Math 110. Fall-2011 Final Exam:

1. Express $\det(\text{adj}(A))$ in terms of $\det A$, where A is an $n \times n$-matrix.

2. Solve system of linear equations:

$$\begin{array}{rcl} x_1 - 2x_2 + 3x_3 - 4x_4 &=& 4 \\ x_2 - x_3 + x_4 &=& -3 \\ x_1 + 3x_2 \qquad - 3x_4 &=& 1 \\ -7x_2 + 3x_3 + x_4 &=& -3 \end{array} .$$

3. Use Sylvester's rule to find inertia indices of quadratic form:

$$x_1x_2 - x_2^2 + x_3^2 + 2x_2x_4 + x_4^2.$$

4. Transform quadratic form $x_1x_2 + x_3x_4$ to the normal form by an orthogonal transformation.

5. Find the Jordan normal form of matrix:

$$\begin{bmatrix} 0 & 3 & 3 \\ -1 & 8 & 6 \\ 2 & -14 & -10 \end{bmatrix}.$$

6. Can a non-zero anti-symmetric matrix be nilpotent? If "yes" give an example, if "no" provide a proof.

7. Classify all linear operators in $\mathbb{R}^2$ up to linear changes of coordinates.

8. Find all those values of $a_1, \ldots, a_n$ for which the following matrix is nilpotent:

$$\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ . & . & . & . & . \\ 0 & \ldots & 0 & 0 & 1 \\ -a_n & -a_{n-1} & \ldots & -a_2 & -a_1 \end{bmatrix}.$$

9. Find out if the following quadratic hypersurfaces in $\mathbb{C}^4$ can be transformed into each other by linear inhomogeneous changes of coordinates:

$$z_1z_2 + z_2z_3 + z_3z_4 = 1 \quad \text{and} \quad z_1^2 + z_2^2 + z_3^2 + z_4^2 = z_1 + z_2 + z_3 + z_4.$$

10. Prove that any orthogonal transformation in $\mathbb{R}^4$ with the determinant equal to -1 has an invariant 3-dimensional subspace.

11. For a linear map $A : \mathbb{K}^n \to \mathbb{K}^n$, prove that if the null-space of the cofactor matrix $adj(A)$ is non-zero, then it contains the range of A.

12. Let $Q(\mathbf{x}) := \sum_{i,j=1}^n q_{ij} x_i x_j$ be a quadratic form with *integer* coefficients $q_{ij} = q_{ji}$. Prove that $\det[q_{ij}]$ of the coefficient matrix does not change under *integer* changes of variables, i.e. linear changes $\mathbf{x} = C\mathbf{x}'$ where C is an integer square matrix invertible over $\mathbb{Z}$.

13. Prove that $\operatorname{rk}(A + B) \leq \operatorname{rk} A + \operatorname{rk} B$ for any $m \times n$-matrices A and B.

14. Given three 3-dimensional linear subspaces $\mathcal{U}, \mathcal{V}, \mathcal{W}$, such that $\mathcal{U} \cap \mathcal{V}$ and $\mathcal{V} \cap \mathcal{W}$ have dimension 2 each. Prove that $\dim(\mathcal{U} \cap \mathcal{V} \cap \mathcal{W}) > 0$.

15. In $\mathbb{R}^6$, find a subspace of maximal dimension on which the quadratic form $x_1x_2 + x_3x_4 + x_5x_6$ is positive definite.

16. Find an orthonormal basis of eigenvectors of the operator U in the standard Hermitian space $\mathbb{C}^5$ given by the permutation of the coordinates: $(z_1, z_2, z_3, z_4, z_5) \mapsto (z_3, z_4, z_5, z_1, z_2)$.

17. State Courant–Fischer's minimax principle about eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ of a quadratic form S in the standard Euclidean space $\mathbb{R}^n$, and deduce from it that the eigenvalues do not decrease when a positive definite quadratic form is added to S.

18. Find the Jordan canonical form of e^N where N is a regular nilpotent operator on $\mathbb{C}^n$.

19. Prove that for $n > 1$, a regular nilpotent operator $N : \mathbb{C}^n \to \mathbb{C}^n$ does not have a cubic root (i.e. an operator M such that $M^3 = N$).