Answers to homework problems

HW1

1.4. The discriminant surface is a swallow-tail. One way to see the connection is to notice that the graph of the optical distance function (e.g. to the parabola $y = x^2$ from exercise 1.2) consists of lines in space (because along each ray, the optical distance changes linearly) tangent to the curve with the cusp, which is the lift of the caustic from the plane into space by its arc length function. The discriminant has a similar description. Namely, the curve $(x-t)^3(x+3t) = x^4 - 6t^2x^2 + 8t^3x - 3t^4$ consisting of the polynomials x^4+Ax^2+Bx+C with a triple root is the "semi-cubical parabola" $(B/8)^2 = (-A/6)^3$ in the (A, B)-plane (with the same cusp as the caustic) lifted into space by $(C/3) = (-A/6)^2$. The tangent line to this curve at $(x-t)^3(x+3t)$ is spanned by the t-derivative $-12t(x-t)^2$; adding it with an arbitrary coefficient yields all the polynomials with at least a double root at $x = t$ (when $t \neq 0$). When $t = 0$, such polynomials form the line x^4+sx^2 "tangent" to our cuspidal curve at the very cusp, x^4.

This "coincidence" is a part of the general *theory of singularities of caustics and wave fronts*. According to this theory, in 2D geometrical optics, the graph of the optical distance function to a *generic* initial wave front can have only those singularities which appear at various point of our discriminant surface.

2.4. Equating the sectorial velocities $|L|/2m$ and the total energies $E(< 0)$ at the perigee $(i = 1)$ and apogee $(i = 2)$, we find

$$v_i r_i = \frac{|L|}{m}, \ \frac{mv_i^2}{2} - \frac{G}{r_i} = E,$$

where $i = 1, 2$, and (eliminating v_i): $|L|^2/2m - Gr_i - Er_i^2 = 0$. From the Vieta theorem we have:

$$r_1 r_2 = \frac{|L|^2}{-2mE}, \quad r_1 + r_2 = \frac{G}{-E}.$$

On the other hand, from the geometry of an ellipse with the major semiaxis a and minor b, we have $a = (r_1 + r_2)/2$ and $b = \sqrt{r_1 r_2}$. The period of revolution

$$T = \frac{\text{area of the ellipse}}{\text{the sectorial velocity}} = \frac{\pi ab}{|L|/2m} = \sqrt{\frac{\pi^2 m G^2}{2(-E)^3}},$$

i.e. $|L|$ cancels out.

Since $m = G/M\gamma$, and $G/(-E) = 2a$, we obtain Kepler's 3rd law: *the square of the periods are proportional to the cubes of the major semiaxes of the elliptic orbits*, with the proportionality coefficient $4\pi^2/M\gamma$, where M is the mass of the attracting center, and γ is the universal gravitational constant.

HW2

2.6. In short, rotations of the cube permute its four diagonals.

In a bit more detail: the previous sentence defines a map $f : G \to S_4$ from the group G of rotations of the cube to the group S_4 of permutations on the set of cube's diagonals. Obviously, f is a *group homomorphism*, i.e. it maps compositions of rotations to composition of permutations: $f(gg') = f(g)f(g')$ for all $g, g' \in G$ (and therefore $f(e) = \mathrm{id}$, $f(g^{-1}) = (f(g))^{-1}$)). Next, $|S_4| = 4! = 24$, and $|G| = 24$ too. Namely, a rotation g can map face of the cube to any of the cub's 6 faces, and every rotation g' mapping the first face into the same face as g does can be uniquely written as the composition $g(g^{-1}g')$ of g with a rotation preserving the first face. Since there are 4 elements of G preserving the first face (namely, rotations through 0, 90, 180 and 270 degrees around the axis through the center of the face), the total number of elements in G is 6×4. (Similar arguments based on counting edges or vertices of the cube give, of course, the same result: 12×2 or 8×3.) Therefore, to prove that f is a bijection, it suffices to show that f is one-to-one: $f(g) = f(g') \;\Rightarrow\; g = g'$, or, equivalently, (taking $h = g^{-1}g'$) $\mathrm{id} = f(h) \;\Rightarrow\; h = e$. Thus, let h be a rotation preserving each diagonal. The assumption that some diagonals are reversed leads to a contradiction: picking 3 diagonals of which an odd number is reversed for the axes of a (non-Cartesian) coordinate system, we find that $\det h = -1$ (i.e. h transforms right gloves into left ones), so h is not a rotation. Therefore h is the identity transformation.

3.4. First assume $\mathrm{Re}\,\beta > 0$ and compute $I_k := \int_{-\infty}^{\infty} e^{-\beta y^2} y^k dy$. For $k = 2l-1$, $I_k = 0$ since the integrand is an odd function. For $k = 2l > 0$ we integrate by parts:

$$\int_{-\infty}^{\infty} y e^{-\beta y^2} y^{2l-1} dy = \frac{2l-1}{2\beta} \int_{-\infty}^{\infty} e^{-\beta y^2} y^{2l-2} dy.$$

Writing $2l - 1/2\beta$ as $2l(2l-1)/4\beta l$ and continuing inductively, we find $I_{2l} = I_0 \times (2l)!/4^l \beta^l l!$, where $I_0 = \sqrt{\pi/\beta}$ (as is well-known).

In our problem, we put $x = \sqrt{\lambda} y$ and have

$$\int e^{i(x^2 - x^3)/\lambda} dx = \sqrt{\lambda} \int e^{iy^2} e^{-i\sqrt{\lambda} y^3} dy =$$

$$\sqrt{\lambda} \int e^{iy^2} \left(1 - i\sqrt{\lambda} y^3 - \frac{1}{2}\lambda y^6 + \mathcal{O}(\lambda^{3/2})\right)\, dy =$$

$$e^{\pi i/4}\sqrt{\pi\lambda}\left(1 - \frac{6!}{4^3 i 3!}\frac{\lambda}{2} + \mathcal{O}(\lambda^2)\right) = e^{\pi i/4}\sqrt{\pi\lambda}\left(1 + \frac{15i}{16}\lambda + \mathcal{O}(\lambda^2)\right).$$

In the above computation, the (divergent!) integral $\int e^{iy^2} y^6 dy$ is evaluated by replacing i in the exponent with $\beta = i - \epsilon$, applying the result of our auxiliary computation with $l = 3$, and then passing to the limit $\epsilon = 0$, i.e., in other words, formally using the result, obtained for $\mathrm{Re}\,\beta > 0$, when $\mathrm{Re}\,\beta = 0$.

`Remark.` The above approach gives an algorithm of formally writing out terms of the entire asymptotical series, and shows that its coefficients depend on the Taylor coefficients of the amplitude and phase functions at the (non-degenerate) critical point of the phase, but do not depend at all on the interval of integration as long as it contains the critical point. A motivation for this algorithm, that is, an explanation of what the asymptotical series thus produced has with the oscillating integral.

In fact, the oscillating integral (which in our case has the amplitude function $a(x) = 1$) is supposed to have a compactly supported amplitude. Having this in mind, introduce the "hat" function $\sigma(x)$: an infinitely differentiable even function equal to 1 on some interval $[-a, a]$ around the critical point $x = 0$, and equal to 0 outside some greater interval $[-b, b]$ ($b > a > 0$). Now, consider the auxiliary integrals as they would occur in the honest computation:

$$\tilde{I}_k(\lambda) := \int_{-\infty}^{\infty} e^{ix^2/\lambda} x^k \sigma x dx,$$

where $m \leq k$. Integrating by parts as above, we find

$$\begin{aligned}\tilde{I}_k(\lambda) &= \frac{i\lambda}{2} \int_{-\infty}^{\infty} e^{ix^2/\lambda} \frac{d}{dx}(x^{k-1}\sigma(x))dx \\ &= \frac{i(k-1)\lambda}{2} \tilde{I}_{k-2}(\lambda) + \int_{-\infty}^{\infty} e^{ix^2} x^{k-1} \frac{d\sigma(x)}{dx} dx.\end{aligned}$$

Note that $d\sigma(x)/dx$ is a smooth function *vanishing* on the interval $[-a, a]$ containing the critical point. Therefore, as λ tends to 0, the last integral tends to 0 faster than any power of λ. Denoting the class of such functions by $\mathcal{O}(\lambda^\infty)$, we conclude from our inductive integration by parts that

$$\begin{aligned}\tilde{I}_{2l-1}(\lambda) &= 0 &&+\mathcal{O}(\lambda^\infty) \\ \tilde{I}_{2l}(\lambda) &= \tfrac{(i\lambda)^l (2l)!}{4^l l!}\ \tilde{I}_0(\lambda) &&+\mathcal{O}(\lambda^\infty).\end{aligned}$$

Here $\tilde{I}_0(\lambda)$ is not a constant but a function of λ. However

$$\tilde{I}_0(\lambda) = \int_{-b}^{b} e^{ix^2/\lambda} \sigma(x) dy = \int_{-a}^{a} e^{ix^2/\lambda} dx + \mathcal{O}(\lambda^\infty),$$

because the intervals $[a, b]$ and $[-b, -a]$ don't contain the critical point. Finally, the last integral

$$\int_{-a}^{a} e^{ix^2/\lambda} dx = \int_{-\infty}^{\infty} e^{ix^2/\lambda} dx + \mathcal{O}(\lambda^\infty),$$

i.e. differs from the value $e^{\pi i/4}\sqrt{\pi\lambda}$ of Fresnel's integral by a function of class $\mathcal{O}(\lambda^\infty)$, because the rays $[a, \infty]$ and $[-\infty, -a]$ also don't contain the critical point.

Thus, the formal algorithm of writing out the asymptotics of an oscillating integral yields a series, the finite part of which to order in λ less than $N + \frac{1}{2}$ differs from the integral (as a function of λ) by an error of class $\mathcal{O}(\lambda^{N+1/2})$.

HW3

3.5. When $u(t,x) = U(x \pm ct)$, then, by the chain rule, $u_{tt} = (\pm c)^2 U''(x \pm ct)$, while $u_{xx} = U''(x \pm ct)$, so that the wave equation $u_{tt} = c^2 u_{xx}$ holds. The solution $U(x - ct)$ takes at $x = x_0 + ct$ on the same value as the initial condition $U(x)$ takes at the point x_0. In other words, the graph of $U(x-ct)$ at the moment t is obtained from the graph of $U(x)$ by the shift through the distance ct to the right. Thus, the solution describes the running wave of shape $U(x)$ moving with the speed c toward the positive direction of the x-axis. Likewise, $U(x + ct)$ describes such a running wave moving to the left. Note that if $U = const$, then both $U(x \pm ct)$ are the same constant solution (running nowhere).

A general solution $u(t,x)$ satisfying the initial conditions $u(0,x) = w(x)$, $u_t(0,x) = v(x)$ is the superposition $u(t,x) = U_+(x + ct) + U_-(x - ct)$ of two running waves, one moving to the left the other to the right. To fit the initial conditions, we must have $u(0,x) = U_+(x) + U_-(x) = w(x)$ (and hence $U'_+ + U'_- = w'$) and $u_t(0,x) = c(U'_+(x) - U'_-(x)) = v(x)$. Therefore $U'_\pm = (w' \pm v/c)/2$. This determines both functions $U_\pm$ uniquely up to additive integration constants:

$$U_\pm(x) = C_\pm + \frac{1}{2}\int_0^x \left(w'(y) \pm \frac{v(y)}{c}\right) dy.$$

Clearly, adding a constant to U_+ and subtracting the same constant from U_- doesn't change the resulting superposition $u(t,x)$. So, only the sum of the constants matters, and must satisfy $C_+ + C_- = w(0)$. Thus, we can take

$$U_\pm(x) = \frac{1}{2}w(x) \pm \frac{1}{2c}\int_0^x v(y)dy.$$

4.3. Recall that $\hat{p}^2 = -\hbar^2\partial^2/\partial q^2$, $\hat{q}^2 = q^2$. For any ψ, we have

$$-\hbar^2\frac{\partial^2}{\partial q^2}q^2\psi = -q^2\frac{\partial^2}{\partial q^2}\psi - 4\hbar^2 q\frac{\partial}{\partial q}\psi - 2\hbar^2\psi.$$

Therefore the commutator $[\hat{p}^2, \hat{q}^2] = 4(\hbar/i)\hat{q}\hat{p} - 2\hbar^2$.

On the other hand, the Poisson bracket $\{p^2, q^2\} = 4pq$. Trying to quantize pq, we encounter the ordering issue:

$$\hat{p}\hat{q} = \frac{\hbar}{i}\frac{\partial}{\partial q}q = q\frac{\hbar}{i}\frac{\partial}{\partial q} + \frac{\hbar}{i} = \hat{q}\hat{p} + \frac{\hbar}{i}.$$

Either $4\hat{p}\hat{q}$ or $4\hat{q}\hat{p}$ qualifies as a quantization of $4pq$, and neither coincides with $i[\hat{p}^2, \hat{q}^2]/\hbar$, but their arithmetic average (which also qualifies) does:

$$\frac{1}{2}(\hat{p}\hat{q} + \hat{q}\hat{p}) = \frac{i}{\hbar}[\hat{p}^2, \hat{q}^2].$$

So, perhaps, this arithmetic average is the most satisfying choice as the quantization of $\{p^2, q^2\}$, since then we have

$$\widehat{\{p^2, q^2\}} = \frac{i}{\hbar}[\widehat{p^2}, \widehat{q^2}].$$

HW4

5.7. We are looking for solutions to stationary Schrödinger equation $-\hbar^2\psi''/2m = E\psi$ satisfying the periodicity condition $\psi(q+L) = \psi(q)$. The eigenfunctions of the differentiation operator d/dq have the form e^{ikq} and satisfy the periodicity condition when $kL = 2\pi n$, $n = 0, \pm1, \pm1, \dots$, i.e. when $k = 2\pi n/L$. The functions $\psi_n := e^{2\pi inq/L}$ serve as the eigenfunctions of the Hamiltonian operator at the energy level $E_n = 2\pi^2\hbar^2n^2/mL^2$. That is, when $n = 0$ the eigenspace is 1-dimensional (spanned by $\psi_0 = 1$), and when $n > 0$, the eigenspace is 2-dimensional (spanned by $\psi_{\pm n}$). A general L-periodic initial condition $\Psi(q, 0)$ in the time-dependent Schrödinger equation can be expanded into the usual Fourier series for L-periodic functions:

$$\Psi(q,0) = \sum_{n=-\infty}^{\infty} C_n(0)e^{2\pi inq/L}, \quad \text{where } C_n(0) = \frac{1}{L}\oint \Psi(q,0)e^{-2\pi inq/L}dq,$$

and the time evolution $\Psi(q, t)$ of the quantum state is described by the evolution of the Fourier coefficients:

$$C_n(t) = C_n(0)e^{-iE_nt/\hbar}, \quad \text{where } E_n = \frac{2\pi^2\hbar^2n^2}{mL^2}.$$

Here $\oint$ means integration over the circle, i.e. any interval of length L. Therefore, when $\Psi(q, 0)$ is the delta-function on the circle concentrated at $q = 0$, i.e. $\Psi(q,0) = \sum_{m=-\infty}^{\infty}\delta(q - mL)$, all the Fourier coefficients

$$C_n(0) = \int_a^{a+L} e^{-2\pi inq/L}\sum_{m=-\infty}^{\infty}\delta(q - mL)\,dq = \frac{1}{L}.$$

Thus

$$\Psi(q,t) = \frac{1}{L}\sum_{n=-\infty}^{\infty} e^{-2\pi^2n^2\hbar it/mL^2+2\pi inq/L} = \frac{1}{L}\,\theta\left(\frac{q}{L}; -\frac{2\hbar}{mL^2}\right),$$

where $\theta(z;\tau)$ is the famous Jacobi theta-function. It was discovered by Fourier in the context of heat propagation in a circle, which differs from our situation only by replacing time t with it (the so-called *Wick's rotation*).

Finally, the phase area A_n of the cylinder enclosed within the energy level $E_n = p^2/2m = 2\pi^2\hbar^2n^2/mL^2$, i.e. with $-2\pi n\hbar/L \le p \le 2\pi n\hbar/L$, equals $4\pi n\hbar$. Thus, the phase cylinder (infinite in the p-direction) is partitioned by the classical trajectories $p = 2\pi n\hbar$, $n = 0, \pm1, \pm2, \dots$ into a sequence of finite cylindrical chunks of the same area $2\pi\hbar$.

5.10. A positive energy level E in the finite well problem (with $E = V_0$ corresponding to the top of the well) is represented by $E + V_0$ if the well is understood as a dip with the bottom $E = -V_0$ and top at $E = 0$. With this change of notation, the level E_1 is determined as the "leftmost" solution of the equation

$$\tan\mu\sqrt{V_0+E} = \sqrt{\frac{V_0}{V_0+E}-1}\ \left(= \frac{\sqrt{-E}}{\sqrt{V_0+E}}\right), \text{ where } \mu := \sqrt{\frac{m}{2}}\frac{a}{\hbar}.$$

When $V_0 a$ is kept constant and equal to $\hbar^2/2md$, we have $V_0 = \hbar^2/2mda \to \infty$ and $\mu^2 V_0 = a/4d \to 0$ as $a \to 0$, while $\mu^2 V_0^2$ remains constant. Rewriting our equation as

$$E = -(V_0 + E)\tan^2 \mu\sqrt{V_0 + E} = -V_0^2\mu^2 \left(1 + \frac{E}{V_0}\right)^2 \frac{\tan^2\left[\mu\sqrt{V_0}\sqrt{1 + \frac{E}{V_0}}\right]}{\left[\mu\sqrt{V_0}\sqrt{1 + \frac{E}{V_0}}\right]^2},$$

and using that $(\tan x)/x \to 1$ as $x \to 0$, we conclude that as $a \to 0$, the function of E on the right tends to the required constant $V_0^2\mu^2 = -\hbar^2/8md^2$. Therefore so does the value of E on the left when determined from the equation.

HW5

5.12. The eigenfunctions have the form

$$\psi(q) = \begin{cases} Ae^{ikq} + Be^{-ikq} & q < 0 \\ Ce^{-\kappa q} & q > 0 \end{cases},$$

where $k = \sqrt{2mE}/\hbar$ and $\kappa = \sqrt{2m(V_0 - E)}/\hbar$, and the continuity of ψ and ψ' at $q = 0$ require $A + B = C$, $ik(A - B) = \kappa C$, i.e. $A = (1 - i\kappa/k)C/2$, and $B = (1 + i\kappa/k)C/2$. Thus, we have a simple continuous spectrum: for every energy level $0 < E < V_0$ (and in fact for $E = V_0$ too), the eigenfunctions form a 1-dimensional space. The probability current under the barrier ($q > 0$) is $j(q) := \frac{i\hbar}{2m}(\psi_q\psi^* - \psi\psi_q^*) = 0$ (e.g. because $e^{-\kappa q}$ is real), and for $q < 0$ (from p. 42 of the book) $j(q) = \frac{\hbar k}{m}(|A|^2 - |B|^2) = 0$ too (since $|A| = |B|$) in agreement with "preservation of probability" principle. The transmission probability $j_{trans}/j_{inc} = 0$ and the reflection probability $j_{ref}/j_{inc} = |B|^2/|A|^2 = 1$.

Comparing to the problem of the finite well potential $V(q) = 0$ for $-a < q < 0$ and $V(q) = V_0$ for $q > 0$ or $q < -a$ (which in the limit $a \to \infty$ turns into the step potential), we find that the eigenfunctions at an energy level E have the same form as above for $q > -a$, but for $q < -a$ are given by $\psi(q) = De^{\kappa q}$. The function *per se* would tend to that in the step problem as $a \to \infty$. However, the continuity condition for ψ and ψ' at $q = -a$ introduce two more homogeneous linear relations between A, B, C and (only one more indeterminate) D. As we know from the analysis of the finite well problem in the book (which differs only by the location of the well from what we dealing with now), the relations can be satisfied only for discrete values of E found from the relation $\sqrt{V_0/E - 1} = \tan \mu\sqrt{E}$ or $= -\cot \mu\sqrt{E}$, where $\mu = a\sqrt{2m}/2\hbar$. As a increases, the period of tan and $-\cot$ decreases, and the discrete energy levels (see Figure 11 on p. 36) fill the range $0 < E < V_0$ more and more densely. Thus, when the well is "very wide", the spacings between the (still discrete) energy levels become practically indiscernible, and so the system behaves as if the spectrum were continuous — in the same sense a

pencil (or any other macroscopic object) is perceived as continuous while in fact it is formed by finitely many (of order 10^9) atoms per inch.

5.14. We have

$$\begin{aligned}\frac{d}{dt}\mathcal{E}[a,b] &= \frac{1}{2}\int_a^b \left(c^{-2}u_{tt}u_t^* + c^{-2}u_{tt}^*u_t + u_{xt}u_x^* + u_{xt}^*u_x\right)\, dx \\ &= \frac{1}{2}\int_a^b \left(u_{xx}u_t^* + u_{xx}^*u_t + u_{xt}u_x^* + u_{xt}^*u_x\right)\, dx \\ &= \frac{1}{2}\int_a^b \frac{d}{dx}\left(u_xu_t^* + u_x^*u_t\right)\, dx = j(a) - j(b),\end{aligned}$$

if $j := -(u_xu_t^* + u_x^*u_t)/2$ as suggested.

For a solution $u(x,t) = e^{i(kx-\omega t)} + Ae^{-i(kx+\omega t)}$ for $x < 0$ and $u(x,t) = Be^{i(k'x-\omega t)}$ for $x > 0$, where $\omega/k' = c_+$ and $\omega/k = c_-$ (as in problem 5.13), from continuity of u and u_x at $x = 0$ we have: $1 + A = B$ and $k(1 - A) = k'B$, i.e. $1 - A^2 = c_-B^2/c_+$. On the other hand, the energy current $j := -(u_tu_x^* + u_xu_t^*)/2$ equals: $|B|^2\omega k' = B^2\omega^2/c_+$ for the transmitted wave $e^{i(k'x-\omega t)}$ (at any $x > 0$), and for the the combination of the incident and reflected wave we have

$$\begin{aligned}&-\frac{1}{2}(-i\omega)\left(e^{i(kx-\omega t)} + Ae^{-i(kx+\omega t)}\right)(-ik)\left(e^{-i(kx-\omega t)} - A^*e^{i(kx+\omega t)}\right) + \\ &-\frac{1}{2}(ik)\left(e^{i(kx-\omega t)} - Ae^{-i(kx+\omega t)}\right)(i\omega)\left(e^{-i(kx-\omega t)} + A^*e^{i(kx+\omega t)}\right)\end{aligned}$$

which after some simplifications (and taking into account that A is real) yields $(1 - A^2)\omega^2/c_-$ (at any $x < 0$). Since the total energy current across the boundary $x = 0$ must vanish (for, the energy cannot be generated at this point out of nothing), the currents at $x = 0^-$ and at $x = 0^+$ must coincide: $(1 - A^2)\omega^2/c_- = B^2\omega^2/c_+$.

HW6

6.8. The expectation values of $\hat{q}$ and $\hat{p}$ at the state $\psi = e^{-q^2/2\sigma^2}$ vanish:

$$\langle\psi|\hat{q}|\psi\rangle = \int_{-\infty}^{\infty} qe^{-q^2/\sigma^2}dq = 0,\ \langle\psi|\hat{p}|\psi\rangle = -i\hbar\int_{-\infty}^{\infty}\frac{-q}{\sigma^2}e^{-q^2/\sigma^2}dq = 0.$$

Therefore the squares of the standard deviations of $\hat{q}$ and $\hat{p}$ coincide with the expectation values of $\hat{q}^2$ and $\hat{p}^2$ respectively. Namely, $(\Delta\hat{q})^2 = \|\hat{q}\psi\|^2/\|\psi\|^2$ and $(\Delta\hat{p})^2 = \|\hat{p}\psi\|^2/\|\psi\|^2$, where $\hat{q}\psi = qe^{-q^2/2\sigma^2}$, and $\hat{p}\psi = -i\hbar\frac{d}{dq}e^{-q^2/2\sigma^2} = i\hbar(q/\sigma^2)e^{-q^2/2\sigma^2}$. Integrating by parts, we find:

$$\int_{-\infty}^{\infty} q^2e^{-q^2/\sigma^2}dq = \frac{\sigma^2}{2}\int_{-\infty}^{\infty} e^{-q^2/\sigma^2}dq = \frac{\sigma^2}{2}\|\psi\|^2.$$

Therefore $\Delta\hat{q} = \sigma/\sqrt{2}$, $\Delta\hat{p} = \hbar/\sigma\sqrt{2}$, and $\Delta\hat{p}\cdot\Delta\hat{q} = \hbar/2$. The abstract uncertainty principle guarantees that $\Delta\hat{p}\cdot\Delta\hat{q} \geq \hbar/2$ (since $i[\hat{p},\hat{q}] = \hbar$). The example at hands shows that, generally speaking, the inequality cannot be improved.

6.13. The evolution of expectation values of quantum observables is described by differential equations $d\bar{A}/dt = \frac{i}{\hbar}\overline{[\hat{H}, A]}$. Take the quantized hamiltonian $\hat{H} = (a\hat{p}^2 + b\hat{p}\hat{q} + b\hat{q}\hat{p} + c\hat{q}^2)/2$. Using the relation $\hat{p}\hat{q} = \hat{q}\hat{p} + \hbar/i$, we find

$$\frac{i}{\hbar}[\hat{H}, \hat{q}] = a\hat{p} + b\hat{q}, \quad \frac{i}{\hbar}[\hat{H}, \hat{p}] = -b\hat{p} - c\hat{q}.$$

Therefore $d\bar{q}/dt = a\bar{p} + b\bar{q} = H_p(\bar{p}, \bar{q})$, $d\bar{p}/dt = -b\bar{p} - c\bar{q} = -H_q(\bar{p}, \bar{q})$, which coincides with the classical Hamilton equations for the hamiltonian H.

HW7

7.5. Functions $\psi_k := e^{ikq}, k = 0, \pm 1, \pm 2, \dots$ form a basis of eigenstates, two per energy level $E_{|k|} = k^2\hbar^2/2m$ for $|k| \neq 0$, and one for $|k| = 0$. Therefore the ground state of 3 identical fermions is proportional to the determinant $\psi_0 \wedge \psi_1 \wedge \psi_{-1}$:

$$\frac{1}{2i}\begin{vmatrix} 1 & 1 & 1 \\ e^{iq_1} & e^{iq_2} & e^{iq_3} \\ e^{-iq_1} & e^{-iq_2} & e^{-iq_3} \end{vmatrix} = \frac{e^{i(q_1-q_2)} - e^{i(q_2-q_1)}}{2i} + \frac{e^{i(q_3-q_1)} - e^{i(q_1-q_3)}}{2i}$$
$$+\frac{e^{i(q_2-q_3)} - e^{i(q_3-q_2)}}{2i} = \sin(q_1 - q_2) + \sin(q_3 - q_1) + \sin(q_2 - q_3).$$

Its energy level is $0 + \hbar^2/2m + \hbar^2/2m = \hbar^2/m$. The next energy level $0 + \hbar^2/2m + 2^2\hbar^2/2m = 5\hbar^2/2m$ contains 4 independent eigenstates $\psi_0 \wedge \psi_{\pm 1} \wedge \psi_{\pm 2}$.

7.6. Functions $\psi_k := e^{2\pi ikq/L}$ form the basis of eigenstates of a free particle on the circle of length L, and have energy levels $E_{|k|} = 2\pi^2\hbar^2k^2/mL^2$. The states of 2 identical fermions are the coefficients $\psi_k \wedge \psi_l := \psi_k(q_1)\psi_l(q_2) - \psi_l(q_1)\psi_k(q_2)$ $(k < l)$ of the double-series in Grassmann variables:

$$\left(\sum_{k=-\infty}^{\infty} \psi_k \xi_k\right) \otimes \left(\sum_{l=-\infty}^{\infty} \psi_l \xi_l\right) = \sum_{-\infty < k < l < \infty} (\psi_k \wedge \psi_l)\xi_k \xi_l.$$

The energy level of $\psi_k \wedge \psi_l$ equals $2\pi^2\hbar^2(k^2 + l^2)/mL^2$. Therefore the number $N(E)$ of independent 2-fermion states with energy $< E$ equals the number of integer lattice points (k, l) satisfying inequalities $k^2 + l^2 < mL^2E/2\pi^2\hbar^2$ and $k < l$, i.e. in the interior of semi-disk of radius $R = \sqrt{mE/2}L/\pi\hbar$ and area $\frac{1}{2}\pi R^2$. As E tends to ∞, R tends to ∞ too, and the ratio $N(E)/\frac{1}{2}\pi R^2$ tends to 1.[1] Thus, multiplying and dividing $N(E)/E$ by $\frac{1}{2}\pi R^2$, and noticing that $\frac{1}{2}\pi R^2/E = mL^2/4\pi\hbar^2$ doesn't depend on E, we conclude that $\lim_{E\to\infty} N(E)/E = mL^2/4\pi\hbar^2$.

[1] If this is not obvious, note that the difference $|\frac{1}{2}\pi R^2 - N(E)|$ comes from unit squares, centered at the lattice points, which lie partly inside and partly outside the semi-disk. They are covered by the semi-annulus $|\sqrt{x^2 + y^2} - R| \leq \sqrt{2}$ $(x \leq y)$ and the strip $|x - y| \leq 1/\sqrt{2}$ $(|x + y| \leq \sqrt{2}R)$. The areas of these regions $(2\sqrt{2}\pi R$ and $2R)$ divided by $\frac{1}{2}\pi R^2$ tend to 0 as R tends to ∞.

HW 8.

8.6. Collecting all terms of order r^{k-1} in the differential equation on page 71, we find (from the last displayed formula on that page) that the coefficients v_k of solution $v(r) = \sum_{k>l} v_k r^k$ must satisfy

$$k(k+1)v_{k+1} - 2\lambda k v_k + 2mQ\hbar^{-2}v_k - l(l+1)v_{k+1} = 0,$$

i.e.

$$v_{k+1} = 2\frac{mQ\hbar^{-2} - \lambda k}{k(k+1) - l(l+1)}v_k.$$

Taking $v_{l+1} \neq 0$, we obtain from this a unique sequence of values for v_k with $k > l+1$, which are all non-zero unless $\lambda = mQ\hbar^{-2}/n$ for some $n > l$. When this is the case, v_{n+1} and all subsequent v_k turn out to be 0, and the solution v polynomial.

8.10. The ground state (quantum numbers $n = 1, l = 0$) in the hydrogen model problem is proportional to $\psi = e^{-\lambda r}$, where $\lambda = mQ/\hbar^2 = me^2/4\pi\epsilon_0\hbar^2$. The corresponding probability density is proportional $|\psi|^2 = e^{-2r^2}dxdydz = e^{-2\lambda r}r^2 \sin\phi dr d\phi d\theta$ (in spherical coordinates). It achieves maximum where $e^{-2\lambda r}r^2$ does, i.e. at $r_0 = 1/\lambda$. The expected value of r is computed as

$$\bar{r} = \frac{\int_{\mathbb{R}^3} re^{-2\lambda r}dxdydz}{\int_{\mathbb{R}^3} e^{-2\lambda r}dxdydz} = \frac{\int_0^\infty r^3 e^{-2\lambda r}dr}{\int_0^\infty r^2 e^{-2\lambda r}dr} = \frac{3}{2\lambda} = \frac{3}{2}r_0.$$

The ratio was computed by integrating by parts in the numerator once. The classical orbit with the total angular momentum $|L|^2 = l(l+1) = 0$ (since $l = 0$), and hence $L := q \times p = \vec{0}$, is a degenerate ellipse: the trajectory of a "stone" falling to the center with the initial velocity 0 from the altitude r determined by the value of energy: $-Q/r = E_1 = -mQ^2/2\hbar^2$, i.e. $r = 2/\lambda = 2r_0$. The actual value of r can be found (in meters) as

$$\frac{8\pi\epsilon_0\hbar^2}{m_e e^2} \approx \frac{8 \times 3.14 \times 8.85 \cdot 10^{-12} \times (1.055 \cdot 10^{-34})^2}{9.11 \cdot 10^{-31} \times (1.60 \cdot 10^{-19})^2} \approx 1.06 \cdot 10^{-10}.$$

HW9.

8.14. In both — gravitational and electrostatic — Kepler problems, the classical Hamiltonian has the form $p^2/2m_e - Q/r$, where $Q = m_e m_n \gamma$ in the gravitational case (here $\gamma \approx 6.67 \times 10^{-11} m^3/kg \cdot s^2$ is the universal gravitational constant) and $Q = e^2/4\pi\epsilon_0$ is the electrostatic one. The Bohr radius (as found in the solution of **8.10**) equals $1/\lambda = \hbar^2/m_e Q$ (in the notation of page 72). Thus, the Bohr radius of "neutrogen" is

$$\frac{\hbar^2}{m_n m_e^2 \gamma} \approx \frac{(1.05 \times 10^{-34})^2}{1839 \cdot (9.11 \times 10^{-31})^3 (6.67 \times 10^{-11})} \approx 1.19 \times 10^{29} m.$$

The size of the Universe < 100 billion light years or $\approx 9.46 \times 10^{26} < 10^{27} m$ is two decimal orders smaller.

9.9. The map $\mathbb{H} \ni x \mapsto q_1 x q_2^{-1}$, where $q_1, q_2 \in Sp_1 = SU_2$ are unit quaternions, preserves the norm, $\|q_1 x q_2^{-1}\| = \|q_1\| \cdot \|x\| \cdot \|q_2\|^{-1} = \|x\|$, and hence defines an orthogonal transformation in $\mathbb{H} = \mathbb{R}^4$. Since the space $Sp_1 \times Sp_1 \equiv S^3 \times S^3$ is connected, the group homomorphism $Sp_1 \times Sp_1 \to O_4$ lands in that connected component of O_4 where the unit element (I, I) lands, i.e. in SO_4. The *kernel* of this homomorphism (i.e. the inverse image of the identity) consists of those pairs (q,q_2) for which $q_1 x q_2^{-1} = x$ for all x, i.e. (taking $x = 1$) for the pairs (q, q) where q commutes with all x, and is therefore real, i.e. consist of two elements: $\pm(1, 1)$. Thus, we have a 2-to-1 homomorphism $SU_2 \times SU_2 \to SO_4$ which factors through an injective homomorphism $f : (SU_2 \times SU_2)/(\pm(I, I)) \to SO_4$. To show that the inclusion is surjective, we note that both the source and the target space have dimension 6 (as it is not hard to check). Considered as a smooth map, a Lie group homomorphism must have constant rank: $f(g_0 g) = f(g_0)f(g)$ implying that the rank of the differential $d_e f$ of f at the identity e coincides with the rank of the differential $d_{g_0} f$ at g_0. Therefore our f has everywhere maximal rank, 6, i.e. (by the Inverse Function Theorem) it is locally invertible, and thus has an open image. Since $SU_2 \times SU_2$ is compact, the image must also be closed, i.e. must coincide with the whole connected component SO_4 of the target space. We conclude that f factors through $SU_2 \times SU_2/(\pm(I, I)$ to an isomorphism with SO_4.

HW 10.

9.19. By the Clebsch-Gordan rule, $V_{l+1/2} = V_{1/2}V_l - V_{l-1/2}$, where $l = 1/2, 1, 3/2, \ldots$ (while for $l = 0$ we have, of course: $V_{1/2} = V_{1/2}V_0$ where $V_0 = 1$, the unit element in the representation ring $\mathcal{R}$). This gives a recursion relation expressing V_{l+1} as a monic polynomial of degree $2l = 0, 1, 2, 3, \ldots$ in $V := V_{1/2}$. For instance, $V_1 = V^2 - 1$, $V_{3/2} = V^3 - 2V$, $V_2 = V^4 - 3V^2 + 1$, $V_{5/2} = V^5 - 4V^3 + 3V$, $V_3 = V^6 - 5V^4 + 6V^2 - 1$. In the coefficients of these polynomials, it is not hard to recognize the terms of Pascal's triangle, albeit alternating in signs, and read not from the same row of Pascal's triangle, but along a slant line. (A control test: the sum of the terms of Pascal's triangle along these slant lines yield the Fibonacci sequence: $1, 1, 2, 3, 5, 8, \ldots.$) Namely:

$$V_l = \sum_{k=0}^{\lfloor l \rfloor} (-1)^k \binom{2l-k}{k} V^{2l-2k}.$$

This conjecture (which obviously holds true for $l = 0$) follows by induction from the defining property of Pascal's triangle; namely

$$(-1)^k \binom{2(l+1/2)-k}{k} = (-1)^k \binom{2l-k}{k} - (-1)^{k-1} \binom{2(l-1/2)-(k-1)}{k-1}.$$

Another interpretation of this polynomial is obtained by noting that in V_l,

$$\operatorname{tr} e^{-iS_z t/\hbar} = \sum_{k=0}^{2l} e^{i(l-k)t} = \frac{e^{i(l+1/2)t} - e^{-i(l+1/2)t}}{e^{it/2} - e^{-it/2}} = \frac{\sin(l+1/2)t}{\sin t/2}.$$

This is an even 4π-periodic trigonometric function, and can be expressed as the above polynomial of $\sin t/\sin(t/2) = 2\cos(t/2)$.

9.20. We have: $S_x = (J_+ + J_-)/2$, $S_y = (J_+ - J_-)/2i$, and therefore

$$\begin{aligned} S_xS_y + S_yS_x &= \frac{1}{4i}\left(J_+^2 - J_+J_- + J_-J_+ - J_-^2 + J_+^2 + J_+J_- - J_-J_+ - J_-^2\right) \\ &= \frac{J_x^2 - J_-^2}{2i} = \frac{\hbar^2}{2i}\left(y^2\frac{\partial^2}{\partial x^2} - x^2\frac{\partial^2}{\partial y^2}\right). \end{aligned}$$

In the basis x^2, xy, y^2 of V_1, the matrix of this operator assumes the form $\begin{bmatrix} 0 & 0 & i\hbar^2 \\ 0 & 0 & 0 \\ -i\hbar^2 & 0 & 0 \end{bmatrix}$. The spectrum of this (Hermitian) matrix consists of the roots $\lambda = 0, \hbar^2, -\hbar^2$ of its characteristic polynomial $\lambda^3 - \hbar^4\lambda$.

Alternatively, $S_x/i\hbar, S_y/i\hbar, S_z/i\hbar$ commute according the cross-product scheme: $\mathbf{i}\times\mathbf{j} = \mathbf{k}$, etc. for a right-handed basis in $\mathbb{R}^3$. Therefore, identifying V_1 with the complexification of the standard representation of SO_3 in $\mathbb{R}^3$, we have $S_x\mathbf{v} = i\hbar(\mathbf{i}\times\mathbf{v})$ and $S_y\mathbf{v} = i\hbar(\mathbf{j}\times\mathbf{v})$ for any $\mathbf{v}\in\mathbb{R}^3$. Then

$$(S_x, S_y)\mathbf{v} = -\hbar^2\left[\mathbf{i}\times(\mathbf{j}\times\mathbf{v}) + \mathbf{j}\times(\mathbf{i}\times\mathbf{v})\right].$$

In the basis $\mathbf{v} = \mathbf{i}, \mathbf{j}, \mathbf{k}$ this operator has matrix $\begin{bmatrix} 0 & -\hbar^2 & 0 \\ -h^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ with the same characteristic polynomial $\lambda^3 - \hbar^4\lambda$.

HW11.

10.3. The Hamilton equations with the hamiltonian $\mathcal{H} := c^2(p\cdot p)/2 - E^2/2$ in the extended phase space with coordinates $q, \tau, p, -E$ have the form:

$$\dot q := \mathcal{H}_p = c^2 p,\ \dot p := -\mathcal{H}_q = 0,\ \dot\tau := -\mathcal{H}_E = E,\ \dot E := \mathcal{H}_\tau = 0.$$

Solutions lying in the level set $\mathcal{H} = -m^2c^4/2$ (where necessarily $E\neq 0$) are:

$$q(t) = c^2p(0)t + q(0),\ p(t) = p(0),\ \tau(t) = E(0)t + \tau(0),\ E(t) = E(0).$$

Using p, E, q_0, τ_0 for $p(0), E(0), q(0), \tau(0)$, and eliminating $t = (\tau - \tau_0)/E$ (where $E^2 = c^2\|p\|^2 + m^2c^4$), we find

$$\frac{q-q_0}{\tau-\tau_0} = c\frac{p}{\|p\|} \text{ if } m = 0, \text{ and } \frac{q-q_0}{\tau-\tau_0} = \frac{p/m}{\sqrt{1+\|p/m\|^2/c^2}} \text{ if } m > 0.$$

These are straight lines in space-time, the graphs of trajectories in space passing through q_0 at the moment τ_0 in the direction of $p/\|p\|$ with speed c when $m = 0$, and speed $\|p/m\|/\sqrt{1+\|p/m\|^2/c^2} = c/\sqrt{1+m^2c^2/\|p\|^2} < c$ when $m > 0$.

10.9. Trying harmonic running waves $\Psi_\pm(q,t) = \psi_\pm e^{i(k\cdot q-\omega t)}$, where $q=(x,y,z)$, $k=(k_x,k_y,k_z)$, and $\psi_\pm \in V_{1/2} \cong \mathbb{C}^2$ are constant spinors, in the role of solutions to the Dirac system on page 97, we obtain:

$$\hbar\omega\psi_\pm = \pm mc^2\psi_\pm + c\hbar(k_x\sigma_x + k_y\sigma_y + k_z\sigma_z)\psi_\mp.$$

These relations allow one to express $\psi_\pm$ through each other, given the values of energy $E=\hbar\omega$ and momentum $p=\hbar k$ of the running wave:

$$(E \mp mc^2)\psi_\pm = c(p_x\sigma_x + p_y\sigma_y + p_z\sigma_z)\psi_\mp.$$

Eliminating ψ_- or ψ_+, we find (using $(p_x\sigma_x+p_y\sigma_y+p_z\sigma_z)^2 = p_x^2+p_y^2+p_z^2$):

$$(E^2 - m^2c^4)\psi_\pm = c^2(p\cdot p)\psi_\pm,$$

i.e. the linear relations between ψ_+ and ψ_- are consistent if and only if $E^2 = c^2(p\cdot p) + m^2c^4$(or, equivalently, $\omega^2 = c^2(k\cdot k) + m^2c^4/\hbar^2$). Explicitly, taking ψ_+ to be an arbitrary non-zero vector in $\mathbb{C}^2$, we have

$$\Psi_+ = \psi_+ e^{i(p\cdot q-Et)/\hbar},\ \Psi_- = \frac{c}{E+mc^2}\begin{bmatrix} p_z & p_x-ip_y \\ p_x+ip_y & -p_z\end{bmatrix}\psi_+ e^{i(p\cdot q-Et)/\hbar},$$

where $E = \pm\sqrt{c^2(p\cdot p)+m^2c^4}$. Equivalently — but not additionally! — taking arbitrary non-zero $\psi_- \in \mathbb{C}^2$, we have

$$\Psi_+ = \frac{c}{E-mc^2}\begin{bmatrix} p_z & p_x-ip_y \\ p_x+ip_y & -p_z\end{bmatrix}\psi_- e^{i(p\cdot q-Et)/\hbar},\ \Psi_- = \psi_- e^{i(p\cdot q-Et)/\hbar}.$$

Thus, for each non-zero momentum vector $p \in \mathbb{R}^3$, and each of the two values of the square root $E = \pm\sqrt{c^2(p\cdot p)+m^2c^4}$, there is a complex 2-dimensional space of running-wave solutions.

HW12

11.1. Since the potential energy of an oxygen molecule of mass m at the height h is mgh, where $g \approx 9.8\ kg\cdot m/s^2$ is the gravitational acceleration, the probability of finding the atom at an altitude h is, up to normalization, is given by the Gibbs distribution $e^{-mgh/kT}$. Respectively, the density changes with altitude according to the barometric law $\rho(h) = \rho(0)e^{-mgh/kT}$ (within the range of altitudes where both g and temperature T remain constant).

Alternatively, this can be derived form the ideal gas law $PV = kNT$, where N is the number of atoms in a volume V, which can be rewritten as the formula for density: $\rho = mN/V = mP/kT$. Namely, a layer of air of width Δh contains $\Delta h \times S \times \rho(h)$ amount of oxygen per area S, whose weight $\Delta h \times S \times gmP(h)/kT$ is supported by the difference $S\times[P(h+\Delta h) - P(h)]$ of pressure forces. This leads to the differential equation $dP(h)/dh = -mgP(h)/kT$, or (in terms of density) $d\rho(h)/dh = -mg\rho(h)/kT$. Its solution $\rho(h) = \rho(0)e^{-mgh/kT}$ coincides with the Gibbs distribution.

The altitude where the density is half that at the sea level is found from $e^{-mgh/kT} = 1/2$, i.e.

$$h = \frac{kT}{mg}\log 2 \approx \frac{1.38\cdot 10^{-23}\times 300}{32\times 1.67\cdot 10^{-27}\times 9.8}\times 0.69 \approx 5.5\cdot 10^3\ m.$$

This is, roughly, the altitude of Mt. Elbrus ($5,642\ m = 18,510\ ft$).

11.8. From the Fermi gas model (at $T \approx 0$) $E_F = (3\pi^2 N/V)^{2/3}\hbar^2/2m_e$. The model applies to the "free" electrons, i.e. valence electrons in our alloy, which for the metals in question are typically 2 per atom. So, taking into account the mass density $8\cdot 10^3\ kg/m^3$ and the weight of one atom $60\times 1.67\cdot 10^{-27}\ kg \approx 10^{-25}\ kg$, we find the electron density $N/V \approx 2\times 8\cdot 10^3/10^{-25} = 1.6\cdot 10^{29}\ m^{-3}$. We also have $3\pi^2 \approx 30$, and so $(3\pi^2 N/V)^{2/3} \approx (4.8\cdot 10^{30})^{2/3} \approx 2.8\cdot 10^{20}\ m^{-2}$. On the other hand, $\hbar^2/2m_e \approx (1.05\cdot 10^{-34})^2/2\times 0.9\cdot 10^{-30} \approx 0.61\cdot 10^{-38}$. Thus, in joules, $E_F \approx (2.8\times .61)\cdot 10^{-18} \approx 1.7\cdot 10^{-18}$. Dividing by $k \approx 1.38\cdot 10^{-23}$, we find the Fermi temperature of our alloy $T_F \approx 120,000\ K$ (i.e. quite high relative to the normal conditions of $\approx 300\ K$). Since $1\ eV \approx 1.6\cdot 10^{-19}\ joule$, we obtain the Fermi energy of our alloy $E_F \approx 10\ eV$.

HW13.

12.8. Since $\kappa^{-1}\sinh\kappa = a \to a$ as $\kappa \to 0$, at $E = 0$ we have

$$M = \lim_{\kappa\to 0}\begin{bmatrix} \cosh\kappa a & \frac{1}{\kappa}\sinh\kappa a \\ \kappa\sinh\kappa a & \cosh\kappa a \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\frac{\alpha}{a} & 1 \end{bmatrix} = \begin{bmatrix} 1-\alpha & a \\ -\frac{\alpha}{a} & 1 \end{bmatrix}.$$

At $\alpha = 2$, M is traceless, with eigenvalues $\pm i$. Its eigenvectors are easily found from

$$\begin{bmatrix} -1 & a \\ -\frac{2}{a} & 1 \end{bmatrix}\begin{bmatrix} \psi_\pm(0^-) \\ \psi'_\pm(0^-) \end{bmatrix} = \pm i\begin{bmatrix} \psi_\pm(0^-) \\ \psi'_\pm(0^-) \end{bmatrix} \text{ as } \begin{bmatrix} \psi_\pm(0^-) \\ \psi'_\pm(0^-) \end{bmatrix} = \begin{bmatrix} a \\ 1\pm i \end{bmatrix}.$$

From $\psi'(0^+) - \psi'(0^-) = -2\psi(0)/a$, we find $\psi'_\pm(0^+) = -1\pm i$, while $\psi_\pm(0^+) = \psi_\pm(0^-) = a$. On the interval $(0, a)$, the stationary Schrödineger equation with $E = 0$ (and $V = 0$) reduces to $\psi'' = 0$, whose solutions are linear: $\psi(x) = \psi(0^+) + x\psi'(0^+)$. Therefore

$$\psi_\pm(x) = a - x \pm ix \ \text{ for } 0 \le x \le a, \text{ while } \ \psi_\pm(x + na) = (\pm i)^n\psi_\pm(x).$$

The eigenspace of the Schrödinger operator at $E = 0$ consists of arbitrary complex linear combinations of $\psi_\pm$.

12.11. The energy of a photon of wave length λ is $\hbar\omega = 2\pi\hbar c/\lambda$, where c is the speed of light. Since $2\pi\hbar c \approx 20\cdot 10^{-26}\ J\cdot m$, for λ in the visible range (400 – $700\ nm$), this gives the energy range between $5\cdot 10^{-19}$ and $3\cdot 10^{-19}$ joules. Since $1\ eV \approx 1.6\cdot 10^{-19}$ joules, this gives the energy range 2–3 eV. This is greater than the energy gap $1.1\ eV$ between valency and conduction bands for silicon, and smaller than 5 times that ($5.5\ eV$) for diamond. Thus, visible light cannot excite the valent electrons in diamond, and hence passes through it unabsorbed, while in silicon, it interacts with the electrons.

12.16. The KdV soliton $\phi(x-vt)$ is described by the function $\phi(x) = (v/2)/\cosh^2(\sqrt{v}x/2)$ satisfying $\psi''+3\phi^2 = v\phi$. Take $v=1$, and interpret the ODE as the stationary Schrödinger equation $-\psi''-3\phi\psi = -\psi$, where $\psi=\phi$, the potential energy is $3\phi = 3/2\cosh^2(x/2)$ and the eigenvalue $E=-1$. On the other hand, the ODE for ϕ is obtained by integrating $\phi''' + 6\phi\phi' = v\phi'$, which (for $v=1$) can also be interpreted as the Schrödinger equation $-\psi'' - 6\phi\psi = -\psi$ with $\psi=\phi'$, the potential energy $6\phi = 3/\cosh^2(x/2)$ and the eigenvalue $E=-1$.

HW14.

13.3. For $\mathcal{H}(u) := \oint (u^3 - u_x^2/2)\ dx$, we have: $\delta\mathcal{H}/\delta u = u_{xx} + 3u^2$, and $\frac{d}{dx}\delta\mathcal{H}/\delta u = u_{xxx} + 6uu_x$. Take $\mathcal{F}_\phi(u) := \oint \phi\ u\ dx$, where ϕ is a fixed 1-periodic function of x. Then $\delta\mathcal{F}_\phi/\delta u = \phi$. The evolution of a classical observable $\mathcal{F}$ under the Hamiltonian flow with Hamilton function $\mathcal{H}$ is governed by the differential equation $\dot{\mathcal{F}} = \{\mathcal{H},\mathcal{F}\}$. Obviously, $\frac{d}{dt}\mathcal{F}_\phi(u) = \oint \phi\ u_t\ dx$. Therefore

$$\oint \phi\ u_t\ dx = -\{\mathcal{F},\mathcal{H}\} := -\oint \frac{\delta\mathcal{F}}{\delta u}\frac{d}{dx}\frac{\delta\mathcal{H}}{\delta u}\ dx = -\oint \phi\ [u_{xxx} + 6uu_x]\ dx$$

for all ϕ. Thus, $u_t = -u_{xxx} - 6uu_x$.

13.8. Specializing Wick's theorem to the case of one variable ($D=1$), $b_{11} = 1/i$, $t_a = 0$ for $a \neq 3$ and $t_3 = \lambda/i$, and setting $\hbar = 1$, we find that for each connected graph Γ (equipped with the only possible decoration), its weight $(i\hbar)^{E(\Gamma)}(i/\hbar)^{V(\Gamma)}W(\Gamma)/|Sym(\Gamma)| = \lambda^{V(\Gamma)}/|Sym(\Gamma)|$ when each vertex of Γ has valence 3, and $=0$ otherwise. This justifies the asymptotical formula $\sum_V G_V\lambda^V$. For a 3-valent graph, $3V = 2E$, implying that V must be even. There are two 3-valent graphs with 2 vertices: Γ_1 where the vertices are connected by 3 edges, and Γ_2 where they are connected by one edge, but are equipped with a loop each. We have: $|Sym(\Gamma_1)| = 2\times 3!$ and $|Sym(\Gamma_2)| = 2\times 2^2$, and hence $G_2 = 1/12 + 1/8 = 5/24$. Alternatively,

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}+\lambda\frac{x^3}{6}}dx &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\left[1+\frac{\lambda x^3}{6}+\frac{\lambda^2x^6}{72}+o(\lambda^2)\right]dx \\ &= 1+0+\frac{5\cdot 3\cdot 1}{72}\lambda^2+o(\lambda^3) = \exp\left[\frac{5\lambda^2}{24}+o(\lambda^3)\right].\end{aligned}$$