. Since the precision guaranteed by our measurement is about 0.5 $mm/0.5\ m = 0.001$, we reasonably *conjecture* that all the 12 ratios are probably the same and approximately equal to 0.943 ± 0.001.

Here is how the *musical root*, q, is born. If the distance to fret 1 is q times the distance to fret 0, and the distance to fret 2 is q times the distance to fret 1, and so on, then the distance to fret 12 is q^{12} times the distance to fret

0. Since fret 12 is the midpoint of the string, we conclude that $q^{12} = 0.5$, i.e. q is the 12th root of 0.5. Figuring this out is a good exercise for a 6–7th grader, but even a greater challenge is to find a way of computing q on our virtual calculator. When the excitement, caused by the news that function x^y with $y = 1/12$ spits out the 12th root of x, cools down, the musical root is found (with the precision, provided by our virtual calculator, of 32 decimal places after the dot):

$$q = 0.5^{1/12} = 0.94387431268169349664191315666753\ldots$$

In order to test our conjecture that positions of guitar frets are governed by powers of q, we compute the average of our 12 experimental ratios, and find (after dropping 25 rightmost digits invariably displayed by the calculator):

$$0.9438746\ldots$$

My son takes this result for granted ("Good precision!") but I am stunned. Since $\sqrt{12}$ is between 3 and 4, taking the average of our 12 measurements should be expected to improve the accuracy from 10^{-3} to, say, $3 \cdot 10^{-4}$, but how on Earth could it become $3 \cdot 10^{-7}$, a thousand times better than expected?!

Anyhow, it remains to compare powers of the musical root (i.e. relative wave lengths of the 12 notes of the same octave) with corresponding fractions shown on Figure 5:

$$\begin{array}{ll}
q^1 = 0.94387\ldots & 15/16 = 0.9375 \\
q^2 = 0.89089\ldots & 8/9 = 0.88888\ldots \\
q^3 = 0.84089\ldots & 5/6 = 0.83333\ldots \\
q^4 = 0.79370\ldots & 4/5 = 0.8 \\
q^5 = 074915\ldots & 3/4 = 0.75 \\
q^6 = 0.70710\ldots & ? \\
q^7 = 0.66741\ldots & 2/3 = 0.66666\ldots \\
q^8 = 0.62996\ldots & 5/8 = 0.625 \\
q^9 = 0.59460\ldots & 3/5 = 0.6 \\
q^{10} = 0.56123\ldots & 5/9 = 0.55555\ldots \\
q^{11} = 0.52073\ldots & 8/15 = 0.53333\ldots \\
q^{12} = 0.5 & 1/2 = 0.5
\end{array}$$

We see now how to reconcile two conflicting features of the musical scale. Wave lengths (or frequencies) of successive notes produced by a properly tuned musical instrument form a geometric progression. As a result, *transposing* a tune from one key to another (i.e. translating it up or down by any number of half-steps) preserves ratios between frequencies of participating notes. When the *common ratio* of the geometric progression equals the musical root $q = 0.5^{1/12}$, relative wave length of the notes (left column) are not too far from the fractions (right column) made of prime numbers 2,3, and 5. This explains why the intervals between the notes sound consonant: they are not ideal, but the error is small enough, so that our ear does not catch the difference.

The Well-Tempered Clavier

Even a brief look at the Wikipedia article [3] convinces that the actual theory and practice of musical tuning is a sophisticated subject with a long and interesting history.

The system we discovered with the amusical foot is called *just intonation*, a term referring to the frequency ratios being simple fractions. While some intervals of such scale sound ideal to the ear, some others sound terribly off-tune. This makes it hard to compose music in some of the 24 (12 major and 12 minor) musical keys, and therefore to *modulate* (i.e. move back and forth) between several keys within the same musical piece. In the past, one of the attempted solutions to this problem on keyboard instruments was introducing, for some notes, duplicate keys and tuning them slightly differently.

The modern solution to the problem is provided by the system based on the musical root and geometric progression. It is called *equal temperament*, (where the second word refers to any way of "tempering" with just intervals). A monumental work by J. S. Bach, *The Well Tempered Clavier*, explores advantages of the novel tuning system. The work consists of 24 pieces written one in each of the major and minor keys. It demonstrates that, with the modern system of tuning, all the keys are equally usable. It may therefore come as a surprise[1] that the tuning system, called *well temperament* and meant by J. S. Bach, is different from what we use today. The equal temper-

[1] Especially to those who've done the exercise from Gelfand–Shen's *Algebra* [4] asking one to "listen to the Well Tempered Clavier and enjoy it."

ament system became standard only in the 19th century, although we find Galileo's father Vincenzo Galilei (1520–1591) among its early proponents.

Consumer Science Reports

To the Science Fair in my son's school, I went with a hefty doze of skepticism. I already knew that, inspired by the alleged ability of one great scientist to explain everything to a six-years-old, the teacher gave each young Einstein five minutes to present his or her project to the class of kindergartners. The message was clear: You don't need to know much in order to start a science project, nor you are expected to learn anything new as a result of it, and so no time, special training, or intellectual effort is required to understand what you've done. Yet, the Big Questions struck me with their uniform and painfully familiar style; apparently, I didn't know that *Consumer Reports* has become a leading scientific journal.

Here is my sampler of topics, two practical and two "pure" ones:
Which baseball bat hits farther: wooden, plastic, or aluminum?" Which brand of toilet tissue absorbs more liquid? The most expensive won.
Which kind of apples dries up faster? My favorite one (as it was openly sarcastic): *Which brand of soda produces the most spectacular fountain when Mentos are dropped into the bottle?*

A disturbing thought was on my mind as I was leaving the event: Yes, indeed, they *are* just little kids, but when they grow up, they *will* remember, and they will help *their* kids (and students) to answer (and ask) the same Big Questions, and so it will proceed.

For Adults Only

It remains to find out how on Earth could precision improve from 10^{-3} to $3 \cdot 10^{-7}$ as a result of averaging our 12 ratios. Let $a_0, a_1, \ldots, a_{12}$ denote our measurements of string's lengths to frets $0, 1, \ldots, 12$. The ratios in question were: $a_1/a_0, a_2/a_1, \ldots, a_{12}/a_{11}$. If the mean taken were the *geometric* (rather than arithmetic) one, the result would have been equal to the musical root $q = 2^{-1/12}$ *exactly*. This is because

$$\frac{a_1}{a_0} \cdot \frac{a_2}{a_1} \cdots \frac{a_{12}}{a_{11}} = \frac{a_{12}}{a_0} = \frac{1}{2}.$$

The arithmetic mean is greater than the geometric one. Therefore the error had to be positive (as it was). Moreover, when the numbers whose average is taken are close to the mean with the error of order δ, the arithmetic mean exceeds the geometric one by a quantity of order δ^2. More precisely, when 12 numbers $q(1+\delta_i)$, being close to q with the errors $|\delta_i| < 10^{-3}$, have the geometric mean q, then

$$1 = \prod_1^{12}(1+\delta_i) = 1 + \sum_i \delta_i + \sum_{i<j} \delta_i\delta_j + \text{ error of order } 10^{-7}$$

(namely, $\approx \binom{12}{3}10^{-9}$). We find that the arithmetic mean $\frac{1}{12}\sum\delta_i$, instead of being of order 10^{-3}, must be smaller than $\frac{1}{12}\binom{12}{2}10^{-6} < 10^{-5}$, and hence its square is already negligibly small. Using this, and the identity

$$\sum_{i<j}\delta_i\delta_j = \frac{1}{2}\left(\sum_i \delta_i\right)^2 - \frac{1}{2}\sum_i \delta_i^2,$$

we finally conclude that, modulo an error of order 10^{-8},

$$\frac{1}{12}\sum_{i=1}^{12}\delta_i \approx \frac{1}{24}\sum_{i=1}^{12}\delta_i^2 < 5\cdot 10^{-7}.$$

Thus error $3\cdot 10^{-7}$ of our computation was just about the size the theory predicts!

It becomes clearer now how our scientific calculator works, or could work. Suppose we want to compute the 12th (or any other) root of a given positive number a, using only the four arithmetic operations, and may be the integer power function (but not the power $1/12$ of course). Let x denote some approximation to $a^{1/12}$. Then the 12 numbers: $x, x, \ldots, x$ (11 times), and a/x^{11}, have the geometric mean exactly equal to $a^{1/12}$. Therefore their arithmetic mean, $(11x + ax^{-11})/12$, will be a much better approximation to $a^{1/12}$ than x was. Thus, we arrive at the following algorithm of computing $a^{1/12}$. Start with an approximation x_0 and generate a sequence of approximations:

$$x_{n+1} = \frac{11x_n + ax_n^{-11}}{12}, \quad n = 0, 1, 2, \ldots$$

Following the previous discussion, one can conclude that if the error of the initial approximation is small enough, the successive errors will roughly *square* in each iteration. Here is the output of this algorithm (realized on our virtual scientific calculator) with $a = 0.5$ and $x_0 = 0.9$ ($\delta \approx 0.05$)

$$
\begin{aligned}
x_1 &= 0.95777648105520561635751190885374\ldots & (\delta < 2 \cdot 10^{-2}) \\
x_2 &= 0.94493217630548010807655654564121\ldots & (\delta < 2 \cdot 10^{-3}) \\
x_3 &= 0.94388080204093916058494154597692\ldots & (\delta < 10^{-5}) \\
x_4 &= 0.94387431292707352810542390029257\ldots & (\delta < 3 \cdot 10^{-10}) \\
x_5 &= 0.94387431268169349699276758114248\ldots & (\delta < 5 \cdot 10^{-20}) \\
x_6 &= 0.94387431268169349664191315666753\ldots & (\delta < 10^{-32})
\end{aligned}
$$

In Chapter 3 of Rudin's *Principles of Mathematical Analysis* [5], exercise no. 16 asks to investigate the convergence of an algorithm for computation of square roots, based on the formula: $x_{n+1} = \frac{1}{2}(x_n + ax_n^{-1})$. I often assign this exercise to my students for homework, but only now I begin to truly understand and appreciate it myself.

References

[1] *Science Content Standards for California Public Schools*, at http://www.cde.ca.gov/be/st/ss/documents/sciencestnd.pdf

[2] T. S. Kuhn. *The Structure of Scientific Revolutions.* 1st. ed., Univ. of Chicago Pr., Chicago, 1962.

[3] *Musical Tuning*, at http://en.wikipedia.org/wiki/Musical_tuning

[4] I.M. Gelfand, A. Shen. *Algebra.* Birkhäuser, Boston, 1993.

[5] W. Rudin. *Principles of Mathematical Analysis,* 3rd edition. McGraw Hill, 1976.