Riemann–Roch Theorems in Gromov–Witten Theory

by

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To Mum, John, and Ed

To Dad, Eileen, and Jonathan

with love.

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Abstract

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Gromov–Witten invariants of a compact almost-Kähler manifold X are intersection numbers in moduli spaces of stable maps to X. These spaces, introduced by Kontsevich, are compactifications of spaces of pseudo-holomorphic maps from marked Riemann surfaces to X. Gromov–Witten invariants encode information about the enumerative geometry of X — roughly speaking, they count the number of curves in X which pass through various cycles and satisfy certain conditions on their complex structure. These invariants have important applications in both symplectic topology and enumerative algebraic geometry.

In this dissertation we use various Riemann–Roch theorems, together with Givental's formalism of quantized quadratic Hamiltonians, to develop tools for computing Gromov–Witten invariants and their generalizations. As a consequence, we obtain a new proof of the Mirror Conjecture of Candelas, de la Ossa, Green and Parkes, concerning the genus-0 Gromov–Witten invariants of quintic hypersurfaces in $\mathbb{C}P^4$.

Following Kontsevich, we introduce a notion of Gromov–Witten invariant twisted by a holomorphic vector bundle E over X and an invertible multiplicative characteristic class $\mathbf{c}$. Special cases of this construction are closely related to Gromov–Witten invariants of hypersurfaces and to local Gromov–Witten invariants (these measure the contribution to the Gromov–Witten invariants of a space Y coming from curves in a neighbourhood of a submanifold X, where the normal bundle to X in Y is E). We express all twisted Gromov–Witten invariants, of all genera, in terms of untwisted Gromov–Witten invariants. This result (Theorem 1) is a consequence of the Grothendieck–Riemann–Roch formula applied to the universal family of stable maps.

As an application, we obtain the Quantum Lefschetz Hyperplane Principle (Theorem 2 and Corollary 5). This determines genus-0 Gromov–Witten invariants of a large class of complete intersections in terms of genus-0 Gromov–Witten invariants of the ambient space. It is more general than earlier versions, due to Givental, Kim, Lian–Liu–Yau, Bertram, Lee and Gathmann, as it applies to complete intersections of arbitrary Fano index and does not require "restriction to the small parameter space". In particular, this gives a new proof of the Mirror Conjecture of Candelas *et al.*. We also establish "non-linear Serre duality" in a very general form.

Tangent-twisted Gromov–Witten invariants are intersection numbers involving characteristic classes of virtual tangent bundles to moduli spaces of stable maps. They give a rich supply of symplectic invariants of X. We determine all tangent-twisted Gromov–Witten invariants, of all genera, in terms of untwisted Gromov–Witten invariants. A key step is to interpret tangent-twisted Gromov–Witten invariants in terms of Gromov–Witten invariants with values in complex cobordism. We extend Givental's quantization formalism to the cobordism-valued setting, and combine this with various Riemann–Roch calculations to give a formula (Theorem 3) expressing cobordism-valued Gromov–Witten invariants in terms of usual (cohomological, untwisted) Gromov–Witten invariants. This determines all Gromov–Witten invariants with values in any complex-oriented cohomology theory in terms of cohomological Gromov–Witten invariants. Theorem 3 reduces "quantum extraordinary cohomology" to quantum cohomology, and in this sense can be regarded as a "quantum" version of the Hirzebruch–Riemann–Roch theorem.

Professor Alexander Givental
Dissertation Committee Chair

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Chapter 0

Introduction

In this chapter, we state the main theorems proved in the later chapters and deduce some simple corollaries of them. In order that the presentation be self-contained, we give some definitions both here and in later chapters. Proofs of those results concerning twisted Gromov–Witten invariants can be found in Chapter 1. The results concerning quantum extraordinary cohomology and quantum cobordism are proved in Chapter 2.

Gromov–Witten invariants

Let X be a compact Kähler manifold. Moduli spaces of stable maps, introduced by Kontsevich [37], are compactifications of spaces of holomorphic maps from marked Riemann surfaces to X. Gromov–Witten invariants are certain integrals over moduli spaces of stable maps. They encode information about the enumerative geometry of X — roughly speaking, they count the number of curves in X which pass through various cycles and satisfy certain conditions on their complex structure. These invariants have been the subject of much recent interest in connection with the mathematical implications of mirror symmetry.

Denote by $X_{g,n,d}$ the moduli space of stable maps [8, 37] of degree $d \in H_2(X;\mathbb{Z})$ from n-pointed, genus-g curves to X. This space is compact, and a Riemann–Roch calculation shows that it has “expected dimension”

$$\mathrm{vdim} = n + (1-g)(D-3) + \int_d c_1(TX)$$

where D is the complex dimension of X. The spaces $X_{g,n,d}$ can be quite ill-behaved — they may be singular, and may not have the expected dimension — but we can always equip them with a virtual fundamental class $[X_{g,n,d}] \in H_{2\,\mathrm{vdim}}(X_{g,n,d};\mathbb{Q})$ of the expected dimension. There are natural maps

$$\mathrm{ev}_i : X_{g,n,d} \to X \qquad i = 1, 2, \ldots, n$$

given by evaluation at the ith marked point and line bundles

$$L_i \to X_{g,n,d} \qquad i = 1, 2, \ldots, n$$

called universal cotangent lines. The fiber of L_i at the stable map $f : C \to X$ is the cotangent line to the curve C at the ith marked point. We denote the first Chern class of the line bundle L_i by ψ_i.

The genus-g Gromov–Witten potential

$$\mathcal{F}_X^g(t_0, t_1, \ldots) = \sum_{\substack{d \in H_2(X;\mathbb{Z}) \\ n \geq 0}} \frac{Q^d}{n!} \int_{[X_{g,n,d}]} \bigwedge_{i=1}^{i=n} \Big(\sum_{k_i \geq 0} \mathrm{ev}_i^\star \, t_{k_i} \wedge \psi_i^{k_i} \Big) \tag{GW}$$

is a generating function for genus-g Gromov–Witten invariants. Here Q^d is the representative of d in the group ring of $H_2(X;\mathbb{Z})$ — this separates the contributions of curves of different degrees — and $t_0, t_1, \ldots \in H^\star(X;\Lambda)$ are cohomology classes on X. We take the coefficient ring Λ to be a Novikov ring $\mathbb{C}[\![Q]\!]$, which is a completion of the semigroup ring of degrees of holomorphic curves in X. We regard $\mathcal{F}_X^g$ as a formal function of $\mathbf{t}(z) = t_0 + t_1 z + \ldots \in H^\star(X;\Lambda)[z]$ which takes values in Λ.

Givental’s quantization formalism

The structure of genus-0 Gromov–Witten theory is well-understood, following work of Dijkgraaf and Witten [14], Dubrovin [15], Kontsevich and Manin [38], and Barannikov [3]. Genus-0 Gromov–Witten invariants satisfy many universal identities: the string equation, the dilaton equation, the topological recursion relations and the celebrated WDVV equations. A recent insight of Givental [30, 12, 25] is that this structure admits a very simple interpretation in terms of symplectic geometry and the theory of loop groups. It turns out that the totality of Gromov–Witten invariants in genus 0 can be encoded by a Lagrangian submanifold $\mathcal{L}_X$ of a certain symplectic vector space $\mathcal{H}$, and that the universal identities mentioned above are equivalent to the assertion that $\mathcal{L}_X$ takes a very special form — see the Proposition below and [25]. Many natural operations in Gromov–Witten theory, such as applying the string equation or (as we will see below) "twisting" Gromov–Witten potentials in various ways, correspond to elements of a loop group of symplectic transformations of $\mathcal{H}$. The effect of such an operation on genus-0 Gromov–Witten invariants can be concisely described in terms of the corresponding symplectic transformation S: it replaces the Lagrangian submanifold $\mathcal{L}_X$ with $S(\mathcal{L}_X)$.

This point of view also gives insight into the structure of higher-genus Gromov–Witten invariants, about which very little is currently known. Higher genus Gromov–Witten theory can be regarded as a quantization of genus-0 Gromov–Witten theory, in the following sense. Introduce the total descendent potential

$$\mathcal{D}_X = \exp\Big(\sum_{g\geq 0} \hbar^{g-1}\mathcal{F}^g_X\Big)$$

which is a generating function for Gromov–Witten invariants of all genera. Consider an element S of the loop group which, as above, corresponds to some operation in Gromov–Witten theory. The process of geometric quantization associates to S a differential operator $\hat{S}$. The total descendent potential $\mathcal{D}_X$ can be regarded as a function on the symplectic vector space $\mathcal{H}$, and the effect of the operation corresponding to S on Gromov–Witten invariants of all genera is to replace the generating function $\mathcal{D}_X$ by $\hat{S}(\mathcal{D}_X)$. The results in this dissertation are phrased in terms of this "quantization formalism", so we now describe this language in more detail.

Introduce the supervector space

$$\mathcal{H} = H^\star(X;\Lambda)(\!(z^{-1})\!)$$

of cohomology-valued Laurent series in $1/z$, where the indeterminate z is regarded as even. (In fact we will need to consider a completion of this space — see section 1.3.2 for details.)

We equip $\mathcal{H}$ with the even Λ-valued symplectic form

$$\Omega(f,g) = \frac{1}{2\pi i}\oint (f(-z), g(z))\,dz$$

where $(\cdot\,,\cdot)$ is the Poincaré pairing on $H^\star(X)$ and the contour of integration winds once anticlockwise about the origin. The polarization

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$$

by Lagrangian subspaces

$$\mathcal{H}_+ = H^\star(X;\Lambda)[z]$$
$$\mathcal{H}_- = z^{-1}H^\star(X;\Lambda)[\![z^{-1}]\!]$$

identifies $\mathcal{H}$ with the cotangent bundle $T^\star\mathcal{H}_+$. (Here we need to complete $\mathcal{H}_+$ too; see section 1.3.2 again.) We regard the genus-0 Gromov–Witten potential $\mathcal{F}_X^0$ and the total descendent potential $\mathcal{D}_X$, which are functions of $\mathbf{t}$, as functions on $\mathcal{H}_+$ by setting

$$\mathbf{q}(z) = \mathbf{t}(z) - z$$

where $\mathbf{q}(z) = q_0 + q_1 z + \ldots$ is a co-ordinate on $\mathcal{H}_+$. In other words

$$q_0 = t_0 \qquad q_1 = t_1 - 1 \qquad q_2 = t_2 \qquad q_3 = t_3 \qquad \ldots$$

This identification is called the dilaton shift. Via the dilaton shift, the genus-0 Gromov–Witten potential generates (the germ near $\mathbf{q}(z) = -z$ of) a Lagrangian submanifold $\mathcal{L}_X$ of $\mathcal{H}$:

$$\mathcal{L}_X = \{(\mathbf{p},\mathbf{q}) : \mathbf{p} = d_{\mathbf{q}}\mathcal{F}_X^0\}$$
$$\subset T^\star\mathcal{H}_+ \cong \mathcal{H}$$

Proposition ([12]). *$\mathcal{L}_X$ is (the germ of) a Lagrangian cone with vertex at the origin such that each tangent space L to $\mathcal{L}_X$ is tangent to $\mathcal{L}_X$ exactly along zL. In other words,*

$$L \cap \mathcal{L}_X = zL$$

and the tangent space to $\mathcal{L}_X$ at all points of zL is the same Lagrangian subspace L.

$\mathcal{L}_X$ is therefore ruled by the family of isotropic subspaces

$$\{zL : L \text{ is a tangent space to } \mathcal{L}_X\}$$

It is clear from the proof of the Proposition given in section 1.5 that this family is of (finite) dimension $\dim H^\star(X)$.

As mentioned above, and proved in [25], the Proposition is equivalent to various universal identities between genus-0 Gromov–Witten invariants and hence holds whenever Dubrovin's axioms for a genus-0 topological field theory coupled to gravity are satisfied. The proof that we give below is geometric in character, however, and so only applies to those Frobenius structures which come from Gromov–Witten theory.

It remains to describe the action of symplectic transformations of $\mathcal{H}$ on Gromov–Witten invariants. The process of geometric quantization associates to an infinitesimal symplectomorphism $A : \mathcal{H} \to \mathcal{H}$ a differential operator $\widehat{A}$, in the following way. Consider the quadratic Hamiltonian

$$h_A(f) = \tfrac{1}{2}\,\Omega(Af, f)$$

Choose Darboux co-ordinates $\{p^\alpha, q^\beta\}$ adapted to the polarization (so that $\mathcal{H}_+$ is given by $p^1 = p^2 = \ldots = 0$ and $\mathcal{H}_-$ is given by $q^1 = q^2 = \ldots = 0$), express h_A in terms of these co-ordinates and define $\widehat{A}$ to be the quantization $\widehat{h_A}$ of h_A, where

$$\widehat{q^\alpha q^\beta} = \frac{q^\alpha q^\beta}{\hbar} \qquad \widehat{q^\alpha p^\beta} = q^\alpha \frac{\partial}{\partial q^\beta} \qquad \widehat{p^\alpha p^\beta} = \hbar \frac{\partial}{\partial q^\alpha}\frac{\partial}{\partial q^\beta}$$

and $\widehat{\cdot}$ is linear. This quantization procedure gives a projective representation of the Lie algebra of infinitesimal symplectomorphisms as differential operators. We call transformations of the form $S = \exp(A)$, where A is an infinitesimal symplectomorphism of the form

$$A = \sum_{m \in \mathbb{Z}} A_m z^m \qquad A_m \in \operatorname{End}(H^\star(X))$$

elements of the loop group, and define

$$\hat{S} = \exp(\widehat{A})$$

Such quantizations $\hat{S}$ act (projectively) on the total descendent potential $\mathcal{D}_X$, which we regard as a formal function of $\mathbf{q}$ via the dilaton shift. Taking the "semi-classical limit" $\hbar \to 0$ in the quantization procedure we find that acting on $\mathcal{D}_X$ by $\hat{S}$

$$\mathcal{D}_X \rightsquigarrow \hat{S}(\mathcal{D}_X)$$

corresponds to applying the (unquantized) linear transformation S to the Lagrangian submanifold $\mathcal{L}_X$:

$$\mathcal{L}_X \rightsquigarrow S(\mathcal{L}_X)$$

Twisted Gromov–Witten invariants

The classical Riemann–Roch formula gives a purely topological expression for the index of the Cauchy–Riemann operator acting on sections of a holomorphic vector bundle over a compact complex curve. Such indices can be regarded as virtual vector spaces

$$\ker \bar{\partial} \ominus \operatorname{coker} \bar{\partial}$$

and in a parametric situation form a virtual vector bundle over the parameter space. Topological invariants of this index bundle take the form of characteristic classes in the cohomology of the parameter space. The parametric situation just described occurs in Gromov–Witten theory, and allows us to enrich our notion of Gromov–Witten invariant. A holomorphic vector bundle E over the target space X restricted to the curves in X yields an index bundle $E_{g,n,d}$ over the moduli space $X_{g,n,d}$. The "fiber" of the virtual vector bundle $E_{g,n,d}$ at the stable map $f : C \to X$ is

$$H^0(C, f^\star E) \ominus H^1(C, f^\star E)$$

Given an invertible multiplicative characteristic class $\mathbf{c}$ of complex vector bundles, we define twisted Gromov–Witten invariants by replacing the virtual fundamental class $[X_{g,n,d}]$ occurring in equation (GW) by the cap product $[X_{g,n,d}] \cap \mathbf{c}(E_{g,n,d})$. If $\mathbf{c}$ is the trivial characteristic class then these are the usual Gromov–Witten invariants of X. Two other important special cases are as follows.

- Suppose that E is a line bundle which is sufficiently positive that $H^1(C, f^\star E) = 0$ for all genus-0 stable maps $f : C \to X$. Such bundles are called convex; examples include those bundles which are spanned fiberwise by global holomorphic sections. If we take the characteristic class $\mathbf{c}$ to be the Euler class[1] then genus-0 twisted Gromov–Witten

[1]The Euler class is not invertible, and so strictly speaking our construction does not apply. However, in the convex case the virtual vector bundle $E_{g,n,d}$ is in fact a bundle, so the Euler class of $E_{g,n,d}$ is well-defined

invariants of X coincide with genus-0 Gromov–Witten invariants of the hypersurface cut out by a generic section of E. Understanding the relationship between twisted and untwisted Gromov–Witten invariants will allow us to prove a very general version of the Quantum Lefschetz Hyperplane Principle (Corollary 5 below), which relates genus-0 Gromov–Witten invariants of complete intersections to those of the ambient space. This implies the celebrated mirror formula, due to Candelas, de la Ossa, Green and Parkes [11], for genus-0 Gromov–Witten invariants of quintic hypersurfaces in $\mathbb{C}P^4$.

- If X is sufficiently negative that $H^0(C, f^\star E) = 0$ for all genus-0 stable maps $f : C \to X$ — such bundles are called concave — then genus-0 Gromov–Witten invariants of X twisted by the S^1-equivariant inverse Euler class of E are closely related to so-called local Gromov–Witten invariants in genus 0. (The S^1-action here rotates the fibers of E and the index bundles $E_{g,n,d}$, and leaves X and the moduli spaces $X_{g,n,d}$ fixed.) Given X a submanifold of the Kähler manifold Y with normal bundle E, local Gromov–Witten invariants measure the contribution to Gromov–Witten invariants of Y coming from curves lying in a neighbourhood of X. In genus 0, such curves of degree $d \neq 0$ in fact lie inside X; this follows from the concavity of E. Local Gromov–Witten invariants participate in the "non-linear Serre duality" of [26, 27], which has been used [27, 35] to establish various enumerative predictions of mirror symmetry, including the mirror formula. Corollary 2 below implies a very general form of non-linear Serre duality, which is formulated as Corollary 1.8.2 on page 81.

Theorem 1 below determines the relationship between twisted and untwisted Gromov–Witten invariants in all genera. It gives an explicit formula for twisted Gromov–Witten invariants in terms of untwisted ones. The formula is written in terms of the quantization formalism, which we extend to the "twisted" setting in the next section. Before doing this, we give precise definitions of the twisted Gromov–Witten potentials.

---

and the Euler-twisted potentials make sense. We can fit this into our general framework by first considering the S^1-equivariant Euler class, which is invertible, and then passing to the non-equivariant limit. The S^1-action we consider here rotates the fibers of E and of $E_{g,n,d}$, and leaves X and the moduli spaces $X_{g,n,d}$ fixed.

Twisted Gromov–Witten potentials

There is a natural map

$$\pi : X_{g,n+1,d} \to X_{g,n,d}$$

given by forgetting the last marked point and contracting any components of the curve on which the resulting map is unstable. This can be regarded as the universal family of stable maps over $X_{g,n,d}$:

$$\begin{array}{ccc} X_{g,n+1,d} & \xrightarrow{\mathrm{ev}_{n+1}} & X \\ {\scriptstyle\pi}\downarrow & & \\ X_{g,n,d} & & \end{array}$$

Given a holomorphic vector bundle E over X, we can pull it back along the map ev_{n+1} and then take the K-theoretic push-forward along π to define a virtual vector bundle

$$E_{g,n,d} = \pi_\star \, \mathrm{ev}_{n+1}^\star \, E$$

over $X_{g,n,d}$. The "fiber" of the virtual bundle $E_{g,n,d}$ at the stable map $f : C \to X$ is

$$H^0(C, f^\star E) \ominus H^1(C, f^\star E)$$

An invertible multiplicative characteristic class $\mathbf{c}$ of complex vector bundles can be written as

$$\mathbf{c}(\,\cdot\,) = \exp\Big(\sum_{k \geq 0} s_k \, \mathrm{ch}_k(\,\cdot\,)\Big)$$

for some choice of $s_0, s_1, \ldots$, where ch_k is the degree-$2k$ component of the Chern character. We regard $s_0, s_1, s_2, \ldots$ as formal parameters, and incorporate them in the ground ring Λ:

$$\Lambda = \mathbb{C}[\![Q]\!] \otimes \mathbb{C}[\![s_0, s_1, s_2, \ldots]\!]$$

The $(\mathbf{c}, E)$-twisted genus-g Gromov–Witten potential

$$\mathcal{F}^g_{\mathbf{c},E}(t_0, t_1, \ldots) = \sum_{\substack{d \in H_2(X;\mathbb{Z}) \\ n \geq 0}} \frac{Q^d}{n!} \int_{[X_{g,n,d}]} \bigwedge_{i=1}^{i=n} \Big(\sum_{k_i \geq 0} \mathrm{ev}_i^\star \, t_{k_i} \wedge \psi_i^{k_i}\Big) \wedge \mathbf{c}(E_{g,n,d})$$

is a generating function for $(\mathbf{c}, E)$-twisted Gromov–Witten invariants. It is a formal function of $\mathbf{t}(z) = t_0 + t_1 z + \ldots \in H^\star(X; \Lambda)[z]$ which takes values in Λ. The $(\mathbf{c}, E)$-twisted total descendent potential of X

$$\mathcal{D}_{\mathbf{c},E} = \exp\Big(\sum_{g \geq 0} \hbar^{g-1} \mathcal{F}^g_{\mathbf{c},E}\Big)$$

is a generating function for $(\mathbf{c}, E)$-twisted Gromov–Witten invariants of all genera.

Extending the quantization formalism to the twisted setting

The Poincaré pairing on the cohomology of the target space X occurs in untwisted Gromov–Witten theory as an intersection index on the moduli space $X_{0,3,0} = X$:

$$(a, b) = \int_{[X_{0,3,0}]} \mathrm{ev}_1^\star a \wedge \mathrm{ev}_2^\star 1 \wedge \mathrm{ev}_3^\star b$$

In the twisted setting this takes the form

$$\begin{aligned}(a, b)_{\mathbf{c},E} &:= \int_{[X_{0,3,0}]} \mathrm{ev}_1^\star a \wedge \mathrm{ev}_2^\star 1 \wedge \mathrm{ev}_3^\star b \wedge \mathbf{c}(E_{0,3,0}) \\ &= \int_X a \wedge b \wedge \mathbf{c}(E)\end{aligned}$$

which suggests that we should base the symplectic form occurring in the quantization formalism on this "twisted Poincaré pairing". Except for this change, and the fact that the ground ring Λ now contains $s_0, s_1, \ldots$, all ingredients of the quantization formalism are exactly as before.

We take the symplectic space to be[2]

$$\mathcal{H}_{\mathbf{c},E} = H^\star(X; \Lambda)((z^{-1}))$$

with the Λ-valued symplectic form

$$\Omega_{\mathbf{c},E}(f, g) = \frac{1}{2\pi i} \oint (f(-z), g(z))_{\mathbf{c},E}\, dz$$

As before, the polarization

$$\mathcal{H}_{\mathbf{c},E} = \mathcal{H}_+^{\mathbf{c},E} \oplus \mathcal{H}_-^{\mathbf{c},E}$$

[2] Again, we suppress some details about completions here: see section 1.3.2.

by Lagrangian subspaces

$$\mathcal{H}_+^{\mathbf{c},E} = H^\star(X;\Lambda)[z]$$
$$\mathcal{H}_-^{\mathbf{c},E} = z^{-1}H^\star(X;\Lambda)[[z^{-1}]]$$

identifies $\mathcal{H}_{\mathbf{c},E}$ with the cotangent bundle $T^\star\mathcal{H}_+^{\mathbf{c},E}$. We regard the twisted Gromov–Witten potentials $\mathcal{F}^0_{\mathbf{c},E}$ and $\mathcal{D}_{\mathbf{c},E}$ as functions on $\mathcal{H}_+^{\mathbf{c},E}$ via the dilaton shift

$$\mathbf{q}(z) = \mathbf{t}(z) - z$$

where $\mathbf{q}(z) = q_0 + q_1 z + \ldots$ is a co-ordinate on $\mathcal{H}_+^{\mathbf{c},E}$. Via the dilaton shift, the twisted genus-0 potential generates (the germ near $\mathbf{q}(z) = -z$ of) a Lagrangian submanifold $\mathcal{L}_{\mathbf{c},E}$ of $\mathcal{H}_{\mathbf{c},E}$:

$$\mathcal{L}_{\mathbf{c},E} = \{(\mathbf{p},\mathbf{q}) : \mathbf{p} = d_{\mathbf{q}}\mathcal{F}^0_{\mathbf{c},E}\}$$

We will want to apply quantized elements of the loop group to the twisted total descendent potential $\mathcal{D}_{\mathbf{c},E}$. We therefore need to identify $\mathcal{D}_{\mathbf{c},E}$ with a function on $\mathcal{H}_+$ (rather than on $\mathcal{H}_+^{\mathbf{c},E}$). We do this via the symplectomorphism

$$\begin{aligned} \varphi : \mathcal{H}_{\mathbf{c},E} &\to \mathcal{H} \\ x &\mapsto \sqrt{\mathbf{c}(E)}\,x \end{aligned}$$

φ maps $\mathcal{H}_+^{\mathbf{c},E}$ isomorphically to $\mathcal{H}_+$.

Quantum Riemann–Roch

The following result determines all twisted Gromov–Witten invariants in terms of untwisted Gromov–Witten invariants. Since certain twisted Gromov–Witten invariants are closely related to local Gromov–Witten invariants and to the Gromov–Witten invariants of hypersurfaces (see the discussion on pages 6 and 7), this will allow us to establish very general versions of non-linear Serre duality and of the Quantum Lefschetz Hyperplane Principle.

Theorem 1. *Let L be a line bundle with first Chern class z. Multiplication by the asymptotic expansion of the infinite product*

$$\prod_{m=1}^{\infty} \mathbf{c}(E \otimes L^{-m})$$

defines a linear symplectomorphism $\triangle : \mathcal{H} \to \mathcal{H}_{\mathbf{c},E}$, *and the quantization* $\widehat{\varphi\triangle}$ *identifies the one-dimensional subspaces spanned by* $\mathcal{D}_X$ *and by* $\mathcal{D}_{\mathbf{c},E}$*:*

$$\langle \mathcal{D}_{\mathbf{c},E} \rangle = \widehat{\varphi\triangle} \langle \mathcal{D}_X \rangle$$

Remark. The transformation $\varphi\triangle : \mathcal{H} \to \mathcal{H}$ is multiplication by the asymptotic expansion of

$$\sqrt{\mathbf{c}(E)} \prod_{m=1}^{\infty} \mathbf{c}(E \otimes L^{-m})$$

We interpret this as follows. Let $\rho_1, \ldots \rho_r$ be the Chern roots of E, and let $s(\cdot)$ be the logarithm of $\mathbf{c}(\cdot)$:

$$s(x) = \sum_{k \geq 0} s_k \frac{x^k}{k!}$$

Then

$$\begin{aligned} \ln\Big(\sqrt{\mathbf{c}(E)} \prod_{m=1}^{\infty} \mathbf{c}(E \otimes L^{-m})\Big) &\sim \sum_{i=1}^{r} \left[\frac{s(\rho_i)}{2} + \sum_{m=1}^{\infty} s(\rho_i - mz) \right] \\ &\sim \sum_{i=1}^{r} \frac{1}{2} \left[\frac{1 + e^{z\partial_x}}{1 - e^{z\partial_x}} s(x) \right] \Bigg|_{x=\rho_i} \\ &= \sum_{i=1}^{r} \left[\sum_{m \geq 0} \frac{B_{2m}}{(2m)!} (z\partial_x)^{2m-1} s(x) \right] \Bigg|_{x=\rho_i} \\ &= \sum_{m \geq 0} \sum_{l \geq 0} \frac{B_{2m}}{(2m)!} s_{l+2m-1} \operatorname{ch}_l(E) z^{2m-1} \end{aligned}$$

Here the B_k are Bernoulli numbers

$$\frac{t}{1 - e^{-t}} = \sum_{k \geq 0} \frac{B_k}{k!} t^k$$

Multiplication by $\operatorname{ch}_l(E) z^{2m-1}$ is an infinitesimal symplectomorphism of $\mathcal{H}$, so $\triangle$ is a linear symplectomorphism. In Chapter 1 we prove (Theorem 1.6.4) that

$$\exp\Big(-\frac{1}{24} \sum_{l>0} s_{l-1} \int_X \operatorname{ch}_l(E) c_{D-1}(T_X)\Big) (\operatorname{sdet} \sqrt{\mathbf{c}(E)})^{-\frac{1}{24}} \mathcal{D}_{\mathbf{c},E} =$$

$$\exp\Big(\sum_{m>0} \sum_{l \geq 0} s_{2m-1+l} \frac{B_{2m}}{(2m)!} (\operatorname{ch}_l(E) z^{2m-1})^{\wedge}\Big) \exp\Big(\sum_{l>0} s_{l-1} (\operatorname{ch}_l(E)/z)^{\wedge}\Big) \mathcal{D}_X$$

where D is the complex dimension of X and sdet is the superdeterminant. Since the factor in front of $\mathcal{D}_{\mathbf{c},E}$ is a non-vanishing scalar function of $\mathbf{s}$, this implies Theorem 1.

Corollary 1. *The Lagrangian submanifolds $\mathcal{L}_{\mathbf{c},E}$ and $\mathcal{L}_X$ satisfy*

$$\mathcal{L}_{\mathbf{c},E} = \triangle \mathcal{L}_X$$

In particular, $\mathcal{L}_{\mathbf{c},E}$ is (the germ of) a Lagrangian cone which satisfies the conclusions of the Proposition on page 4.

Non-linear Serre duality

Let $\mathbf{c}^*$ denote the multiplicative characteristic class

$$\mathbf{c}^*(\,\cdot\,) = \exp\Big(\sum_{k\geq 0}(-1)^{k+1} s_k \operatorname{ch}_k(\,\cdot\,)\Big)$$

so that

$$\mathbf{c}^*(E^*) = \frac{1}{\mathbf{c}(E)}$$

Corollary 2. *The one-dimensional spaces spanned by $\mathcal{D}_{\mathbf{c},E}$ and by $\mathcal{D}_{\mathbf{c}^*,E^*}$ are equal:*

$$\langle \mathcal{D}_{\mathbf{c}^*,E^*} \rangle = \langle \mathcal{D}_{\mathbf{c},E} \rangle$$

Proof. $\mathcal{D}_{\mathbf{c}^*,E^*}$ is obtained from $\mathcal{D}_X$ by the quantization of

$$\sqrt{\mathbf{c}^*(E^*)} \prod_{m=1}^{\infty} \mathbf{c}^*(E^* \otimes L^{-m}) = \frac{1}{\sqrt{\mathbf{c}(E)}} \prod_{m=1}^{\infty} \frac{1}{\mathbf{c}(E \otimes L^m)}$$

Replacing L^m by L^{-m} on the right-hand side of this formula corresponds to replacing z by $-z$. But elements $S(z)$ of the loop group satisfy

$$S^T(-z)S(z) = I$$

where ^{T} denotes the adjoint with respect to the Poincaré pairing, and multiplication by a cohomology class is self-adjoint, so

$$\sqrt{\mathbf{c}^*(E^*)} \prod_{m=1}^{\infty} \mathbf{c}^*(E^* \otimes L^{-m}) = \sqrt{\mathbf{c}(E)} \prod_{m=1}^{\infty} \mathbf{c}(E \otimes L^{-m})$$

Now apply Theorem 1. □

Note that the equality in Corollary 2 involves formal functions of $\mathbf{q}$. The relationship between $\mathcal{D}_{\mathbf{c},E}$ and $\mathcal{D}_{\mathbf{c}^*,E^*}$ as formal functions of $\mathbf{t}$ is described explicitly by Corollary 1.8.1 on page 80.

"Non-linear Serre duality", discovered in [26, 27] in the context of fixed-point localization formulas for genus-0 Gromov–Witten invariants of toric manifolds, is a close relationship between Gromov–Witten invariants twisted by the S^1-equivariant Euler class of a bundle E (where S^1 rotates the fibers of E) and those twisted by the inverse equivariant Euler class of E^* (equipped with the dual S^1-action). More concretely, it gives a close relationship between genus-0 Gromov–Witten invariants of hypersurfaces and certain local Gromov–Witten invariants (see page 7). If the characteristic class $\mathbf{c}$ in Corollary 2 is the S^1-equivariant Euler class then $\mathbf{c}^*$ is almost equal to the S^1-equivariant inverse Euler class. Corollary 2 therefore implies a very general version of non-linear Serre duality, which applies to any compact Kähler target space X and to twisted Gromov–Witten invariants in arbitrary genus. This is formulated as Corollary 1.8.2 on page 81.

Quantum Lefschetz Hyperplane Principle

As mentioned above, if E is a positive line bundle then the genus-0 Gromov–Witten invariants of X twisted by the Euler class of E coincide with the genus-0 Gromov–Witten invariants of the hypersurface cut out by a generic section of E. We now analyze the consequences of Corollary 1 in the case where $\mathbf{c}$ is the S^1-equivariant Euler class. By passing to the non-equivariant limit, we will be able to prove a very general version of the Quantum Lefschetz Hyperplane Principle (Corollary 5), which relates genus-0 Gromov–Witten invariants of complete intersections to those of the ambient space. Our argument hinges on the fact, explained in the next-but-one section, that the cone $\mathcal{L}_X$ is entirely determined by a generic family

$$\tau \mapsto J(\tau) \qquad \tau \in H^\star(X)$$

of elements of $\mathcal{L}_X$. In the next-but-one section we also exhibit such a family, which we call the J-function of X, and in addition give a family which determines the twisted cone $\mathcal{L}_{\mathbf{e},E}$, called the twisted J-function. The J-function encodes all genus-0 Gromov–Witten invariants of X and the twisted J-function encodes all genus-0 twisted Gromov–Witten invariants.

Using Corollary 1 and the J-function of X we build another family, the I-function, and prove that this also determines $\mathcal{L}_{\mathbf{e},E}$ (Theorem 2). Since the I-function and the twisted J-function determine the same cone, we can write one in terms of the other (Corollary 4). On passing to the non-equivariant limit, this gives the Quantum Lefschetz Hyperplane Principle (Corollary 5). We then show that this implies the earlier mirror theorems of [26, 4, 35, 47, 9, 43, 21], and in particular give a new proof of the mirror formula for genus-0 Gromov–Witten invariants of the quintic threefold [11].

The quantization formalism

In order to take the characteristic class $\mathbf{c}$ by which we twist to be the S^1-equivariant Euler class $\mathbf{e}$, we need to base our ground ring Λ on the coefficient ring $H^\star(BS^1;\mathbb{C})$ of S^1-equivariant cohomology theory. We identify $H^\star(BS^1;\mathbb{C})$ with $\mathbb{C}[\lambda]$, where λ is the first Chern class of the universal bundle over $\mathbb{C}P^\infty$, and take

$$\Lambda = \mathbb{C}[[Q]](\sqrt{\lambda})[\ln \lambda]$$

We adjoin the $\sqrt{\lambda}$ and $\ln \lambda$ to ensure that the asymptotic expansion in Corollary 1 has coefficients in Λ. We extend the quantization formalism to this situation exactly as on pages 9–10: the only change is the new ground ring Λ.

Corollary 3. *Multiplication by the asymptotic expansion*

$$\Gamma_E(z) \sim \prod_{m=1}^{\infty} \mathbf{e}(E \otimes L^{-m})$$

gives a linear symplectomorphism $\Box : (\mathcal{H}, \Omega) \to (\mathcal{H}_{\mathbf{e},E}, \Omega_{\mathbf{e},E})$ *and*

$$\Box \mathcal{L}_X = \mathcal{L}_{\mathbf{e},E}$$

The series $\Gamma_E(z)$ is closely related to the asymptotic expansion of the gamma function:

$$\Gamma_E(z) \sim \frac{1}{\mathbf{e}(E)} \prod_{i=1}^{r} \frac{1}{\sqrt{2\pi z}} \int_0^\infty e^{\frac{-x+(\lambda+\rho_i)\ln x}{z}}\, dx$$

where $\rho_1, \ldots, \rho_r$ are the Chern roots of E. This observation is the key to the proof of Theorem 2 below.

J-functions and slices of the cones

Recall from the discussion on pages 4 and 5 that the Lagrangian cone $\mathcal{L}_X$ which encodes genus-0 Gromov–Witten invariants of X is ruled by a $(\dim H^\star(X))$-dimensional family of subspaces

$$\{zL : L \text{ is a tangent space to } \mathcal{L}_X\}$$

Given a family

$$\tau \mapsto J(\tau) \qquad \tau \in H^\star(X)$$

of elements of $\mathcal{L}_X$ which is transverse to the ruling, the cone $\mathcal{L}_X$ is therefore swept out by

$$\{zL_\tau : \tau \in H^\star(X)\} \quad \text{where} \quad L_\tau = T_{J(\tau)}\mathcal{L}_X$$

In other words

$$\mathcal{L}_X = \bigcup_{\tau \in H^\star(X)} zL_\tau$$

Fix a basis $\{\phi_1, \ldots, \phi_N\}$ for $H^\star(X;\mathbb{C})$. Since the family $\tau \mapsto J(\tau)$ is transverse to the ruling, the derivatives $\partial_1 J(\tau, -z), \ldots, \partial_N J(\tau, -z)$ in the directions $\phi_1, \ldots, \phi_N \in H^\star(X;\Lambda)$ form a basis for the tangent space L_τ over[3] $\Pi = \Lambda[z]$. In this sense, the family $\tau \mapsto J(\tau)$ generates the whole cone $\mathcal{L}_X$. We say that such a family is a slice of the cone $\mathcal{L}_X$.

One such slice is given by the intersection of $\mathcal{L}_X$ with the affine subspace

$$-z + z\mathcal{H}_- \subset \mathcal{H}$$

We call the function parameterizing this slice the J-function of X. A formula for it in terms of genus-0 Gromov–Witten invariants is as follows. Let $g_{\alpha\beta} = (\phi_\alpha, \phi_\beta)$, and let $g^{\alpha\beta}$ be the entries of the matrix inverse to that with entries $g_{\alpha\beta}$. The J-function $J_X(\tau, -z)$ is the $\mathcal{H}$-valued function of $\tau \in H^\star(X;\Lambda)$ defined by

$$\begin{aligned} J_X(\tau, -z) &= -z + \tau + \sum_{n,d} \frac{Q^d}{n!} \left(\int_{[X_{g,n+1,d}]} \left(\bigwedge_{i=1}^{n} \mathrm{ev}_i^\star \tau \right) \wedge \frac{\phi_\alpha}{-z - \psi_{n+1}} \right) g^{\alpha\beta} \phi_\beta \\ &\in -z + \tau + \mathcal{H}_- \end{aligned}$$

Here we interpret

$$\frac{1}{-z - \psi_{n+1}} = -\frac{1}{z} + \frac{\psi_{n+1}}{z^2} - \frac{\psi_{n+1}^2}{z^3} + \ldots$$

[3]Again, we ignore some completion issues here: the relevant completion Π of $\Lambda[z]$ is described on page 74.

and sum over repeated Greek indices.

Corollary 1 implies that the Lagrangian cone $\mathcal{L}_{\mathbf{e},E} \subset \mathcal{H}_{\mathbf{e}}$ is ruled in exactly the same way as $\mathcal{L}_X$. A slice of $\mathcal{L}_{\mathbf{e},E}$ is given by the intersection of $\mathcal{L}_{\mathbf{e},E}$ with the affine subspace $-z + z\mathcal{H}_-^{\mathbf{e}}$. The $\mathcal{H}$-valued function of $\tau \in H^\star(X;\Lambda)$ which parameterizes this intersection is called the twisted J-function $J_{\mathbf{e},E}(\tau,-z)$. We can write it in terms of genus-0 twisted Gromov–Witten invariants as follows. Let $g^{\mathbf{e}}_{\alpha\beta} = (\phi_\alpha, \phi_\beta)_{\mathbf{e}}$, and let $g_{\mathbf{e}}^{\alpha\beta}$ be the entries of the matrix inverse to that with entries $g^{\mathbf{e}}_{\alpha\beta}$. Then

$$\begin{aligned} J_{\mathbf{e},E}(\tau,-z) &= -z + \tau + \sum_{n,d} \frac{Q^d}{n!}\left(\int_{[X_{g,n+1,d}]} \left(\bigwedge_{i=1}^{n} \mathrm{ev}_i^\star \tau\right) \wedge \frac{\phi_\alpha}{-z-\psi_{n+1}} \wedge \mathbf{e}(E_{0,n+1,d})\right) g_{\mathbf{e}}^{\alpha\beta}\phi_\beta \\ &\in -z + \tau + \mathcal{H}_-^{\mathbf{e}} \end{aligned}$$

Another slice of $\mathcal{L}_{\mathbf{e},E}$

Theorem 2. *Let $\rho_1, \ldots, \rho_r$ be the Chern roots of E. Define an $\mathcal{H}_{\mathbf{e},E}$-valued function of $t \in H^\star(X;\Lambda)$ by*

$$I(t,z) = \prod_{i=1}^{r} \left(\frac{\displaystyle\int_0^\infty e^{x/z} J_X(z, t + (\lambda+\rho_i)\ln x)\, dx}{\displaystyle\int_0^\infty e^{\frac{x-(\lambda+\rho_i)\ln x}{z}}\, dx} \right)$$

where the integrals represent their stationary phase asymptotics as $z \to 0$. Then the family

$$t \mapsto I(t,-z) \qquad\qquad t \in H^\star(X;\Lambda)$$

of elements of $\mathcal{H}_{\mathbf{e},E}$ lies on the Lagrangian cone $\mathcal{L}_{\mathbf{e},E}$.

In fact the family $t \mapsto I(t,-z)$ is a slice of $\mathcal{L}_{\mathbf{e},E}$: it is transverse to the ruling of $\mathcal{L}_{\mathbf{e},E}$ by

$$\{zL : L \text{ is a tangent space to } \mathcal{L}_{\mathbf{e},E}\}$$

and so the derivatives $\partial_1 I(t,-z), \ldots, \partial_N I(t,-z)$ in the directions $\phi_1, \ldots, \phi_N \in H^\star(X;\Lambda)$ form a basis for the tangent space

$$L_t = T_{I(t,-z)}\mathcal{L}_{\mathbf{e},E}$$

over Π. The subspace zL_t meets the affine subspace $-z + z\mathcal{H}^{\mathbf{e}}_-$ at a unique point

$$-z + \tau(t) + \mathcal{H}^{\mathbf{e}}_-$$

and this defines a map $t \mapsto \tau(t)$, which we call the mirror map.

Corollary 4. *The unique intersection of zL_t with the affine subspace $-z + z\mathcal{H}^{\mathbf{e}}_-$ coincides with $J_{\mathbf{e},E}(\tau(t), -z)$. In other words,*

$$J_{\mathbf{e},E}(\tau(t), -z) = I(t, -z) + \sum_{\alpha=1}^{N} C^{\alpha}(t, z)\, z\, \partial_{\alpha} I(t, -z)$$

where $C^{\alpha}(t, z)$ are the unique elements of Π such that the right-hand side lies in $-z + z\mathcal{H}^{\mathbf{e}}_-$. The mirror map $t \mapsto \tau(t)$ is determined by the expansion $-z+\tau(t)+O(z^{-1})$ of the right-hand side.

The mirror map and Birkhoff factorization

This procedure of calculating $J_{\mathbf{e},E}(\tau, z)$ from $I(t, z)$ is reminiscent of Birkhoff factorization in the theory of loop groups. In fact, the corresponding procedure applied to first derivatives of I and $J_{\mathbf{e},E}$ really is an example of Birkhoff factorization: let $S_{\mathbf{e},E}(\tau, z)$ be the matrix with columns

$$\partial_1 J_{\mathbf{e},E}(\tau, z), \ldots, \partial_N J_{\mathbf{e},E}(\tau, z)$$

and let $R(t, z)$ be the matrix with columns

$$\partial_1 I(t, z), \ldots, \partial_N I(t, z)$$

Since the families $I(t, -z)$ and $J_{\mathbf{e},E}(\tau, -z)$ are transverse to the ruling of $\mathcal{L}_{\mathbf{e},E}$, the columns of $R(t, -z)$ and $S_{\mathbf{e},E}(\tau(t), -z)$ both form bases for L_t over Π, and so

$$R(t, -z) = S_{\mathbf{e},E}(\tau(t), -z) C(t, z) \tag{BF}$$

for some matrix $C(t, z)$ with entries in Π. But $S_{\mathbf{e},E}(\tau, z)$ is a matrix-valued power series in $1/z$ and $C(t, z)$ is a matrix-valued power series in z, so (BF) is the Birkhoff factorization of $R(t, -z)$ in the loop group of symplectomorphisms of $\mathcal{H}$. The factorization (BF) determines the mirror map, since applying the linear transformation $S_{\mathbf{e},E}(\tau, z)$ to $1 \in H^{\star}(X; \Lambda)$ gives

$$1 + \frac{\tau}{z} \qquad \mathrm{mod}\, \frac{1}{z^2}$$

Complete intersections

Suppose now that the Chern roots $\rho_1, \ldots, \rho_r$ of E are defined over $\mathbb{Z}$ — for example, E could be a direct sum of line bundles. Write

$$J_X(t,z) = \sum_d J_d(t,z) Q^d$$

Applying the string and divisor equations and integrating by parts gives

$$I(t,z) = \sum_d J_d(t,z) Q^d \prod_{i=1}^{r} \frac{\prod_{k=-\infty}^{\rho_i(d)} (\lambda + \rho_i + kz)}{\prod_{k=-\infty}^{0} (\lambda + \rho_i + kz)}$$

Corollary 5 explains how to obtain the twisted J-function $J_{\mathbf{e},E}$ from this "hypergeometric modification" of J_X.

If E is a direct sum of convex line bundles (see page 6) then both $I(t,z)$ and $J_{\mathbf{e},E}(\tau,z)$ admit non-equivariant limits. Let $j : Y \to X$ denote the inclusion into X of the complete intersection cut out by a generic section of E. The non-equivariant limit $\lim_{\lambda \to 0} I(t,z)$ is

$$I_{X,Y}(t,z) = \sum_d J_d(t,z) Q^d \prod_{i=1}^{r} \prod_{k=1}^{\rho_i(d)} (\rho_i + kz)$$

and $\lim_{\lambda \to 0} J_{\mathbf{e},E}(\tau,z)$ is $J_{X,Y}(\tau,z)$ where

$$\mathbf{e}^{\text{non}}(E) J_{X,Y}(u,z) =_{H_2(Y) \to H_2(X)} j_\star J_Y(j^\star u, z)$$

Here $\mathbf{e}^{\text{non}}$ is the non-equivariant Euler class, J_Y is the J-function of Y (defined as on page 15) and the long subscript indicates that the corresponding homomorphism between Novikov rings should be applied to the right-hand side of the equation. We see that $J_{X,Y}$ determines, up to some "blurring" of Novikov variables, all genus-0 Gromov–Witten invariants of Y which involve cohomology classes coming from X.

Corollary 5. *The series $I_{X,Y}(t,-z)$ and $J_{X,Y}(\tau,-z)$ determine the same cone. In particular, $J_{X,Y}(\tau,-z)$ is determined from $I_{X,Y}(t,-z)$ by the "Birkhoff factorization" procedure followed by the mirror map $t \mapsto \tau$ as described in Corollary 4.*

This is the Quantum Lefschetz Hyperplane Principle advertised above. It essentially determines the genus-0 Gromov–Witten invariants of the complete intersection Y in terms of the genus-0 Gromov–Witten invariants of the ambient space X. It applies to convex complete intersections of arbitrary Fano index, and without restriction on the parameter $\tau \in H^\star(X)$. We can combine Corollary 5 with the Birkhoff factorization procedure outlined on page 17 to compute gravitational descendents of Y — these are Gromov–Witten invariants which involve powers of the universal cotangent line classes ψ_i. We will see below that in the case where Y is either Fano or Calabi–Yau and where τ is restricted to lie in the "small parameter space" $H^{\leq 2}(X)$, Corollary 5 gives the earlier Quantum Lefschetz theorems of Givental, Batyrev *et al.*, Kim, Lian *et al.*, Bertram, Lee and Gathmann [26, 4, 35, 47, 9, 43, 21].

Example: a quintic 3-fold in $\mathbb{C}P^4$

Take $X = \mathbb{C}P^4$, $E = \mathcal{O}(5)$ and $j : Y \to X$ the inclusion of the hypersurface cut out by a generic section of E. Let P be the hyperplane class generating $H^\star(X;\mathbb{C}) = \mathbb{C}[P]/(P^5)$. The restriction of the J-function of X to $H^2(X)$ is known to be [26]

$$J_X(tP, z) = z e^{tP/z} \sum_{d \geq 0} \frac{Q^d e^{dt}}{\prod_{k=1}^{d} (P + kz)^5}$$

so

$$\begin{aligned} I_{X,Y}(tP, z) &= z e^{tP/z} \sum_{d \geq 0} Q^d e^{dt} \frac{\prod_{k=1}^{5d} (5P + kz)}{\prod_{k=1}^{d} (P + kz)^5} \\ &= f(t) z + P g(t) + O(z^{-1}) \end{aligned}$$

where

$$\begin{aligned} f(t) &= \sum_{d \geq 0} Q^d e^{dt} \frac{(5d)!}{(d!)^5} \\ g(t) &= t f(t) + 5 \sum_{d \geq 0} Q^d e^{dt} \frac{(5d)!}{(d!)^5} \sum_{k=d+1}^{k=5d} \frac{1}{k} \end{aligned}$$

According to Corollary 5, $I_{X,Y}$ and $J_{X,Y}$ determine the same cone. But $I_{X,Y}(tP, -z)$ determines the same cone as

$$\frac{I_{X,Y}(tP, -z)}{f(t)} = -z + P \frac{g(t)}{f(t)} + O(z^{-1})$$

and

$$J_{X,Y}(\tau P, -z) = -z + \tau P + O(z^{-1})$$

so if

$$\tau(t) = \frac{g(t)}{f(t)}$$

then

$$J_{X,Y}(\tau(t)P, -z) = \frac{I_{X,Y}(tP, -z)}{f(t)}$$

This is the celebrated quintic mirror formula of Candelas, de la Ossa, Green and Parkes.

More generally:

Corollary 6. *If E is a direct sum of convex line bundles such that $c_1(E) \leq c_1(X)$ then the restriction of $I_{X,Y}(t,z)$ to the small parameter space $H^{\leq 2}(X;\Lambda)$ takes the form*

$$I_{X,Y}(t,z) = zF(t) + \sum_i G^i(t)\phi_i + O(z^{-1})$$

where the $\{\phi_i\}$ are a basis for $H^{\leq 2}(X)$ and $F(t)$, $G^i(t)$ are scalar-valued functions such that $F(t)$ is invertible. The restriction of $J_{X,Y}(\tau, z)$ to $H^{\leq 2}(X;\Lambda)$ is given by

$$J_{X,Y}(\tau, z) = \frac{I_{X,Y}(t,z)}{F(t)}$$

where

$$\tau = \sum_i \frac{G^i(t)}{F(t)}\phi_i$$

Thus we recover the Quantum Lefschetz theorems of [26, 4, 35, 47, 9, 43, 21].

Quantum cobordism

Recall from the discussion on page 2 that even though the moduli space $X_{g,n,d}$ may be singular and may not have the dimension predicted by Riemann–Roch, we can always equip it with a virtual fundamental class $[X_{g,n,d}]$ of the expected dimension. This coincides with the usual fundamental class when $X_{g,n,d}$ is smooth and of the expected dimension. We can similarly equip $X_{g,n,d}$ with a virtual vector bundle $\mathcal{T}^{vir}_{g,n,d} \in K^0(X_{g,n,d})$, called the virtual tangent bundle, which coincides with the usual tangent bundle when $X_{g,n,d}$ is smooth and of the expected dimension. This gives another way to enrich our notion of Gromov–Witten invariant: we can twist by characteristic classes of the virtual tangent bundle. Given an invertible multiplicative characteristic class $\mathbf{c}$ of complex vector bundles, we define tangent-twisted Gromov–Witten invariants by replacing the virtual fundamental class $[X_{g,n,d}]$ occurring in equation (GW) by the cap product $[X_{g,n,d}] \cap \mathbf{c}(\mathcal{T}^{vir}_{g,n,d})$.

The virtual tangent bundle is

$$\mathcal{T}^{vir}_{g,n,d} = \mathcal{T} - \mathcal{N}$$

where the tangent sheaf $\mathcal{T}$ and the obstruction sheaf $\mathcal{N}$ fit into the exact sequence

$$0 \to \mathrm{Aut}(C) \to H^0(C, f^\star TX) \to \mathcal{T} \to \mathrm{Def}(C) \to H^1(C, f^\star TX) \to \mathcal{N} \to 0$$

of sheaves on $X_{g,n,d}$ (see [31, 13]). Here we denote sheaves on $X_{g,n,d}$ by their fibers at the stable map $f : C \to X$. $\mathrm{Aut}(C)$ is the vector space of holomorphic vector fields on C which vanish at the marked points and $\mathrm{Def}(C)$ is the space of infinitesimal deformations of the complex structure on C. The virtual tangent bundle $\mathcal{T}^{vir}_{g,n,d}$ therefore consists of two parts, one

$$H^0(C, f^\star TX) \ominus H^1(C, f^\star TX)$$

coming from variations of the map $f : C \to X$, where the complex structure on C is fixed, and the other

$$\mathrm{Def}(C) \ominus \mathrm{Aut}(C)$$

coming from variations of the complex structure on C. We can describe the contribution of the first part of $\mathcal{T}^{vir}_{g,n,d}$ to tangent-twisted Gromov–Witten invariants using Theorem 1, since

$$H^0(C, f^\star TX) \ominus H^1(C, f^\star TX) = (TX)_{g,n,d}$$

The remaining part, coming from deformations of complex structure on the domain curve, contributes to tangent-twisted Gromov–Witten invariants in a rather complicated way.

Theorem 3 below expresses tangent-twisted Gromov–Witten invariants in terms of untwisted Gromov–Witten invariants. The key geometrical argument, which is contained in section 2.5.3, identifies the virtual tangent bundle $\mathcal{T}^{vir}_{g,n,d}$ as the sum of three parts. One of these parts is $(TX)_{g,n,d}$, another has a simple description in terms of universal cotangent lines L_i and the third is supported entirely on the boundary of the moduli space $X_{g,n,d}$. Since the boundary of $X_{g,n,d}$ is made up of products of "smaller" moduli spaces $X_{g',n',d'}$, this allows us to write down recursion relations which determine tangent-twisted Gromov–Witten invariants in terms of untwisted Gromov–Witten invariants. The key combinatorial step, which allows us to solve these recursion relations, is to interpret tangent-twisted Gromov–Witten invariants in terms of Gromov–Witten invariants with values in complex cobordism. The idea of defining Gromov–Witten invariants with values in cobordism goes back to Gromov [32], who constructed invariants of symplectic manifolds in the form of bordism classes in certain spaces of (pseudo)holomorphic curves; it was later pursued by Kontsevich [37] and Morava [52, 51]. Extending the quantization formalism to this cobordism-valued setting allows us to express the relationship between tangent-twisted and untwisted Gromov–Witten invariants in a very simple form. In the next section we give a brief introduction to complex cobordism. We then define cobordism-valued Gromov–Witten invariants and extend the quantization formalism to deal with them. Finally, we state Theorem 3 and discuss some of its consequences.

Complex cobordism

The complex cobordism of a topological space Y is the extraordinary cohomology of Y with values in the Thom spectrum MU. If Y is a complex manifold of dimension n then the cobordism group $MU^i(Y)$, which is defined in terms of homotopy classes of maps to Thom spaces $MU(k)$ of universal bundles over $BU(k)$

$$MU^i(Y) = \lim_{j \to \infty} [\Sigma^j Y, MU(i+j)]$$

can be described more concretely in Poincaré-dual terms. The Pontryagin–Thom construction identifies $MU^i(Y)$ with the complex bordism group $MU_{2n-i}(Y)$ — this plays the role of the Poincaré isomorphism between complex cobordism and complex bordism — and complex bordism groups admit the following geometric description [55, 66]. A weakly complex manifold M is a smooth real manifold together with a complex vector bundle over M whose underlying real vector space is of the form $TM \oplus \mathbb{R}^N$. We identify complex structures on $TM \oplus \mathbb{R}^N$ which are homotopic, and identify the complex structure on $TM \oplus \mathbb{R}^N$ with the obvious complex structure on $TM \oplus \mathbb{R}^N \oplus \mathbb{R}^2$. The complex bordism group $MU_i(Y)$ is the free Abelian group on the set of continuous maps $M \to Y$, where M is a closed weakly complex manifold of real dimension i, modulo the relations

$$\begin{aligned} &\left[M_1 \coprod M_2 \to Y\right] = [M_1 \to Y] + [M_2 \to Y] \\ &[\partial W \to Y] = 0 \end{aligned}$$

Here W is a weakly complex manifold with boundary and ∂W inherits a weakly complex structure in the obvious way.

We consider cobordism groups with complex coefficients, so the coefficient ring of the theory is

$$\begin{aligned}\Omega^\star_{MU} &= MU^\star(pt) \otimes \mathbb{C} \\ &\cong \mathbb{C}[p_1, p_2, \ldots]\end{aligned}$$

where p_i is the degree $(-2i)$ class represented by $\mathbb{C}P^i \to pt$. We can define cobordism-valued characteristic classes of complex vector bundles exactly as we do for usual cohomology. The cobordism-valued first Chern class of the bundle $\mathcal{O}(1)$ over $\mathbb{C}P^n$ is Poincaré-dual to the inclusion $\mathbb{C}P^{n-1} \to \mathbb{C}P^n$ of a hyperplane section. If u is the cobordism-valued first Chern class of the universal bundle ξ over $\mathbb{C}P^\infty$ then

$$MU^\star(\mathbb{C}P^\infty) \cong \Omega^\star_{MU}[[u]]$$

Much as for K-theory, there is a multiplicative natural transformation from cobordism to cohomology which gives ring isomorphisms

$$\mathrm{ch}_{MU} : MU^\star(X) \otimes \mathbb{C} \to H^\star(X; \Omega^\star_{MU})$$

for all X. This is called the Chern–Dold character. The image of $u \in MU^\star(\mathbb{C}P^\infty)$ under the Chern–Dold character is a formal power series

$$u(z) = z + a_2 z^2 + a_3 z^3 + \ldots$$

where z is the (cohomological) first Chern class of the universal line bundle ξ.

Given a proper map of complex manifolds $\pi : Y \to Z$ there is a pushforward

$$\pi_\star : MU^i(Y) \to MU^{i+\dim Z - \dim Y}(Z)$$

which (in Poincaré-dual terms) sends $[f : M \to Y]$ to $[\pi f : M \to Z]$. We can compute the push-forward to a point in terms of cohomology via the Riemann–Roch formula

$$\pi_\star(\alpha) = \int_Y \mathrm{ch}_{MU}(\alpha)\, \mathrm{Td}_{MU}(TY) \tag{RR}$$

Here α is a cobordism class on Y, π is the map from Y to a point, and Td_{MU} is the multiplicative $H^\star(\cdot; \Omega^\star_{MU})$-valued characteristic class which takes the value

$$\mathrm{Td}_{MU}(\xi) = \frac{z}{u(z)}$$

on the universal line bundle. If we write

$$\mathrm{Td}_{MU}(\,\cdot\,) = \exp\Big(\sum_{k>0} s_k\, \mathrm{ch}_k(\,\cdot\,)\Big)$$

then $s_1, s_2, \ldots$ give another set of generators for $\Omega^\star_{MU}$:

$$\Omega^\star_{MU} = \mathbb{C}[s_1, s_2, \ldots]$$

Cobordism-valued Gromov–Witten invariants

We base the ground ring Λ on the coefficient ring of complex cobordism theory, taking

$$\Lambda = \mathbb{C}[\![Q]\!] \otimes \mathbb{C}[\![s_1, s_2, \ldots]\!]$$

Using the Riemann–Roch formula (RR) we can define cobordism-valued Gromov–Witten invariants in purely cohomological terms. The genus-g cobordism potential of X, which is a generating function for cobordism-valued Gromov–Witten invariants, is defined to be

$$\mathcal{F}^g_{MU}(t_0, t_1, \ldots) = \sum_{\substack{d \in H_2(X;\mathbb{Z}) \\ n \geq 0}} \frac{Q^d}{n!} \int_{[X_{g,n,d}]} \bigwedge_{i=1}^{i=n} \Big(\sum_{k_i \geq 0} \mathrm{ev}_i^\star(\mathrm{ch}_{MU}\, t_{k_i}) \wedge u(\psi_i)^{k_i}\Big) \wedge \mathrm{Td}_{MU}(\mathcal{T}^{vir}_{g,n,d})$$

Here $t_0, t_1, \ldots \in MU^\star(X;\Lambda)$ are cobordism classes on X and, as before, ψ_i is the (cohomological) first Chern class of the ith universal cotangent line L_i. We regard $\mathcal{F}^g_{MU}$ as a formal function of $\mathbf{t} = t_0 + t_1 u + \ldots \in MU^\star(X;\Lambda)[u]$ which takes values in Λ. The total cobordism potential of X

$$\mathcal{D}_{MU} = \exp\Big(\sum_{g \geq 0} \hbar^{g-1} \mathcal{F}^g_{MU}\Big)$$

is a generating function for cobordism-valued Gromov–Witten invariants of all genera.

Since any invertible multiplicative characteristic class is a scalar multiple of

$$\mathrm{Td}_{MU}(\,\cdot\,) = \exp\Big(\sum_{k>0} s_k\, \mathrm{ch}_k(\,\cdot\,)\Big)$$

for appropriate values of $s_1, s_2, \ldots$, the total cobordism potential encodes all tangent-twisted Gromov–Witten invariants.

The quantization formalism

The symplectic space $\mathcal{H}$ associated to usual Gromov–Witten theory consists of cohomology-valued Laurent series in $1/z$

$$\mathcal{H} = H^\star(X;\Lambda)(\!(z^{-1})\!)$$

and the symplectic form is based on the Poincaré pairing

$$\Omega(f_1, f_2) = \frac{1}{2\pi i}\oint (f_1(-z), f_2(z))\, dz$$

We can regard z here as the first Chern class of the universal line bundle ξ over $\mathbb{C}P^\infty$.

We take the symplectic space $\mathcal{U}$ associated to cobordism-valued Gromov–Witten theory to consist of cobordism-valued Laurent series[4] in $1/u$

$$\mathcal{U} = MU^\star(X;\Lambda)(\!(u^{-1})\!)$$

equipped with the symplectic form[5]

$$\Omega_{MU}(f_1, f_2) = \frac{1}{2\pi i}\oint (f_1(u(-z)), f_2(u(z)))_{MU}\, dz$$

based on the Poincaré pairing in cobordism theory

$$(a, b)_{MU} = \int_X \mathrm{ch}_{MU}(a) \wedge \mathrm{ch}_{MU}(b) \wedge \mathrm{Td}_{MU}(TX)$$

We can regard u as the cobordism-valued first Chern class of the universal line bundle ξ.

Define Laurent series $v_k(u)$, $k = 0, 1, 2, \ldots$ by

$$\frac{1}{u(-x-y)} = \sum_{k\geq 0} (u(x))^k v_k(u(y))$$

where we expand the left-hand side in the region where $|x| < |y|$. We prove in section 2.3.2 that (appropriate completions of) the subspaces

$$\begin{aligned} \mathcal{U}_+ &= MU^\star(X;\Lambda)[u] \\ \mathcal{U}_- &= \Big\{ \sum_{n\geq 0} \alpha_n v_n(u) : \alpha_n \in MU^\star(X;\Lambda) \Big\} \end{aligned}$$

---

[4]Once again we suppress some details about completions here: see section 2.3.2.

[5]We can write this more invariantly as

$$\Omega_{MU}(f_1, f_2) = \frac{1}{2\pi i}\oint (f_1(u^\star), f_2(u))_{MU}\, dg(u)$$

where $u^\star$ is the inverse to u in the formal group corresponding to complex cobordism [55, 2]. Here $g(u)$ is the power series inverse to $u(z)$, so $dg(u)$ is the invariant differential on the formal group.

are Lagrangian. The polarization

$$\mathcal{U} = \mathcal{U}_+ \oplus \mathcal{U}_-$$

identifies the symplectic space $(\mathcal{U}, \Omega_{MU})$ with the cotangent bundle $T^\star\mathcal{U}_+$.

Let

$$u^\star = -u + b_1 u^2 + b_2 u^3 + \dots$$

be the cobordism-valued first Chern class[6] of the Hopf bundle ξ^{-1} over $\mathbb{C}P^\infty$. We regard the cobordism potentials $\mathcal{F}^0_{MU}$ and $\mathcal{D}_{MU}$ as formal functions on $\mathcal{U}_+$ via the dilaton shift

$$\mathbf{q}(u) = \mathbf{t}(u) + u^\star$$

where $\mathbf{q}(u) = q_0 + q_1 u + q_2 u^2 + \dots$ is a co-ordinate on $\mathcal{U}_+$. Via the dilaton shift and the identification $\mathcal{U} \cong T^\star\mathcal{U}_+$, the genus-0 cobordism potential generates (the germ near $\mathbf{q}(u) = u^\star$ of) a Lagrangian submanifold $\mathcal{L}_{MU}$ of $\mathcal{U}$:

$$\mathcal{L}_{MU} = \{(\mathbf{p}, \mathbf{q}) : \mathbf{p} = d_{\mathbf{q}} \mathcal{F}^0_{MU}\}$$

The quantum Hirzebruch–Riemann–Roch theorem

We want to compare the total cobordism potential $\mathcal{D}_{MU}$, which is a function on $\mathcal{U}_+$, with the total descendent potential $\mathcal{D}_X$, which is a function on $\mathcal{H}_+$. We define the quantum Chern–Dold character to be the map

$$\begin{aligned} &\mathrm{qch} : \mathcal{U} \to \mathcal{H} \\ &\sum_{n\in\mathbb{Z}} \alpha_n u^n \mapsto \sqrt{\mathrm{Td}_{MU}(TX)} \sum_{n\in\mathbb{Z}} \mathrm{ch}_{MU}(\alpha_n)(u(z))^n \end{aligned}$$

This is a symplectomorphism from $\mathcal{U}$ to $\mathcal{H}$. It maps $\mathcal{U}_+$ isomorphically to $\mathcal{H}_+$, and we regard $\mathcal{D}_{MU}$ as a function on $\mathcal{H}_+$ via this identification.

Although the quantum Chern–Dold character maps $\mathcal{U}_+$ to $\mathcal{H}_+$, it does not map $\mathcal{U}_-$ to $\mathcal{H}_-$. We can, however, find a symplectomorphism $\blacktriangledown : \mathcal{H} \to \mathcal{H}$ which sends $\mathcal{H}_+$ to $\mathcal{H}_+$ and sends $\mathrm{qch}(\mathcal{U}_-)$ to $\mathcal{H}_-$. A simple formula for $\blacktriangledown$ in terms of the power series $u(z)$, together with a discussion of its representation-theoretic meaning, can be found in section 2.3.2.

[6] As the reader may have noticed, this does not lie in the ring of Laurent series $MU^\star(X;\Lambda)((u^{-1}))$. It does, however, lie in the appropriate completion $\mathcal{U}$ of this ring — see section 2.3.2.

Theorem 3. *Applying the quantized operator* $\widehat{\blacktriangledown}$ *to the total cobordism potential* $\mathcal{D}_{MU}$ *yields the Gromov–Witten potential of* X *twisted by the characteristic class* Td_{MU} *and the bundle* TX*:*

$$\widehat{\blacktriangledown}\mathcal{D}_{MU} = \mathcal{D}_{\mathrm{Td}_{MU},TX}$$

In other words,

$$\langle\mathcal{D}_{MU}\rangle = \widehat{\blacktriangledown}^{-1}\widehat{\blacktriangle}\langle\mathcal{D}_X\rangle$$

where $\blacktriangle : \mathcal{H} \to \mathcal{H}$ *is multiplication by the asymptotic expansion of*

$$\sqrt{\mathrm{Td}_{MU}(TX)}\prod_{m=1}^{\infty}\mathrm{Td}_{MU}(TX \otimes L^{-m})$$

This determines all cobordism-valued Gromov–Witten invariants, and hence all tangent-twisted Gromov–Witten invariants, in terms of the the usual (untwisted, cohomology-valued) Gromov–Witten invariants. Remarkably, the entire contribution to the virtual tangent bundle from deformations of the complex structure on the domain curve is absorbed by the comparison qch between the cohomology-valued and cobordism-valued formalisms, and the change of polarization $\blacktriangledown$. Theorem 3 reduces "quantum cobordism" to quantum cohomology, and hence can be regarded as a "quantum" version of the Hirzebruch–Riemann–Roch theorem.

Corollary 7. $\mathrm{qch}(\mathcal{L}_{MU})$ *coincides with the Lagrangian cone for* (Td_{MU}, TX)*-twisted Gromov–Witten theory, so*

$$\mathrm{qch}(\mathcal{L}_{MU}) = \blacktriangle\mathcal{L}_X$$

In particular, $\mathcal{L}_{MU}$ *is (the germ of) a Lagrangian cone which satisfies the conclusions of the Proposition on page 4.*

When X is a point, $\mathcal{L}_X$ is invariant under $\blacktriangle$.

Corollary 8. *If* $X = pt$ *then* $\mathrm{qch}(\mathcal{L}_{MU}) = \mathcal{L}_X$*.*

Given any complex-oriented extraordinary cohomology theory E we can define quantum E-cohomology much as we did quantum cobordism, replacing the Chern–Dold character ch_{MU} with an appropriate Chern character ch_E and the Todd class Td_{MU} with an appropriate Todd class Td_E. Complex cobordism is the universal complex-oriented cohomology theory, so there is a natural transformation $\theta_E : MU \to E$ from complex cobordism to E. We can compute the "total E-potential" $\mathcal{D}_E$, which is defined (in the obvious way) on page 86, by applying θ_E to the total cobordism potential $\mathcal{D}_{MU}$. Thus Theorem 3 determines all Gromov–Witten invariants with values in an arbitrary complex-oriented extraordinary cohomology theory.

Almost-Kähler manifolds

Gromov–Witten invariants can be defined whenever the target space X is a compact symplectic manifold equipped with an almost-complex structure J which is tamed by the symplectic form. The results described in this chapter, with the exception of Corollaries 5 and 6, go through to this almost-Kähler setting; this is established in Appendix B. Corollaries 5 and 6 rely on a comparison result between algebraic virtual fundamental classes, the almost-Kähler analog of which does not seem to be known.

Chapter 1

Quantum Cohomology

1.1 Introduction

A major goal of this chapter is to understand, at least in genus zero, the relationship between Gromov–Witten invariants of a complete intersection and those of the ambient space. Following Kontsevich [37], we approach this problem by studying not Gromov–Witten invariants of the complete intersection directly, but instead Gromov–Witten invariants of the ambient space twisted by the bundle which determines the complete intersection (see section 1.6.1). Kontsevich originally defined genus-0 Gromov–Witten invariants of sufficiently positive complete intersections in terms of Gromov–Witten invariants of the ambient space twisted by the Euler class; as discussed in section 1.7.1, his definition agrees with the general definitions of [44, 6] in this case.

The main result of this chapter, Theorem 1.6.4, determines the relationship between twisted and untwisted Gromov–Witten invariants in all genera. Our approach, following [53, 17], is to apply the Grothendieck–Riemann–Roch theorem to the universal family over the moduli space of stable maps. In [17], Faber and Pandharipande interpreted Mumford's Grothendieck–Riemann–Roch calculation [53] as giving differential equations satisfied by generating functions for Gromov–Witten invariants twisted by the Euler class and the trivial bundle. Exactly the same approach gives differential equations satisfied by generating functions for more general twisted Gromov–Witten invariants. The main new ingredient in

Theorem 1.6.4 is the quantization formalism [30] outlined in Chapter 0, which allows us to interpret these differential equations in geometric terms — as the quantizations of certain infinitesimal symplectic transformations — and consequently to solve them.

Extracting genus-0 Gromov–Witten invariants corresponds to taking a "semi-classical limit" of the full genus picture. It turns out (Theorem 1.5.3) that the totality of gravitational descendents in genus-0 Gromov–Witten theory can be encoded by a semi-infinite ruled cone $\mathcal{L}_X$ in the cohomology of X with coefficients in the field of Laurent series in $1/z$, and that another such cone corresponds to each twisted theory. Taking the semi-classical limit of Theorem 1.6.4, we find that the twisted and untwisted cones are related by a symplectic transformation. In the case of twistings by the Euler class of a line bundle E, this transformation can be described in terms of the stationary phase asymptotics of the oscillating integral

$$\frac{1}{\sqrt{2\pi z}} \int_0^\infty e^{\frac{-x+(\lambda+\rho)\ln x}{z}} \, dx$$

where ρ is the first Chern class of E. This allows us to derive a Quantum Lefsctetz Hyperplane Principle (Corollary 1.7.5) which is more general than earlier versions [4, 35, 47, 9, 43, 21] in the sense that the restrictions $t \in H^{\leq 2}(X)$ on the space of parameters and $c_1(E) \leq c_1(X)$ on the Fano index are removed.

The material of this chapter represents joint work with Givental, and has previously appeared in the preprint [12].

The chapter is arranged as follows. In section 1.2, we fix notation for moduli spaces of stable maps and Gromov–Witten potentials. Section 1.3 describes the quantization formalism [30] in detail; in particular, Examples 1.3.1.1 and 1.3.3.1 introduce notation which is used throughout the rest of the chapter. Section 1.4 describes the geometry associated to the semi-classical limit of the quantization formalism, and explains the role of various objects familiar from genus-0 Gromov–Witten theory in this geometric framework. In section 1.5 we introduce gravitational ancestors, describe their relationship to gravitational descendents (following Givental [30]) and use them to prove that the ruled cone $\mathcal{L}_X$ which encodes genus-0 gravitational descendents is indeed a ruled cone. In section 1.6, we define twisted Gromov–Witten invariants and describe their relationship to untwisted Gromov–Witten invariants. We use this relationship in section 1.7 to derive the Quantum Lefschetz Hyperplane Principle, and in section 1.8 to derive a very general version of "non-linear Serre duality"

[26, 27].

1.2 Stable maps and Gromov–Witten invariants

1.2.1 Moduli spaces of stable maps

Throughout, let X denote a compact projective complex manifold of complex dimension D. Denote by $X_{g,n,d}$ the moduli space of stable maps [8, 37] of degree $d \in H_2(X;\mathbb{Z})$ from n-pointed, genus g curves to X. This is a compact complex orbifold. In the case where the target space X is a point, it coincides with the Deligne–Mumford space $\overline{\mathcal{M}}_{g,n}$. The space $X_{g,n,d}$ can be equipped [7, 44, 60] with a virtual fundamental class $[X_{g,n,d}] \in H_*(X_{g,n,d};\mathbb{Q})$ of complex dimension $(1-g)(D-3)+n+\langle c_1(TX), d\rangle$.

There are natural maps

$$\mathrm{ev}_i : X_{g,n,d} \to X \qquad i = 1, 2, \ldots, n$$

given by evaluation at the ith marked point,

$$\pi : X_{g,n+1,d} \to X_{g,n,d}$$

given by forgetting the last marked point and contracting any components of the curve on which the resulting map is unstable, and

$$\mathrm{ct} : X_{g,n,d} \to \overline{\mathcal{M}}_{g,n}$$

given by forgetting the map and contracting any unstable components of the curve. The diagram

$$\begin{array}{ccc} X_{g,n+1,d} & \xrightarrow{\mathrm{ev}_{n+1}} & X \\ \pi \big\downarrow & & \\ X_{g,n,d} & & \end{array}$$

is the universal family over $X_{g,n,d}$. The marked points define sections

$$\sigma_i : X_{g,n,d} \to X_{g,n+1,d} \qquad i = 1, 2, \ldots, n$$

of the universal family.

1.2.2 Gromov–Witten potentials

Gromov–Witten invariants are intersection indices of the form

$$\int_{[X_{g,n,d}]} \mathrm{ev}_1^\star \alpha_1 \wedge \psi_1^{k_1} \wedge \ldots \wedge \mathrm{ev}_n^\star \alpha_n \wedge \psi_n^{k_n}$$

where $\alpha_1, \ldots, \alpha_n \in H^*(X)$, $k_1, \ldots, k_n \in \mathbb{N}$ and ψ_i is the first Chern class of the ith universal cotangent line bundle $L_i \to X_{g,n,d}$. If any of the k_i are non-zero, the corresponding Gromov–Witten invariant is called a gravitational descendent invariant. We will use the following correlator notation: given polynomials (or power series)

$$\begin{aligned} a_1(\psi) &= a_1^0 + a_1^1\psi + a_1^2\psi^2 + \ldots \\ a_2(\psi) &= a_2^0 + a_2^1\psi + a_2^2\psi^2 + \ldots \\ &\vdots \\ a_n(\psi) &= a_n^0 + a_n^1\psi + a_n^2\psi^2 + \ldots \end{aligned}$$

in $H^\star(X)[\psi]$ (or $H^\star(X)[\![\psi]\!]$) and $b \in H^\star(X_{g,n,d})$, define

$$\langle a_1, a_2, \ldots, a_n; b\rangle_{g,n,d} = \int_{[X_{g,n,d}]} \Big(\sum_{k_1 \geq 0} \mathrm{ev}_1^\star a_1^{k_1} \wedge \psi_1^{k_1}\Big) \wedge \ldots \wedge \Big(\sum_{k_n \geq 0} \mathrm{ev}_n^\star a_n^{k_n} \wedge \psi_n^{k_n}\Big) \wedge b$$

and

$$\langle a_1, a_2, \ldots, a_n\rangle_{g,n,d} = \langle a_1, a_2, \ldots, a_n; 1\rangle_{g,n,d}$$

The genus-g Gromov–Witten potential

$$\mathcal{F}_X^g = \sum_{n,d} \frac{Q^d}{n!} \langle \mathbf{t}(\psi), \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi)\rangle_{g,n,d}$$

is a generating function for genus-g Gromov–Witten invariants. It is a formal function of $\mathbf{t}(z) = t_0 + t_1 z + \ldots \in H^\star(X;\Lambda)[z]$ taking values in the ring Λ, which is assumed to contain an appropriate Novikov ring $\mathbb{C}[\![Q]\!]$ (see [49]). We will specify Λ more precisely in the next section. The total descendent potential

$$\mathcal{D}_X = \exp\Big(\sum_g \hbar^{g-1} \mathcal{F}_X^g\Big)$$

is a formal function of $\mathbf{t}$ which takes values in $\Lambda[\![\hbar, \hbar^{-1}]\!]$. Despite the presence of both $\hbar$ and $\hbar^{-1}$ in the exponent, it is well-defined : see Lemma 1.3.1 below.

1.3 Givental's quantization formalism

1.3.1 A symplectic vector space

Consider the symplectic (super)vector space

$$\mathcal{H}^0 = H^\star(X; \Lambda_0)(\!(z^{-1})\!)$$

where the indeterminate z is regarded as even, equipped with the (even) symplectic form

$$\Omega(f, g) = \frac{1}{2\pi i} \oint (f(-z), g(z))\, dz$$

Here $\Lambda_0 = \mathbb{C}[\![Q]\!]$ is an appropriate Novikov ring, $(\cdot\,,\cdot)$ denotes the Poincaré pairing on $H^\star(X)$ and the contour of integration winds once anticlockwise about the origin. The polarization of $(\mathcal{H}^0, \Omega)$ by the Lagrangian subspaces

$$\mathcal{H}^0_+ = H^\star(X)[z]$$
$$\mathcal{H}^0_- = z^{-1} H^\star(X)[\![z^{-1}]\!]$$

gives a symplectic identification of $\mathcal{H}^0$ with the cotangent bundle $T^\star\mathcal{H}^0_+$. Pick a homogeneous co-ordinate system $\{q_a\}$ on $\mathcal{H}^0_+$ and let $\{p_a\}$ be the dual co-ordinate system on $\mathcal{H}^0_-$, so that $\{p_a, q_b\}$ forms a Darboux co-ordinate system for Ω:

$$\Omega(f, g) = \sum_a (p_a(f) q_a(g) - (-1)^{\bar{p}_a \bar{q}_a} q_a(f) p_a(g))$$

Example 1.3.1.1 This example introduces notation which we will use throughout Chapter 1 without further comment. Denote the dimension of $H^\star(X)$ by N. Let

$$\{\phi_\alpha : \alpha = 1, \ldots, N\}$$

be a homogeneous basis for $H^\star(X)$ such that $\phi_1 = 1$, and let $g_{\alpha\beta} = (\phi_\alpha, \phi_\beta)$. Write $g^{\alpha\beta}$ for the entries of the matrix inverse to that with entries $g_{\alpha\beta}$. Then

$$\sum_{k \geq 0} q_k^\alpha \phi_\alpha z^k + \sum_{l \geq 0} p_l^\beta g^{\beta\epsilon} \phi_\epsilon (-z)^{-1-l} \tag{1.1}$$

gives such a Darboux co-ordinate system on $(\mathcal{H}^0, \Omega_0)$. Here and throughout this chapter, we use the summation convention for Greek indices but not Roman indices. In other words, we sum over repeated Greek indices. Such indices will always correspond to directions in $H^\star(X)$. We raise (respectively lower) indices with $g^{\alpha\beta}$ (respectively $g_{\alpha\beta}$), so for instance in (1.1) we could write $g^{\beta\epsilon}\phi_\epsilon$ as ϕ^β. ◇

1.3.2 Completions

In what follows, we will need to extend the ground ring Λ_0 in various ways and work with various completions of $\mathcal{H}^0$. For instance, we will often equip the ground ring Λ_0 with the Q-adic topology and replace $\mathcal{H}^0$ by the space

$$\mathcal{H}^1 = \Big\{\sum_{k\in\mathbb{Z}} h_k z^k : h_k \in H^\star(X;\Lambda_0), h_k \to 0 \text{ in the topology of } \Lambda_0 \text{ as } k \to \infty\Big\}$$

Also, we often work with S^1-equivariant Gromov–Witten invariants [26], which take values in $H^\star(BS^1;\mathbb{C})$; here and throughout we identify $H^\star(BS^1;\mathbb{C})$ with $\mathbb{C}[\lambda]$, where λ is the first Chern class of the universal line bundle over $\mathbb{C}P^\infty$. In this situation, we extend the ground ring to $\Lambda_2 = \Lambda_0(\lambda)$, equip Λ_2 with the $(Q, 1/\lambda)$-adic topology, and replace $\mathcal{H}^0$ by the space

$$\mathcal{H}^2 = \Big\{\sum_{k\in\mathbb{Z}} h_k z^k : h_k \in H^\star(X;\Lambda_2), h_k \to 0 \text{ in the topology of } \Lambda_2 \text{ as } k \to \infty\Big\}$$

Also, in section 1.6.3, we will need to extend the ground ring to $\Lambda_3 = \Lambda_0[[s_0, s_1, \ldots]]$. Here we equip Λ_3 with the topology induced from the $(Q, 1/\lambda)$-adic topology on Λ_3 by the map

$$\begin{aligned} &\Lambda_3 \to \Lambda_2 \\ &s_k \mapsto \lambda^{-k} \end{aligned}$$

and replace $\mathcal{H}^0$ by the space

$$\mathcal{H}^3 = \Big\{\sum_{k\in\mathbb{Z}} h_k z^k : h_k \in H^\star(X;\Lambda_3), h_k \to 0 \text{ in the topology of } \Lambda_3 \text{ as } k \to \infty\Big\}$$

Throughout, we will denote the relevant ground ring by Λ, the relevant completions of $\mathcal{H}^0$, $\mathcal{H}^0_+$ and $\mathcal{H}^0_-$ by $\mathcal{H}$, $\mathcal{H}_+$ and $\mathcal{H}_-$ respectively and assume that Ω and the Darboux coordinates $\{p_a, q_b\}$ have been extended to $\mathcal{H}$ in the obvious way. Exactly which completion and ground ring we are using at any point should be clear from context. The completions are necessary to ensure that various symplectic transformations, such as those in Theorem 1.5.1 and Theorem 1.6.4, really do act on $\mathcal{H}$.

1.3.3 Quantization procedure

We associate to each infinitesimal symplectomorphism $A : \mathcal{H} \to \mathcal{H}$ a differential operator $\widehat{A}$ of at most second order via the following (standard) procedure. The infinitesimal symplectomorphism A corresponds to a quadratic Hamiltonian

$$h_A(f) = \frac{1}{2}\,\Omega(Af, f)$$

In Darboux co-ordinates $\{p_a, q_b\}$, we set

$$\widehat{q_a q_b} = \frac{q_a q_b}{\hbar}$$

$$\widehat{q_a p_b} = q_a \frac{\partial}{\partial q_b}$$

$$\widehat{p_a p_b} = \hbar \frac{\partial}{\partial q_a} \frac{\partial}{\partial q_b}$$

By linearity, this determines a differential operator $\widehat{A}$ acting on functions on $\mathcal{H}_+$. We will also need to quantize certain non-infinitesimal symplectomorphisms. We call transformations of the form $S = \exp(A)$, where A is an infinitesimal symplectomorphism of the form

$$A = \sum_{m \in \mathbb{Z}} A_m z^m \qquad A_m \in \operatorname{End}(H^\star(X))$$

elements of the loop group, and set

$$\hat{S} = \exp(\widehat{A})$$

Define the Fock space $\mathfrak{Fock}$ to be the space of formal functions of $\mathbf{t}(z) = t_0 + t_1 z + \ldots \in H^\star(X; \Lambda)[z]$ which take values in $\Lambda[[\hbar, \hbar^{-1}]]$. We regard this as a space of formal functions in $\mathbf{q}(z) = q_0 + q_1 z + \ldots \in \mathcal{H}_+$ via the identification

$$\mathbf{q}(z) = \mathbf{t}(z) - z$$

which we call the dilaton shift. The dilaton shift identifies the Fock space with a space of formal functions on $\mathcal{H}_+$ near $\mathbf{q} = -z$. The differential operators $\widehat{q_a q_b}$, $\widehat{q_a p_b}$, and $\widehat{p_a p_b}$ act on $\mathfrak{Fock}$ via this identification. Note, however, that the quantizations $\widehat{A}$ may contain infinite sums of such operators and so do not in general act on $\mathfrak{Fock}$. Each time that we apply the quantization of an infinitesimal symplectomorphism (or of an element of the loop group) to an element of $\mathfrak{Fock}$, we will therefore need to check that the result is well-defined. Many of these verifications have very little geometrical content; these are relegated to Appendix A.

Example 1.3.3.1 Consider an infinitesimal symplectomorphism of $\mathcal{H}$ of the form

$$A = B z^m$$

where the matrix entries of $B \in \mathrm{End}(H^\star(X))$ with respect to the basis of Example 1.3.1.1 are $B^\alpha{}_\beta$. Here and throughout the rest of the chapter, set

$$\partial_{\alpha,k} = \frac{\partial}{\partial q_k^\alpha}$$

A straightforward calculation shows that

$$\text{if } m < 0 \qquad \text{then} \qquad \widehat{A} = \frac{1}{2\hbar}\sum_k (-)^{k+m} B_{\alpha\beta} q_k^\beta q_{-1-k-m}^\alpha - \sum_k B^\alpha{}_\beta q_k^\beta \partial_{\alpha,k+m} \tag{1.2}$$

and

$$\text{if } m \geq 0 \qquad \text{then} \qquad \widehat{A} = -\sum_k B^\alpha{}_\beta q_k^\beta \partial_{\alpha,k+m} + \frac{\hbar}{2}\sum_k (-)^k B^{\alpha\beta}\partial_{\beta,k}\partial_{\alpha,m-1-k} \tag{1.3}$$

In particular, this shows that infinitesimal symplectomorphisms of the form

$$\sum_{-\infty < m \leq N} A_m z^m \qquad A_m \in \mathrm{End}(H^\star(X))$$

have quantizations which act on $\mathfrak{Fock}$.

Denote the expression

$$\sum_k B^\alpha{}_\beta q_k^\beta \partial_{\alpha,k+m}$$

occurring in (1.2) and (1.3) by ∂_A. We have

$$\begin{aligned}
\partial_A \mathbf{q} &= \Big(\sum_k B^\alpha{}_\beta q_k^\beta \partial_{\alpha,k+m}\Big)\Big(\sum_l q_l^\gamma \phi_\gamma z^l\Big) \\
&= \Big[\sum_k B^\alpha{}_\beta q_k^\beta \phi_\alpha z^{k+m}\Big]_+ \\
&= [A\mathbf{q}]_+
\end{aligned}$$

Also, $\partial_{\beta,k}\partial_{\alpha,m-1-k}$ is the bivector field corresponding to

$$\phi_\beta \psi_+^k \otimes \phi_\alpha \psi_-^{m-1-k} \in H^\star(X)[\psi_+] \otimes H^\star(X)[\psi_-] \cong \mathcal{H}_+ \otimes \mathcal{H}_+$$

For m odd and positive, we have

$$\sum_{k=0}^{m-1} (-)^k \psi_+^k \psi_-^{m-1-k} = \frac{\psi_+^m + \psi_-^m}{\psi_+ + \psi_-}$$

and consequently we can interpret the term

$$\sum_k (-)^k B^{\alpha\beta} \partial_{\beta,k} \partial_{\alpha,m-1-k}$$

occurring in (1.3) as the bivector field $\partial \otimes_A \partial$ corresponding to

$$\left[\frac{A(\psi_+) + A(\psi_-)}{\psi_+ + \psi_-} \right]_+ \in \mathrm{End}(H^\star(X))[\![\psi_+, \psi_-]\!]$$

where we identify $\mathrm{End}(H^\star(X))[\![\psi_+, \psi_-]\!]$ with $\mathcal{H}_+ \otimes \mathcal{H}_+$ via the metric. The $[\cdot]_+$ here, which denotes the part involving non-negative powers of both ψ_+ and ψ_-, ensures that this interpretation is (vacuously) correct for m odd and negative also. ◇

Example 1.3.3.2 The string equation (see *e.g.* [54]) asserts that for $(g, n, d) \neq (0, 3, 0)$, $(1, 1, 0)$ we have

$$\langle \mathbf{t}_1(\psi), \ldots, \mathbf{t}_{n-1}(\psi), 1 \rangle_{g,n,d} = \sum_{i=1}^{n-1} \left\langle \mathbf{t}_1(\psi), \ldots, \left[\frac{\mathbf{t}_i(\psi)}{\psi} \right]_+, \ldots, \mathbf{t}_{n-1}(\psi) \right\rangle_{g,n-1,d}$$

Thus

$$\sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{(n-1)!} \langle \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi), 1 \rangle_{g,n,d} = \sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{(n-1)!} \left\langle \left[\frac{\mathbf{t}(\psi)}{\psi} \right]_+, \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi) \right\rangle_{g,n,d}$$
$$+ \frac{1}{2\hbar} \langle \mathbf{t}(\psi), \mathbf{t}(\psi), 1 \rangle_{0,3,0} + \langle 1 \rangle_{1,1,0}$$

and so

$$-\frac{1}{2\hbar} t_0^\alpha g_{\alpha\beta} t_0^\beta - \sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{(n-1)!} \left\langle \left[\frac{\mathbf{t}(\psi) - \psi}{\psi} \right]_+, \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi) \right\rangle_{g,n,d} = 0$$

But we can write this as

$$-\frac{1}{2\hbar} q_0^\alpha g_{\alpha\beta} q_0^\beta - \sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{(n-1)!} \left\langle \left[\frac{\mathbf{q}(\psi)}{\psi} \right]_+, \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi) \right\rangle_{g,n,d} = 0$$

or in other words as

$$-\frac{1}{2\hbar} q_0^\alpha g_{\alpha\beta} q_0^\beta - \partial_{1/z} \left(\sum_g \hbar^{g-1} \mathcal{F}_X^g \right) = 0$$

Thus the string equation is

$$\widehat{\left(\frac{1}{z}\right)} \mathcal{D}_X = 0$$

◇

Example 1.3.3.3 Let $\rho \in H^2(X)$. Multiplication by ρ defines a transformation of $H^\star(X)$ which is self-adjoint with respect to the Poincaré pairing, so multiplication by ρ/z is an infinitesimal symplectomorphism of $\mathcal{H}$. The divisor equation (see *e.g.* [54]) asserts that for $(g,n,d) \neq (0,3,0)$, $(1,1,0)$ we have

$$\begin{aligned}\langle \mathbf{t}_1(\psi), \ldots, \mathbf{t}_{n-1}(\psi), \rho\rangle_{g,n,d} =& \langle \rho, d\rangle \langle \mathbf{t}_1(\psi), \ldots, \mathbf{t}_{n-1}(\psi)\rangle_{g,n-1,d} \\ &+ \sum_{i=1}^{n-1} \left\langle \mathbf{t}_1(\psi), \ldots, \left[\rho \frac{\mathbf{t}_i(\psi)}{\psi}\right]_+, \ldots, \mathbf{t}_{n-1}(\psi)\right\rangle_{g,n-1,d}\end{aligned}$$

Thus

$$\begin{aligned}\sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{(n-1)!} \langle \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi), \rho\rangle_{g,n,d} =& \sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{(n-1)!} \left\langle \left[\rho \frac{\mathbf{t}(\psi)}{\psi}\right]_+, \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi)\right\rangle_{g,n,d} \\ &+ \sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{n!} \langle \rho, d\rangle \langle \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi)\rangle_{g,n,d} \qquad (1.4) \\ &+ \frac{1}{2\hbar} \langle \mathbf{t}(\psi), \mathbf{t}(\psi), \rho\rangle_{0,3,0} + \langle \rho \rangle_{1,1,0}\end{aligned}$$

Let $\{Q_i\}$ be the generators of the Novikov ring corresponding to some choice of basis for $H_2(X;\mathbb{Z})$, and let ρ_i be the co-ordinates of ρ with respect to the dual basis. We can rewrite (1.4) as

$$\begin{aligned}-\frac{1}{2\hbar}(t_0\rho, t_0) - \sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{(n-1)!} \left\langle \left[\rho \frac{\mathbf{q}(\psi)}{\psi}\right]_+, \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi)\right\rangle_{g,n,d} = \\ \sum_i \rho_i Q_i \frac{\partial}{\partial Q_i} \Big(\sum_g \hbar^{g-1} \mathcal{F}_X^g\Big) - \frac{1}{24} \int_X \rho \wedge c_{D-1}(TX)\end{aligned}$$

The left-hand side of this equation is

$$-\frac{1}{2\hbar}(t_0\rho, t_0) - \partial_{\rho/z} \Big(\sum_g \hbar^{g-1} \mathcal{F}_X^g\Big)$$

and so we can write the divisor equation as

$$\widehat{\left(\frac{\rho}{z}\right)} \mathcal{D}_X = \Big(\sum_i \rho_i Q_i \frac{\partial}{\partial Q_i} - \frac{1}{24} \int_X \rho \wedge c_{D-1}(TX)\Big) \mathcal{D}_X$$

◇

Lemma 1.3.1. *$\mathcal{D}_X$ is well-defined as a formal function of $\mathbf{t}(z)$ taking values in $\Lambda[\![\hbar, \hbar^{-1}]\!]$.*

Proof. Let the $(\hbar, \mathbf{t}, Q)$-degree of a monomial

$$Q^d \hbar^{g-1} (t_{i_1}^{\alpha_1})^{j_1} \dots (t_{i_n}^{\alpha_n})^{j_n} \tag{1.5}$$

be $(g-1, j_1 + \dots + j_n, d)$. Since $j_1 + \dots + j_n$ is the degree of (1.5) with respect to the Euler vector field

$$\sum_j t_j^\alpha \frac{\partial}{\partial t_j^\alpha}$$

this quantity has invariant meaning. Monomials in

$$\sum_{g \geq 0} \hbar^{g-1} \mathcal{F}_X^g$$

of $(\hbar, \mathbf{t}, Q)$-degree (a, b, c) correspond to non-zero Gromov–Witten invariants coming from the moduli space $X_{a+1,b,c}$. Since the moduli spaces $X_{0,0,0}$ and $X_{1,0,0}$ are empty, if $c = 0$ then at least one of a and b is strictly positive. Also, since each moduli space $X_{g,n,d}$ is finite-dimensional, there are only finitely many such monomials of any given degree.

A monomial of degree (a, b, c) arises in

$$\mathcal{D}_X = \exp\Big(\sum_{g \geq 0} \hbar^{g-1} \mathcal{F}_X^g\Big)$$

only if we can find monomials in

$$\sum_{g \geq 0} \hbar^{g-1} \mathcal{F}_X^g$$

of degrees (a_1, b_1, c_1), $\dots$, (a_n, b_n, c_n) such that

$$a_1 + \dots + a_n = a \qquad b_1 + \dots + b_n = b \qquad c_1 + \dots + c_n = c$$

In view of the above, there are only finitely many choices for the $\{(a_i, b_i, c_i)\}$. But there are only finitely many monomials of each given degree in

$$\sum_{g \geq 0} \hbar^{g-1} \mathcal{F}_X^g$$

and so there are only finitely many monomials of each given degree in $\mathcal{D}_X$. $\square$

1.3.4 Cocycle

The quantization procedure gives only a projective representation of the Lie algebra of infinitesimal symplectomorphisms. For infinitesimal symplectomorphisms F and G we have

$$[\widehat{F}, \widehat{G}] = \{F, G\}^\wedge + \mathcal{C}(h_F, h_G)$$

where $\{\cdot\,,\cdot\}$ is the Lie bracket, $[\cdot\,,\cdot]$ is the supercommutator, h_F (respectively h_G) is the quadratic Hamiltonian corresponding to F (respectively G), and $\mathcal{C}$ is the cocycle defined by

$$\begin{aligned}&\mathcal{C}(p_a p_b, q_a q_b) = \delta_{ab} + (-1)^{\bar{q}_a \bar{p}_b}\\ &\mathcal{C} = 0 \quad \text{on any other pair of quadratic Darboux monomials}\end{aligned}$$

We will often abuse notation and write $\mathcal{C}(F, G)$ for $\mathcal{C}(h_F, h_G)$.

Example 1.3.4.1 Let $A, B \in \mathrm{End}(H^\star(X))$ be self-adjoint with respect to the Poincaré pairing, so that A/z and Bz define infinitesimal symplectomorphisms of $\mathcal{H}$. Then

$$\mathcal{C}(A/z, Bz) = \tfrac{1}{2}\,\mathrm{str}(AB)$$

In the even case, for example

$$\begin{aligned}\mathcal{C}(A/z, Bz) &= \mathcal{C}\Big(-\frac{1}{2}\,A_{\alpha\beta} q_0^\beta q_0^\alpha - \sum_k A^\alpha{}_\beta q_k^\beta p_{k-1}^\alpha, -\sum_l B^\mu{}_\nu q_l^\nu p_{l+1}^\mu + \frac{1}{2}\,B^{\mu\nu} p_0^\nu p_0^\mu\Big)\\ &= \tfrac{1}{4}\,A_{\alpha\beta} B^{\mu\nu} \mathcal{C}(p_0^\nu p_0^\mu, q_0^\beta q_0^\alpha)\\ &= \tfrac{1}{4}\,A_{\alpha\beta} B^{\mu\nu} (\delta_{\nu\beta}\delta_{\mu\alpha} + \delta_{\mu\beta}\delta_{\nu\alpha})\\ &= \tfrac{1}{4}\,(A_{\alpha\beta} B^{\alpha\beta} + A_{\alpha\beta} B^{\beta\alpha})\\ &= \tfrac{1}{2}\,A^\alpha{}_\beta B^\beta{}_\alpha\\ &= \tfrac{1}{2}\,\mathrm{str}(AB)\end{aligned}$$

The general case is entirely analogous, but involves more minus signs. ◇

If we write

$$\begin{aligned}\mathfrak{sp}_+ &= \{\text{infinitesimal symplectomorphisms } A = \textstyle\sum_{m>0} A_m z^m,\ A_m \in \mathrm{End}(H^\star(X))\}\\ \mathfrak{sp}_- &= \{\text{infinitesimal symplectomorphisms } A = \textstyle\sum_{m<0} A_m z^m,\ A_m \in \mathrm{End}(H^\star(X))\}\end{aligned}$$

then quadratic Hamiltonians corresponding to operators in $\mathfrak{sp}_+$ contain no $q_a q_b$ terms, and quadratic Hamiltonians corresponding to operators in $\mathfrak{sp}_-$ contain no $p_a p_b$ terms. The cocycle therefore vanishes when restricted to $\mathfrak{sp}_+$ or to $\mathfrak{sp}_-$, and so the quantization procedure gives genuine representations of these subalgebras. As a consequence, given an element S of one of the groups

$$\mathrm{Sp}_+ = \{\text{symplectomorphisms } A = \textstyle\sum_{m\geq 0} A_m z^m,\ A_m \in \mathrm{End}(H^\star(X)),\ A_0 = I\}$$
$$\mathrm{Sp}_- = \{\text{symplectomorphisms } A = \textstyle\sum_{m\leq 0} A_m z^m,\ A_m \in \mathrm{End}(H^\star(X)),\ A_0 = I\}$$

(which have Lie algebras $\mathfrak{sp}_+$, $\mathfrak{sp}_-$ respectively) we can define its quantization $\hat{S}$ to be $\exp \widehat{A}$ where $S = \exp A$. We record here explicit formulae for the quantization of elements of Sp_- and Sp_+:

Proposition 1.3.2 ([30]). *Consider a symplectomorphism of $\mathcal{H}$ of the form*

$$S(z) = I + S_1/z + S_2/z^2 + \ldots \qquad \in \mathrm{End}(H^\star(X))[\![z^{-1}]\!]$$

Define a quadratic form on $\mathcal{H}_+$ by

$$W_S(\mathbf{q}) = \sum_{k,l} (W_{kl} q_k, q_l)$$

where

$$\mathbf{q} = q_0 + q_1 z + \ldots$$

and[1]

$$\sum_{k,l} \frac{W_{kl}}{w^k z^l} = \frac{S^\star(w)S(z) - I}{z + w}$$

Then the quantization of S^{-1} acts on $\mathfrak{Fock}$ by

$$(\hat{S}^{-1}\mathcal{G})(\mathbf{q}) = \exp\left(\frac{W_S(\mathbf{q})}{2\hbar}\right)\mathcal{G}([S\mathbf{q}]_+)$$

where $[S\mathbf{q}]_+$ is the power series truncation of $S(z)\mathbf{q}$.

Proposition 1.3.3 ([30]). *Consider a symplectomorphism of $\mathcal{H}$ of the form*

$$R(z) = I + R_1 z + R_2 z^2 + \ldots \qquad \in \mathrm{End}(H^\star(X))[\![z]\!]$$

Define a quadratic form on $\mathcal{H}_-$ by

$$V_R(\mathbf{p}) = \sum_{k,l} (p_k, V_{kl} p_l)$$

[1] This definition makes sense as $S^\star(-z)S(z) = I$.

where

$$\mathbf{p} = \frac{p_0}{-z} + \frac{p_1}{(-z)^2} + \dots$$

and[2]

$$\sum_{k,l} (-)^{k+l} V_{kl} w^k z^l = \frac{R^\star(w)R(z) - I}{z+w}$$

Then the quantization of R *acts on* $\mathfrak{Fock}$ *by*

$$(\widehat{R}\mathcal{G})(\mathbf{q}) = \left[\exp\left(\frac{\hbar V_R(\partial_{\mathbf{q}})}{2}\right)\mathcal{G}\right](R^{-1}\mathbf{q})$$

where $V_R(\partial_{\mathbf{q}})$ *is the second-order differential operator obtained from* $V_R(\mathbf{p})$ *by replacing* p_k *by differentiation* ∂_k *in the direction of* q_k*.*

1.4 The genus-zero picture

We regard the genus-0 Gromov–Witten potential $\mathcal{F}_X^0$ as a function on $\mathcal{H}_+$ via the dilaton shift. Since the polarization

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$$

identifies $\mathcal{H}$ with the cotangent bundle $T^\star\mathcal{H}_+$, the function $\mathcal{F}_X^0$ defines a Lagrangian submanifold

$$\mathcal{L}_X = \{(\mathbf{p}, \mathbf{q}) : \mathbf{p} = d_{\mathbf{q}}\mathcal{F}_X^0\} \subset \mathcal{H}$$

We will see in the next section that $\mathcal{L}_X$ has some very special properties — it is a homogeneous Lagrangian cone swept out by a finite-dimensional family of isotropic subspaces of $\mathcal{H}$.

Taking the limit $\hbar \to 0$ in the quantization procedure described in the previous section, we find that applying a quantized infinitesimal symplectomorphism $\widehat{A}$ to an element

$$\exp\Big(\sum_{g\geq 0} \hbar^{g-1} f_g(\mathbf{q})\Big) \in \mathfrak{Fock} \tag{1.6}$$

corresponds to changing the Lagrangian submanifold

$$\{(\mathbf{p}, \mathbf{q}) : \mathbf{p} = d_{\mathbf{q}} f_0\} \subset \mathcal{H} \tag{1.7}$$

[2] This definition makes sense as $R^\star(-z)R(z) = I$.

by the Hamiltonian flow of h_A. Exponentiating this statement, we see that applying a quantized symplectic transformation $\exp(\widehat{A})$ to (1.6) corresponds to moving the Lagrangian submanifold (1.7) using the (unquantized) symplectic transformation $\exp(A)$.

We next identify the roles of two objects familiar from genus-zero Gromov–Witten theory — the J-function and the fundamental solution [26, 28, 15] — in our geometric framework. It will be convenient to extend our correlator notation: for polynomials (or power series)

$$\begin{aligned} a_1(\psi) &= a_1^0 + a_1^1\psi + a_1^2\psi^2 + \dots \\ a_2(\psi) &= a_2^0 + a_2^1\psi + a_2^2\psi^2 + \dots \\ &\vdots \\ a_m(\psi) &= a_m^0 + a_m^1\psi + a_m^2\psi^2 + \dots \end{aligned}$$

in $H^\star(X)[\psi]$ (or $H^\star(X)[\![\psi]\!]$) and $\tau \in H^\star(X)$, we set

$$\langle\!\langle a_1, a_2, \dots, a_m \rangle\!\rangle_{g,m}(\tau) = \sum_{n,d} \frac{Q^d}{n!} \langle a_1, a_2, \dots, a_m, \overbrace{\tau, \tau, \dots, \tau}^{n} \rangle_{g,m+n,d}$$

1.4.1 The J-function

The J-function [26, 28] is a formal function of $t \in H^\star(X;\Lambda)$ defined by

$$(J_X(t), a) = (z + t, a) + \left\langle\!\!\left\langle \frac{a}{z-\psi} \right\rangle\!\!\right\rangle_{0,1}(t) \qquad \forall a \in H^\star(X;\Lambda)$$

It takes values in $H^\star(X;\Lambda)((z^{-1}))$. We expand the term $1/(z-\psi)$ which occurs here as a power series in ψ.

Consider the slice

$$\{-z + t + \mathcal{H}_-\} \subset \mathcal{H}$$

which corresponds to setting the descendent variables $t_1, t_2, \dots$ to zero. A point of $\mathcal{L}_X$ which lies on this slice is

$$-z + t + \sum_{n,d} \frac{Q^d}{(n-1)!} \sum_{i,\alpha} \langle \overbrace{t, t, \dots, t}^{n-1}, \phi_\alpha \psi^i \rangle_{0,n,d} \frac{\phi^\alpha}{(-z)^{i+1}}$$

Rewriting this as

$$-z + t + \sum_\alpha \left\langle\!\!\left\langle \frac{\phi_\alpha}{-z-\psi} \right\rangle\!\!\right\rangle_{0,1}(t)\phi^\alpha$$

we see that the point of $\mathcal{L}_X$ above $-z + t \in \mathcal{H}_+$ is $J_X(t, -z)$.

1.4.2 The fundamental solution

The fundamental solution [15, 26]

$$S_\tau(z) = I + S_1/z + S_2/z^2 + \dots \qquad \in \mathrm{End}(H^\star(X))[[z^{-1}]]$$

is defined by

$$(S_\tau(z)u, v) = (u, v) + \left\langle\!\!\left\langle \frac{u}{z-\psi}, v \right\rangle\!\!\right\rangle_{0,2}(\tau) \qquad \forall u, v \in H^\star(X; \Lambda)$$

Note that $S_\tau(z)$ depends on τ. We will see that, for each τ, $S_\tau(z)$ defines a symplectomorphism of $\mathcal{H}$. This will follow from:

Proposition 1.4.1. *Let*

$$S_{\alpha\beta}(z) = (S_\tau(z)\phi_\beta, \phi_\alpha)$$

Then

$$S_{\mu\alpha}(w)g^{\mu\nu}S_{\nu\beta}(z) = (z+w)\left\langle\!\!\left\langle \frac{\phi_\alpha}{w-\psi}, \frac{\phi_\beta}{z-\psi} \right\rangle\!\!\right\rangle_{0,2}(\tau) + g_{\alpha\beta}$$

Proof. The string equation shows that

$$S_{\mu\alpha}(z) = z\left\langle\!\!\left\langle \frac{\phi_\alpha}{z-\psi}, 1, \phi_\mu \right\rangle\!\!\right\rangle_{0,3}(\tau)$$

and so

$$S_{\mu\alpha}(w)g^{\mu\nu}S_{\nu\beta}(z) = zw\left\langle\!\!\left\langle \frac{\phi_\alpha}{w-\psi}, 1, \phi_\mu \right\rangle\!\!\right\rangle_{0,3}(\tau)g^{\mu\nu}\left\langle\!\!\left\langle \phi_\nu, 1, \frac{\phi_\beta}{z-\psi} \right\rangle\!\!\right\rangle_{0,3}(\tau)$$

The argument which proves the WDVV identity (see *e.g.* [54]) also shows that this quantity is equal to

$$zw\left\langle\!\!\left\langle \frac{\phi_\alpha}{w-\psi}, \frac{\phi_\beta}{z-\psi}, \phi_\mu \right\rangle\!\!\right\rangle_{0,3}(\tau)g^{\mu\nu}\langle\!\langle \phi_\nu, 1, 1\rangle\!\rangle_{0,3}(\tau)$$

Schematically:

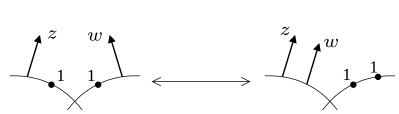


Figure 1.1: A cryptic picture

Applying the string equation again, we see that

$$\langle\langle \phi_\nu, 1, 1 \rangle\rangle_{0,3}(\tau) = \langle \phi_\nu, 1, 1 \rangle_{0,3,0}$$

and so

$$S_{\mu\alpha}(w) g^{\mu\nu} S_{\nu\beta}(z) = zw \left\langle\!\!\left\langle \frac{\phi_\alpha}{w-\psi}, \frac{\phi_\beta}{z-\psi}, 1 \right\rangle\!\!\right\rangle_{0,3}(\tau)$$

A final application of the string equation yields

$$\begin{aligned} S_{\mu\alpha}(w) g^{\mu\nu} S_{\nu\beta}(z) &= zw \left(\left(\frac{1}{z} + \frac{1}{w} \right) \left\langle\!\!\left\langle \frac{\phi_\alpha}{w-\psi}, \frac{\phi_\beta}{z-\psi} \right\rangle\!\!\right\rangle_{0,3}(\tau) + \left\langle \frac{\phi_\alpha}{w-\psi}, \frac{\phi_\beta}{z-\psi}, 1 \right\rangle_{0,3,0} \right) \\ &= (z+w) \left\langle\!\!\left\langle \frac{\phi_\alpha}{w-\psi}, \frac{\phi_\beta}{z-\psi} \right\rangle\!\!\right\rangle_{0,2}(\tau) + g_{\alpha\beta} \end{aligned}$$

□

Corollary 1.4.2. *For each τ, the operator $S_\tau(z)$ is a symplectomorphism of $\mathcal{H}$.*

Proof. In view of our choices in section 1.3.2, $S_\tau(z)$ defines a linear transformation from $\mathcal{H}$ to itself. Putting $w = -z$ in Proposition 1.4.1 gives $S^\star(-z)S(z) = I$. □

1.5 Ancestors and descendents

There is a map

$$\mathrm{ct}_{m+n,m} : X_{g,m+n,d} \to \overline{\mathcal{M}}_{g,m}$$

defined by forgetting the map and the last n marked points and then contracting any unstable components of the resulting marked curve. Denote by $\bar{L}_{m,i}$ the pullback of the ith universal cotangent line over $\overline{\mathcal{M}}_{g,m}$ via $\mathrm{ct}_{m+n,m}$, and let $\bar{\psi}_{m,i} \in H^\star(X_{g,m+n,d})$ be the first Chern class of $\bar{L}_{m,i}$. We further extend our correlator notation as follows: given polynomials (or power series)

$$\begin{aligned} a_1(\psi, \bar{\psi}) &= \sum_{i,j} a_1^{ij} \psi^i \bar{\psi}^j \\ a_2(\psi, \bar{\psi}) &= \sum_{i,j} a_2^{ij} \psi^i \bar{\psi}^j \\ &\vdots \\ a_m(\psi, \bar{\psi}) &= \sum_{i,j} a_m^{ij} \psi^i \bar{\psi}^j \end{aligned}$$

in $H^\star(X)[\psi,\bar\psi]$ (or $H^\star(X)[[\psi,\bar\psi]]$), together with cohomology classes

$$\{b_{m,n,d} \in H^\star(X_{g,m+n,d}) : n \in \mathbb{N}, d \in H_2(X;\mathbb{Z})\}$$

and $\tau \in H^\star(X)$, we set

$$\langle\langle a_1, a_2, \ldots, a_m; \{b_{m,n,d}\}\rangle\rangle_{g,m}(\tau) = \sum_{n,d} \frac{Q^d}{n!} \int_{[X_{g,m+n,d}]} e_{m,1} \wedge e_{m,2} \wedge \ldots \wedge e_{m,m} \wedge \Big(\bigwedge_{i=m+1}^{m+n} \mathrm{ev}_i^\star\,\tau\Big) \wedge b_{m,n,d}$$

where

$$e_{m,k} = \sum_{i,j} (\mathrm{ev}_k^\star\, a_k^{ij}) \psi_k^i \bar\psi^j_{m,k} \qquad\qquad k = 1, 2, \ldots, m$$

Write

$$\langle\langle a_1, a_2, \ldots, a_m\rangle\rangle_{g,m}(\tau) = \langle\langle a_1, a_2, \ldots, a_m; \{1\}\rangle\rangle_{g,m}(\tau)$$

The genus-g ancestor potential [30] is

$$\bar{\mathcal{F}}^g_\tau = \sum_m \frac{1}{m!} \langle\langle \bar{\mathbf{t}}(\bar\psi), \ldots, \bar{\mathbf{t}}(\bar\psi)\rangle\rangle_{g,m}(\tau)$$

where the sum is over m such that $\overline{\mathcal{M}}_{g,m}$ is non-empty. It is a formal function of $\tau \in H^\star(X;\Lambda)$ and $\bar{\mathbf{t}}(z) = \bar t_0 + \bar t_1 z + \ldots \in H^\star(X;\Lambda)[z]$ which takes values in Λ. The total ancestor potential is

$$\mathcal{A}_\tau = \exp\Big(\sum_{g \geq 0} \hbar^{g-1} \bar{\mathcal{F}}^g_\tau\Big)$$

We regard this as a formal function of $\bar{\mathbf{t}}$ depending on the (formal) parameter τ. It is identified with an element of the Fock space via the dilaton shift

$$\mathbf{q}(z) = \bar{\mathbf{t}}(z) - z$$

The argument of Lemma 1.3.1 shows that $\mathcal{A}_\tau$ is an element of $\mathfrak{Fock}$ (which depends on τ). In other words, $\mathcal{A}_\tau$ is well-defined as a formal function of $\bar{\mathbf{t}}$ (and τ) with values in $\Lambda[[\hbar, \hbar^{-1}]]$.

The following Theorem, due to Givental, describes the connection between the total descendent potential $\mathcal{D}_X$ and the ancestor potentials $\mathcal{A}_\tau$. It is essentially a reinterpretation in our framework of a result of Kontsevich and Manin [39]. A similar result was obtained independently, in a different context, by Getzler [22, 23].

Theorem 1.5.1 ([30, 12]). *Let*

$$F^1(\tau) = \mathcal{F}^1_X(\mathbf{t})|_{t_0=\tau, t_1=t_2=\ldots=0}$$

*denote the genus-*1 *non-descendent Gromov–Witten potential of* X*. Then*

$$\mathcal{D}_X = e^{F^1(\tau)} \hat{S}_\tau^{-1} \mathcal{A}_\tau$$

Proof. Proposition A.0.1 in Appendix A shows that the right-hand side is well-defined as a formal function of $\mathbf{t}$ and τ near $\mathbf{t} = 0$, $\tau = 0$. We will see below that it in fact does not depend on τ.

Suppose that g and m are such that $\overline{\mathcal{M}}_{g,m}$ is non-empty. The bundles L_1 and $\bar{L}_{m,1}$ over $X_{g,m+n,d}$ are identified outside the locus D consisting of maps such that the first marked point is situated on a component of the curve which gets collapsed by $\mathrm{ct}_{m+n,m}$.

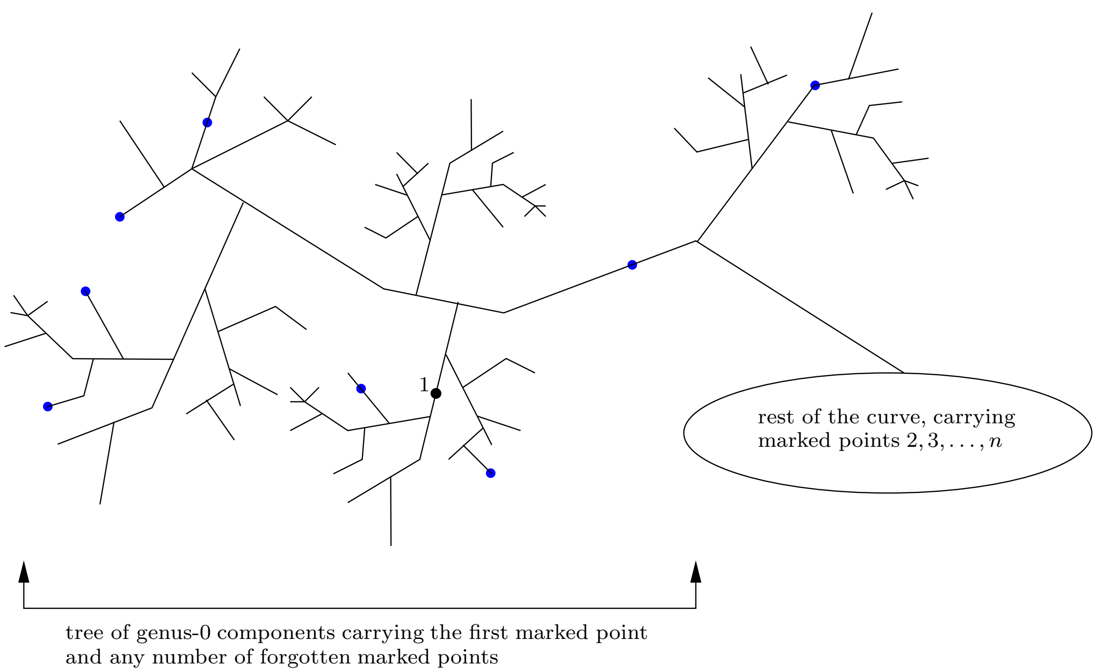


Figure 1.2: The locus where L_1 and $\bar{L}_{m,1}$ differ

This locus D is the image of the gluing map

$$i : \coprod_{\substack{n'+n''=n \\ d'+d''=d}} X_{0,1+\bullet+n',d'} \times_X X_{g,m-1+\circ+n'',d''} \to X_{g,m+n,d}$$

We denote the domain of this map by $Y_{m,n,d}$. The virtual normal bundle to D at a generic point is $\mathrm{Hom}(\bar{L}_{m,1}, L)$, and so D is "virtually Poincaré-dual" to $\psi_1 - \bar{\psi}_{m,1}$ in the sense that

$$[X_{g,m+n,d}] \cap (\psi_1 - \bar{\psi}_{m,1}) = i_\star[Y_{m,n,d}]$$

We will concentrate on the first marked point, so suppress the content of the other marked points from our notation. For any $\theta \in H^\star(X)$, we have

$$\begin{aligned}\langle\!\langle \theta\psi^a\bar{\psi}^b, \ldots\rangle\!\rangle_{g,m}(\tau) &= \langle\!\langle \theta\psi^{a-1}\bar{\psi}^b(\psi - \bar{\psi} + \bar{\psi}), \ldots\rangle\!\rangle_{g,m}(\tau) \\ &= \langle\!\langle \theta\psi^{a-1}\bar{\psi}^{b+1}, \ldots\rangle\!\rangle_{g,m}(\tau) \\ &\quad + \langle\!\langle \theta\psi^{a-1}\bar{\psi}^b, \ldots; \{i_![Y_{m,n,d}]\}\rangle\!\rangle_{g,m}(\tau)\end{aligned}$$

and so

$$\begin{aligned}\langle\!\langle \theta\psi^a\bar{\psi}^b, \ldots\rangle\!\rangle_{g,m}(\tau) =& \langle\!\langle \theta\psi^{a-1}\bar{\psi}^{b+1}, \ldots\rangle\!\rangle_{g,m}(\tau) \\ &+ \langle\!\langle \theta\psi^{a-1}, \phi_\mu\rangle\!\rangle_{0,2}(\tau) g^{\mu\nu} \langle\!\langle \phi_\nu\bar{\psi}^b, \ldots\rangle\!\rangle_{g,m}(\tau)\end{aligned} \tag{1.8}$$

Thus

$$\begin{aligned}\langle\!\langle t_0 + t_1\psi + t_2\psi^2 + \ldots, \ldots\rangle\!\rangle_{g,m}(\tau) = \langle\!\langle t_0, \ldots\rangle\!\rangle_{g,m}(\tau) \\ + \langle\!\langle t_1\bar{\psi}, \ldots\rangle\!\rangle_{g,m}(\tau) + \langle\!\langle t_1, \phi_\mu\rangle\!\rangle_{0,2}(\tau) g^{\mu\nu} \langle\!\langle \phi_\nu, \ldots\rangle\!\rangle_{g,m}(\tau) \\ + \langle\!\langle t_2\psi\bar{\psi}, \ldots\rangle\!\rangle_{g,m}(\tau) + \langle\!\langle t_2\psi, \phi_\mu\rangle\!\rangle_{0,2}(\tau) g^{\mu\nu} \langle\!\langle \phi_\nu, \ldots\rangle\!\rangle_{g,m}(\tau) \\ + \ldots\end{aligned}$$

which, applying (1.8) repeatedly, is

$$\begin{aligned}&\langle\!\langle (t_0 + \langle\!\langle t_1, \phi_\mu\rangle\!\rangle_{0,2}(\tau) g^{\mu\nu}\phi_\nu + \langle\!\langle t_2\psi, \phi_\mu\rangle\!\rangle_{0,2}(\tau) g^{\mu\nu}\phi_\nu + \ldots), \ldots\rangle\!\rangle_{g,m}(\tau) \\ &+ \langle\!\langle (t_1 + \langle\!\langle t_2, \phi_\mu\rangle\!\rangle_{0,2}(\tau) g^{\mu\nu}\phi_\nu + \langle\!\langle t_3\psi, \phi_\mu\rangle\!\rangle_{0,2}(\tau) g^{\mu\nu}\phi_\nu + \ldots)\bar{\psi}, \ldots\rangle\!\rangle_{g,m}(\tau) \\ &+ \langle\!\langle (t_2 + \langle\!\langle t_3, \phi_\mu\rangle\!\rangle_{0,2}(\tau) g^{\mu\nu}\phi_\nu + \langle\!\langle t_4\psi, \phi_\mu\rangle\!\rangle_{0,2}(\tau) g^{\mu\nu}\phi_\nu + \ldots)\bar{\psi}^2, \ldots\rangle\!\rangle_{g,m}(\tau) \\ &+ \ldots\end{aligned}$$

But

$$(t_i + \langle\!\langle t_{i+1}, \phi_\mu\rangle\!\rangle_{0,2}(\tau) g^{\mu\nu}\phi_\nu + \langle\!\langle t_{i+2}\psi, \phi_\mu\rangle\!\rangle_{0,2}(\tau) g^{\mu\nu}\phi_\nu + \ldots)$$

is the coefficient of z^i in $S_\tau\mathbf{t}$, and so

$$\langle\!\langle \mathbf{t}(\psi), \ldots\rangle\!\rangle_{g,m}(\tau) = \langle\!\langle \bar{\mathbf{t}}(\bar{\psi}), \ldots\rangle\!\rangle_{g,m}(\tau)$$

where

$$\bar{\mathbf{t}} = [S_\tau\mathbf{t}]_+$$

Applying the same argument at each marked point, we find that

$$\langle\!\langle \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi) \rangle\!\rangle_{g,m}(\tau) = \langle\!\langle \bar{\mathbf{t}}(\bar{\psi}), \ldots, \bar{\mathbf{t}}(\bar{\psi}) \rangle\!\rangle_{g,m}(\tau)$$

where

$$\bar{\mathbf{t}} = [S_\tau \mathbf{t}]_+$$

If instead, however, we set

$$\bar{\mathbf{q}} = [S_\tau \mathbf{q}]_+$$

then this sets

$$\bar{\mathbf{t}} = [S_\tau \mathbf{t} - S_\tau z]_+ + z$$

We know that, for any v

$$\begin{aligned}
([S_\tau z]_+, v) &= [z(S_\tau 1, v)]_+ \\
&= \left[z(1, v) + z \left\langle\!\!\left\langle \frac{1}{z - \psi}, v \right\rangle\!\!\right\rangle_{0,2}(\tau) \right]_+ \\
&= (z, v) + \sum_{n,d} \frac{Q^d}{n!} \langle 1, v, \tau, \ldots, \tau \rangle_{0,n+2,d} \\
&= (z, v) + \langle 1, v, \tau \rangle_{0,3,0} \qquad \text{(string equation!)} \\
&= (z + \tau, v)
\end{aligned}$$

and so

$$[S_\tau z]_+ = z + \tau$$

Setting

$$\bar{\mathbf{q}} = [S_\tau \mathbf{q}]_+$$

therefore sets

$$\bar{\mathbf{t}} = [S_\tau \mathbf{t}]_+ - \tau$$

But

$$\mathcal{F}_X^g = \sum_{g \geq 0} \frac{1}{m!} \langle\!\langle \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi) \rangle\!\rangle_{g,m}(0)$$

which by Taylor's theorem is equal to

$$\sum_{g \geq 0} \frac{1}{m!} \langle\!\langle \mathbf{t}(\psi) - \tau, \ldots, \mathbf{t}(\psi) - \tau \rangle\!\rangle_{g,m}(\tau)$$

For $g > 1$, therefore, we have shown that

$$\mathcal{F}^g_X(\mathbf{q}) = \overline{\mathcal{F}}^g_\tau(\bar{\mathbf{q}}) \qquad \text{where} \qquad \bar{\mathbf{q}} = [S_\tau \mathbf{q}]_+$$

Note that $\overline{\mathcal{F}}^g_\tau$ depends on τ here, but $\mathcal{F}^g_X$ does not. For $g = 0$ and $g = 1$ the same argument applies but we need also to take care of the discrepancy arising from the "missing" moduli spaces $\overline{\mathcal{M}}_{0,0}$, $\overline{\mathcal{M}}_{0,1}$, $\overline{\mathcal{M}}_{0,2}$, and $\overline{\mathcal{M}}_{1,0}$. Thus:

$$\mathcal{D}_X(\mathbf{q}) = \exp\Big(\frac{1}{\hbar}\langle\!\langle\,\rangle\!\rangle_{0,0}(\tau) + \frac{1}{\hbar}\langle\!\langle \mathbf{t}(\psi) - \tau\rangle\!\rangle_{0,1}(\tau) + \frac{1}{2\hbar}\langle\!\langle \mathbf{t}(\psi) - \tau, \mathbf{t}(\psi) - \tau\rangle\!\rangle_{0,2}(\tau)\Big)$$
$$\exp(\langle\!\langle\,\rangle\!\rangle_{1,0}(\tau)) \exp\Big(\sum_{g\geq 0} \overline{\mathcal{F}}^g_\tau(\bar{\mathbf{q}})\Big)$$

The contribution from $\overline{\mathcal{M}}_{1,0}$ is

$$\langle\!\langle\,\rangle\!\rangle_{1,0}(\tau) = F^1(\tau)$$

The contribution from the missing genus-zero moduli spaces is

$$\langle\!\langle\,\rangle\!\rangle_{0,0}(\tau) + \langle\!\langle \mathbf{t}(\psi) - \tau\rangle\!\rangle_{0,1}(\tau) + \langle\!\langle \mathbf{t}(\psi) - \tau, \mathbf{t}(\psi) - \tau\rangle\!\rangle_{0,2}(\tau)$$

or in other words

$$\begin{aligned}
&\sum_{n,d} \frac{Q^d}{n!}\langle \tau, \ldots, \tau\rangle_{0,n,d} + \sum_{n,d} \frac{Q^d}{n!}\langle \mathbf{t}, \tau, \ldots, \tau\rangle_{0,n+1,d} \\
&\qquad - \sum_{n,d} \frac{Q^d}{n!}\langle \tau, \ldots, \tau\rangle_{0,n+1,d} + \frac{1}{2}\sum_{n,d} \frac{Q^d}{n!}\langle \mathbf{t}, \mathbf{t}, \tau, \ldots, \tau\rangle_{0,n+2,d} \\
&\qquad - \sum_{n,d} \frac{Q^d}{n!}\langle \mathbf{t}, \tau, \ldots, \tau\rangle_{0,n+2,d} + \frac{1}{2}\sum_{n,d} \frac{Q^d}{n!}\langle \tau, \ldots, \tau\rangle_{0,n+2,d} \\
&= \frac{1}{2}\langle\!\langle \mathbf{t}, \mathbf{t}\rangle\!\rangle_{0,2}(\tau) + \sum_{n,d} \frac{Q^d}{n!}(1-n)\langle \mathbf{t}, \tau, \ldots, \tau\rangle_{0,n+1,d} \\
&\qquad + \frac{1}{2}\sum_{n,d} \frac{Q^d}{n!}(n-1)(n-2)\langle \tau, \ldots, \tau\rangle_{0,n,d} \\
&= \frac{1}{2}\langle\!\langle \mathbf{t}, \mathbf{t}\rangle\!\rangle_{0,2}(\tau) - \langle\!\langle \mathbf{t}, \psi\rangle\!\rangle_{0,1}(\tau) + \frac{1}{2}\langle\!\langle \psi, \psi\rangle\!\rangle_{0,2}(\tau) \qquad \text{(dilaton equation)} \\
&= \frac{1}{2}\langle\!\langle \mathbf{t} - \psi, \mathbf{t} - \psi\rangle\!\rangle_{0,2}(\tau) \\
&= \frac{1}{2}\langle\!\langle \mathbf{q}, \mathbf{q}\rangle\!\rangle_{0,2}(\tau)
\end{aligned}$$

So

$$\mathcal{D}_X(\mathbf{q}) = e^{F^1(\tau)} e^{(1/2\hbar)\langle\langle \mathbf{q},\mathbf{q}\rangle\rangle_{0,2}(\tau)} \mathcal{A}_\tau([S\mathbf{q}]_+)$$

Applying Propositions 1.3.2 and 1.4.1, we are done. □

Corollary 1.5.2. *If $\bar{\mathcal{L}}_\tau$ is the Lagrangian submanifold*

$$\bar{\mathcal{L}}_\tau = \{(\mathbf{p}, \mathbf{q}) : \mathbf{p} = d_{\mathbf{q}} \overline{\mathcal{F}}_\tau^0\} \subset \mathcal{H}$$

then

$$\bar{\mathcal{L}}_\tau = S_\tau \mathcal{L}_X$$

Recall that we denote the dimension of $H^\star(X)$ by N. We are now in a position to prove

Theorem 1.5.3. *$\mathcal{L}_X$ is a homogeneous Lagrangian cone swept out by an N-dimensional family of isotropic subspaces. More precisely, for each $\mathbf{f} \in \mathcal{L}_X$ the tangent space $L_{\mathbf{f}} = T_{\mathbf{f}}\mathcal{L}_X \subset \mathcal{H}$ satisfies*

$$\mathcal{L}_X \cap L_{\mathbf{f}} = zL_{\mathbf{f}}$$

Proof. That $\mathcal{L}_X$ is a cone follows immediately from the divisor equation. We will deduce the rest of the Theorem from the corresponding statement about $\bar{\mathcal{L}}_\tau$.

We first show that we can choose τ such that $[S_\tau \mathbf{f}]_+ \in z\mathcal{H}_+$. Write $\mathbf{f} = (\mathbf{p}, \mathbf{q})$. We need to set the coefficient of z^0 in

$$(S_\tau \mathbf{q}, v)$$

equal to zero for all $v \in H^\star(X)$. By the string equation,

$$(S_\tau \mathbf{q}, v) = z\left\langle\!\!\left\langle 1, \frac{\mathbf{q}(z)}{z-\psi}, v \right\rangle\!\!\right\rangle_{0,3}(\tau)$$

and so we need to solve

$$\langle\langle 1, \mathbf{q}(\psi), v\rangle\rangle_{0,3}(\tau) = 0 \qquad \text{for all } v \in H^\star(X)$$

Thus we need τ to be a critical point of the function

$$\tau \mapsto \langle\langle 1, \mathbf{q}(\psi)\rangle\rangle_{0,2}(\tau)$$

(which depends on the parameter $\mathbf{q} \in \mathcal{H}_+$). Since when $\mathbf{q} = -z$ this function has a non-degenerate critical point at $\tau = 0$, there is a unique critical point $\tau(\mathbf{q})$ for all $\mathbf{q}$ in a formal neighbourhood of $\mathbf{q} = -z$. Choosing $\tau = \tau(\mathbf{q})$ gives $[S_\tau \mathbf{f}]_+ \in z\mathcal{H}_+$.

For any $\bar{\mathbf{q}} \in z\mathcal{H}_+$ — in other words, for any $\bar{\mathbf{q}}$ such that $\bar{q}_0 = 0$ — the ancestor potential $\bar{\mathcal{F}}^0_\tau$ has zero 2-jet at $\bar{\mathbf{q}}$. This follows from the fact that the dimension of $\overline{\mathcal{M}}_{0,m}$ is $m - 3$. Thus

$$(\bar{\mathbf{q}}, 0) \in \bar{\mathcal{L}}_\tau$$

and

$$T_{(\bar{\mathbf{q}},0)} \bar{\mathcal{L}}_\tau = \mathcal{H}_+$$

In particular,

$$T_{(\bar{\mathbf{q}},0)} \bar{\mathcal{L}}_\tau \cap \bar{\mathcal{L}}_\tau \supseteq z\mathcal{H}_+$$

Since the component of $d\bar{\mathcal{F}}^0_\tau$ in the p_0^0-direction (*i.e.* the $(-g^{0\alpha}\phi_\alpha/z)$-component) is

$$g_{\mu\nu} q_0^\mu q_0^\nu + \text{higher-order terms}$$

we see that

$$\mathcal{H}_+ \cap \bar{\mathcal{L}}_\tau \subseteq z\mathcal{H}_+$$

and so

$$\begin{aligned} T_{(\bar{\mathbf{q}},0)} \bar{\mathcal{L}}_\tau \cap \bar{\mathcal{L}}_\tau &= z\mathcal{H}_+ \\ &= zT_{(\bar{\mathbf{q}},0)} \bar{\mathcal{L}}_\tau \end{aligned}$$

Applying Corollary 1.5.2 completes the proof. □

In particular this implies that the tangent spaces $L_\mathbf{f}$ to $\mathcal{L}_X$ are Lagrangian subspaces closed under multiplication by z. They consequently belong to the Grassmannian corresponding to the twisted loop group $A^{(2)}$. For more on this point of view, see [25].

1.5.1 The J-function and the fundamental solution again

Since the codimension of $zL_\mathbf{f}$ in $L_\mathbf{f}$ is N, Theorem 1.5.3 shows that given a generic N-dimensional slice of $\mathcal{L}_X$:

$$\{J(t) : t \in H^\star(X)\} \subset \mathcal{L}_X$$

the cone is swept out by

$$\{zL_{J(t)} : t \in H^\star(X)\}$$

To see that the J-function $J_X(t,-z)$ gives such a slice, we need to check that the image of $t \mapsto J_X(t,-z)$ is transverse to the ruling by $zL_{J_X(t,-z)}$. The tangent space $L_{J_X(t,-z)}$ is spanned by the vectors

$$\begin{aligned} v_{\alpha,i} &= \phi_\alpha z^i + \sum_j \partial_{\alpha,i}\partial_{b,j}\mathcal{F}^0_X|_{\mathbf{q}=t-z}\frac{\phi^\beta}{(-z)^{j+1}} \qquad\qquad 1 \le \alpha \le N,\, i \in \mathbb{N} \\ &= \phi_\alpha z^i + O(1/z) \end{aligned}$$

and so the ruling is spanned by

$$zv_{\alpha,i} = \phi_\alpha z^{i+1} + O(1) \qquad\qquad 1 \le \alpha \le N,\, i \in \mathbb{N}$$

Since

$$\frac{\partial}{\partial t^\beta} J_X(t,-z) = \phi_\beta + O(1/z)$$

the family $t \mapsto J_X(t,-z)$ is indeed transverse to the ruling.

Also

$$\begin{aligned} S_{\alpha\beta}(-z) &= \partial_{\alpha,0}(J_X(t,-z))_\beta \\ &= g_{\alpha\beta} + O(1/z) \end{aligned}$$

and so the same argument shows that the columns of the matrix $S^\alpha_\beta(-z)$ form a basis for $L_{J_X(t,-z)}/zL_{J_X(t,-z)}$.

1.6 Twisted Gromov–Witten invariants

1.6.1 Twisted Gromov–Witten invariants

Consider the universal family over $X_{g,n,d}$

$$\begin{array}{ccc} X_{g,n+1,d} & \xrightarrow{\mathrm{ev}_{n+1}} & X \\ \Big\downarrow{\scriptstyle \pi} & & \\ X_{g,n,d} & & \end{array}$$

Given a holomorphic vector bundle E over X, set

$$E_{g,n,d} = \pi_\star \operatorname{ev}_{n+1}^\star E \in K^0(X_{g,n,d})$$

Given also an invertible multiplicative characteristic class of complex vector bundles $\mathbf{c}(\cdot)$, define the $(\mathbf{c}, E)$-twisted genus-g Gromov–Witten potential to be

$$\mathcal{F}^g_{\mathbf{c},E} = \sum_{n,d} \frac{Q^d}{n!} \langle \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi); \mathbf{c}(E_{g,n,d}) \rangle_{g,n,d}$$

This is a formal function of $\mathbf{t}(z) = t_0 + t_1 z + \ldots \in H^\star(X;\Lambda)[z]$ which takes values in Λ. The Taylor coefficients of $\mathcal{F}^g_{\mathbf{c},E}$ at $\mathbf{t} = 0$ are called $(\mathbf{c}, E)$-twisted Gromov–Witten invariants. The $(\mathbf{c}, E)$-twisted total descendent potential of X is defined to be

$$\mathcal{D}_{\mathbf{c},E} = \exp\Big(\sum_g \hbar^{g-1} \mathcal{F}^g_{\mathbf{c},E}\Big)$$

The argument of Lemma 1.3.1 shows that this is well-defined as a formal function of $\mathbf{t}$ taking values in $\Lambda[[\hbar, \hbar^{-1}]]$. We identify it with a formal function of $\mathbf{q}$ (near $\mathbf{q} = -\sqrt{\mathbf{c}(E)}\, z$) via the twisted dilaton shift[3]

$$\mathbf{q}(z) = \sqrt{\mathbf{c}(E)}\,(\mathbf{t}(z) - z)$$

Since the intersection pairing arises in Gromov–Witten theory via intersection indices on $X_{0,3,0} \cong X$, when working with the $(\mathbf{c}, E)$-twisted theory we use the twisted intersection pairing

$$(\theta_1, \theta_2)_{\mathbf{c},E} = \int_X \theta_1 \wedge \theta_2 \wedge \mathbf{c}(E)$$

The symplectic space associated with the $(\mathbf{c}, E)$-twisted theory is $(\mathcal{H}_{\mathbf{c},E}, \Omega_{\mathbf{c},E})$, where

$$\mathcal{H}_{\mathbf{c},E} = \mathcal{H}$$

and

$$\Omega_{\mathbf{c},E}(f, g) = \frac{1}{2\pi i} \oint (f(-z), g(z))_{\mathbf{c},E}\, dz$$

The map

$$\begin{aligned} (H^\star(X), (\,\cdot\,,\,\cdot\,)_{\mathbf{c},E}) &\to (H^\star(X), (\,\cdot\,,\,\cdot\,)) \\ x &\mapsto \sqrt{\mathbf{c}(E)}\, x \end{aligned}$$

[3]Note that in the equivariant situation considered below this requires that we further extend the ground ring Λ by $\sqrt{\lambda}$.

identifies the twisted and untwisted intersection pairings, so the map

$$\begin{aligned}(\mathcal{H}_{\mathbf{c},E}, \Omega_{\mathbf{c},E}) &\to (\mathcal{H}, \Omega) \\ \mathbf{x} &\mapsto \sqrt{\mathbf{c}(E)}\,\mathbf{x}\end{aligned}$$

identifies the symplectic spaces $(\mathcal{H}_{\mathbf{c},E}, \Omega_{\mathbf{c},E})$ and $(\mathcal{H}, \Omega)$.

Notation

Any invertible multiplicative characteristic class $\mathbf{c}(\cdot)$ takes the form

$$\mathbf{c}(\cdot) = \exp\Big(\sum_{k\geq 0} s_k \operatorname{ch}_k(\cdot)\Big)$$

We write $\mathbf{s} = (s_0, s_1, s_2, \ldots)$ throughout. We will often suppress the notation for the bundle E and write $\mathbf{s}$ instead of $\mathbf{c}$, for example writing

$$(\mathcal{H}_{\mathbf{s}}, \Omega_{\mathbf{s}}) \qquad \text{instead of} \qquad (\mathcal{H}_{\mathbf{c},E}, \Omega_{\mathbf{c},E})$$

and

$$\mathcal{F}^g_{\mathbf{s}} \qquad \text{instead of} \qquad \mathcal{F}^g_{\mathbf{c},E}$$

Lagrangian cones

The twisted genus-0 potential, regarded as a function on $\mathcal{H}_+$ via the twisted dilaton shift, determines a Lagrangian section

$$\mathcal{L}_{\mathbf{s}} = \{(\mathbf{p}, \mathbf{q}) : \mathbf{p} = d_{\mathbf{q}}\mathcal{F}^0_{\mathbf{s}}\} \subset \mathcal{H}$$

One can think of this as arising in two stages. First, regard $\mathcal{F}^0_{\mathbf{s}}$ as a function on $(\mathcal{H}_{\mathbf{s}})_+ \subset (\mathcal{H}_{\mathbf{s}}, \Omega_{\mathbf{s}})$ via the untwisted dilaton shift. The graph of its differential gives a Lagrangian submanifold $\mathcal{L}^{\text{nat}}_{\mathbf{s}} \subset (\mathcal{H}_{\mathbf{s}}, \Omega_{\mathbf{s}})$ which is identified with the submanifold $\mathcal{L}_{\mathbf{s}} \subset (\mathcal{H}, \Omega)$ via the map

$$\begin{aligned}(\mathcal{H}_{\mathbf{c},E}, \Omega_{\mathbf{c},E}) &\to (\mathcal{H}, \Omega) \\ \mathbf{x} &\mapsto \sqrt{\mathbf{c}(E)}\,\mathbf{x}\end{aligned}$$

The twisted J-function $J_{\mathbf{c},E}$ is defined to be the section of $\mathcal{L}^{\text{nat}}_{\mathbf{c},E}$ over the slice

$$\{-z + t + \mathcal{H}_- : t \in H^\star(X)\}$$

In other words

$$(J_{\mathbf{c},E}(t,z),a)_{\mathbf{c},E} = (z+t,a)_{\mathbf{c},E} + \sum_{n,d} \frac{Q^d}{n!} \left\langle \overbrace{t,\ldots,t}^{n}, \frac{a}{z-\psi}; \mathbf{c}(E_{0,n+1,d}) \right\rangle_{0,n+1,d}$$

If we write $\tilde{g}_{\alpha\beta} = (\phi_\alpha, \phi_\beta)_{\mathbf{c},E}$ and $\tilde{g}^{\alpha\beta}$ for the entries of the matrix inverse to that with entries $\tilde{g}_{\alpha\beta}$ then

$$J_{\mathbf{c},E}(t,z) = z + t + \left\langle\!\!\left\langle \frac{\phi_\alpha}{z-\psi}; \mathbf{c}(E_{0,n+1,d}) \right\rangle\!\!\right\rangle(t)_{0,1} \tilde{g}^{\alpha\beta}\phi_\beta$$

S^1-equivariant Gromov–Witten invariants

In applications below — in particular, in the proofs of the Quantum Lefschetz Hyperplane Principle and Quantum Serre Duality — we will need to take $\mathbf{c}$ equal to the Euler class. This currently falls outside the domain of our construction: the Euler class is multiplicative, but it is not invertible. We get around this problem by turning on the natural S^1-action on all vector bundles: if F is a vector bundle over a space Y then the Euler class of F is not invertible, but the S^1-equivariant Euler class of F, where Y carries the trivial action and F carries the action which rotates fibers, is invertible over $\mathbb{C}(\lambda)$. We therefore often work with S^1-equivariant Gromov–Witten invariants [26], where X and the moduli spaces $X_{g,n,d}$ carry the trivial S^1-action and E and the sheaves $E_{g,n,d}$ carry the S^1-action which rotates fibers, and regard all characteristic classes as S^1-equivariant. This entails extending our ground ring Λ (see section 1.3.2), but otherwise all constructions from previous sections go through word-for-word.

1.6.2 Various preparatory lemmas

In section 1.6.3 below, we will apply the Grothendieck–Riemann–Roch theorem to the universal family $\pi : X_{g,n+1,d} \to X_{g,n,d}$ to determine the relationship between twisted and untwisted Gromov–Witten invariants. We collect here various lemmas of a geometrical character which will be needed in that computation. The first concerns the behaviour of $E_{g,n,d}$ on a certain stratum consisting of nodal curves.

Define the singular locus $\mathcal{Z}$ in the universal family $X_{g,n+1,d}$ to be the locus of nodes of the fibers of π. This has virtual codimension 2 in the universal family. It coincides with the range of the gluing map

$$\widetilde{\mathcal{Z}}_{red} \coprod \widetilde{\mathcal{Z}}_{irr} \xrightarrow{\gamma_{red} \coprod \gamma_{irr}} \mathcal{Z} \xrightarrow{i} X_{g,n+1,d} \tag{1.9}$$

where

$$\widetilde{\mathcal{Z}}_{red} = \coprod_{\substack{g=g_+ + g_- \\ n=n_+ + n_- \\ d=d_+ + d_-}} X_{g_+,n_+ + \bullet,d_+} \times_X X_{0,1+\bullet+\circ,0} \times_X X_{g_-,n_- + \circ,d_-}$$

and

$$\widetilde{\mathcal{Z}}_{irr} = X_{g-1,n+\bullet+\circ} \times_{X\times X} X_{0,1+\bullet+\circ,0}$$

The virtual fundamental class behaves well on this locus, in the sense that the restriction of the virtual fundamental class of $X_{g,n+1,d}$ to $\mathcal{Z}$ coincides with the pushforward of the virtual fundamental class of the domain of (1.9) via the gluing map.

Lemma 1.6.1. *Denote by p_+ and p_- be the projections onto the first and third factors of $\widetilde{\mathcal{Z}}_{irr}$. We have*

$$\gamma_{red}^\star i^\star E_{g,n+1,d} = p_+^\star E_{g_+,n_+ + \bullet,d_+} + p_-^\star E_{g_-,n_- + \circ,d_-} - \mathrm{ev}_\Delta^\star E \tag{1.10}$$

and

$$\gamma_{irr}^\star i^\star E_{g,n+1,d} = E_{g-1,n+\bullet+\circ,d} - \mathrm{ev}_\Delta^\star E \tag{1.11}$$

where ev_Δ is the evaluation map at the point of gluing.

Proof. Since $\pi : X_{g,n+1,d} \to X_{g,n,d}$ is a local complete intersection morphism [58], there is a complex

$$0 \longrightarrow E^0_{g,n,d} \longrightarrow E^1_{g,n,d} \longrightarrow 0$$

of vector bundles on $X_{g,n,d}$ with cohomology sheaves equal to

$$R^0\pi_\star \mathrm{ev}_{n+1}^\star(E) \qquad \text{and} \qquad R^1\pi_\star \mathrm{ev}_{n+1}^\star(E)$$

$E_{g,n,d}$ is defined to be the difference $[E^0_{g,n,d}] - [E^1_{g,n,d}]$; this does not depend on the choice of complex. Consider first the case where $R^0\pi_\star \mathrm{ev}_{n+1}^\star(E) = 0$. Then

$$0 \longrightarrow E^0_{g,n,d} \longrightarrow E^1_{g,n,d}$$

is an exact sequence of vector bundles, and so $R^1\pi_\star \operatorname{ev}_{n+1}^\star(E)$ is a bundle also. We will prove (1.10) in this case by comparing the fibers of the vector bundles

$$-\gamma_{red}^\star i^\star E_{g,n+1,d} \qquad \text{and} \qquad (-p_+^\star E_{g_+,n_++\bullet,d_+}) \oplus (-p_-^\star E_{g_-,n_-+\circ,d_-})$$

at the point

$$(((C_+,\epsilon_+),f_+),((C_-,\epsilon_-),f_-)) \in \widetilde{\mathcal{Z}}_{red}$$

Applying Serre duality, the fibers in question are

$$H^0(C,f^\star E^\vee \otimes \omega_C)^\vee \qquad \text{and} \qquad H^0(C_+,f_+^\star E^\vee \otimes \omega_{C_+})^\vee \oplus H^0(C_-,f_-^\star E^\vee \otimes \omega_{C_-})^\vee$$

where

$$((C,\epsilon),f)$$

is the stable map obtained from the stable maps

$$((C_+,\epsilon_+),f_+) \qquad \text{and} \qquad ((C_-,\epsilon_-),f_-)$$

by gluing, and ω_C, ω_{C_+}, ω_{C_-} are the dualizing sheaves on C, C_+, C_- respectively. But the dualizing sheaf ω_C consists of meromorphic 1-forms on C which are holomorphic away from the nodes and have at most simple poles at the nodes, such that the two residues at each node sum to zero. There is therefore an exact sequence

$$0 \to H^0(C_+,f_+^\star E^\vee \otimes \omega_{C_+}) \oplus H^0(C_-,f_-^\star E^\vee \otimes \omega_{C_-}) \to H^0(C,f^\star E^\vee \otimes \omega_C) \to \operatorname{ev}_\Delta^\star E^\vee \to 0$$

which when dualized gives (1.10). An entirely analogous argument proves (1.11) in this case also.

For the general case, take L to be a positive line bundle, $N \gg 0$ and consider the exact sequence

$$0 \longrightarrow Ker \longrightarrow H^0(X, E \otimes L^N) \otimes L^{-N} \longrightarrow E \longrightarrow 0$$

of vector bundles on X. Write

$A = H^0(X,E\otimes L^N)\otimes L^{-N}$
$B = Ker$

If $d \neq 0$ then for sufficiently large N both $R^0\pi_\star \operatorname{ev}_{n+1}^\star A$ and $R^0\pi_\star \operatorname{ev}_{n+1}^\star B$ vanish. In this case we have

$$E_{g,n,d} = A_{g,n,d} - B_{g,n,d}$$

where the argument above applies to both $A_{g,n,d}$ and $B_{g,n,d}$.

In the remaining case, when $d = 0$, $R^0\pi_\star \operatorname{ev}_{n+1}^\star E$ does not vanish. However, in this case $R^0\pi_\star \operatorname{ev}_{n+1}^\star E$ is the trivial bundle with fiber E and $R^1\pi_\star \operatorname{ev}_{n+1}^\star E$ is also a vector bundle. Our previous argument therefore deals with this case also. This completes the proof. □

A similar argument proves

Lemma 1.6.2.

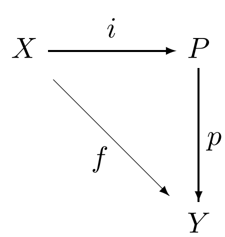
$$\pi^\star E_{g,n,d} = E_{g,n+1,d}$$

1.6.3 A quantum Riemann–Roch theorem

We will determine the relationship between twisted and untwisted Gromov–Witten invariants by applying the Grothendieck–Riemann–Roch theorem to the universal family $\pi : X_{g,n+1,d} \to X_{g,n,d}$ (see [53, 17]). We justify this as follows. Fulton [19] proves the Grothendieck–Riemann–Roch theorem for proper l.c.i. morphisms of schemes $f : X \to Y$

$$\operatorname{ch}(f_\star\alpha) = f_\star(\operatorname{ch}(\alpha)\operatorname{Td}T_f) \qquad \text{for any } \alpha \in K^0(X)$$

The map $f : X \to Y$ is l.c.i. if for some (and hence for any) factorization



with i a closed embedding and p smooth, i is in fact a regular embedding. This means that the normal sheaf of X in P is locally free, which is exactly what we need to define the "virtual tangent bundle"

$$T_f = [i^\star T_{P/Y}] - [N_{X/P}] \in K^0(X)$$

Note that, despite the suggestive terminology, this has no simple relationship to the virtual fundamental class.

We apply this to our situation as follows: the moduli space $X_{g,n,d}$ can be realized [20] as the orbifold (stack) quotient of a subscheme J of a Hilbert scheme by a proper action of the group $G = PGL$. The universal family $\pi : X_{g,n+1,d} \to X_{g,n,d}$ is the quotient by G of the universal family $U \to J$. One way[4] to define the Chow groups (or cohomology groups) of $X_{g,n,d}$ with rational coefficients is as the G-equivariant Chow groups (or cohomology groups) of J. In other words [16] we take a family of finite-dimensional approximations $J_{(N)}$ to the Borel space $EG \times_G J$ and define the Chow groups (or cohomology groups) of $X_{g,n,d}$ to be the limit of the Chow groups (or cohomology groups) of the $J_{(N)}$. Let $U_{(N)}$ be a similar family of finite-dimensional approximations to $EG \times_G U$. We apply Fulton's Grothendieck–Riemann–Roch theorem to the maps

$$\pi_{(N)} : U_{(N)} \to J_{(N)}$$

These maps are l.c.i. since the universal family $U \to J$ is manifestly l.c.i. We find that

$$\mathrm{ch}(\pi_\star \, \mathrm{ev}^\star E) = \pi_\star(\mathrm{ch}(\mathrm{ev}^\star E) \cdot \mathrm{Td}^\vee \, \Omega_\pi) \tag{GRR}$$

where Ω_π is the sheaf of relative differentials of $\pi : X_{g,n+1,d} \to X_{g,n,d}$.

Recall that $\sigma_i : X_{g,n,d} \to X_{g,n+1,d}$ is the section of the universal family defined by the ith marked point. Let ψ_+, ψ_- denote the first Chern classes of the bundles over $\mathcal{Z}$ formed by the cotangent lines at the nodes.

Proposition 1.6.3.

$$[X_{g,n,d}] \cap \mathrm{ch}(E_{g,n,d}) = [X_{g,n,d}] \cap \pi_\star(\mathrm{ev}^\star(\mathrm{ch}(E)) \cdot (\boxed{\textit{codim-}0} + \boxed{\textit{codim-}1} + \boxed{\textit{codim-}2})) \tag{1.12}$$

where

$$\boxed{\textit{codim-}0} = \mathrm{Td}^\vee \, L_{n+1}$$

$$\boxed{\textit{codim-}1} = -\sum_{i=1}^{n} \sigma_{i\star} \left[\frac{\mathrm{Td}^\vee \, (L_i)}{\psi_i} \right]_+$$

$$\boxed{\textit{codim-}2} = i_\star \left[\frac{1}{\psi_+ + \psi_-} \left(\frac{\mathrm{Td}^\vee \, (L_+)}{\psi_+} + \frac{\mathrm{Td}^\vee \, (L_-)}{\psi_-} \right) \right]_+$$

and $[\,\cdot\,]_+$ *denotes the power series truncation of a Laurent series in* ψ_i *or in* ψ_+ *and* ψ_-.

Proof. We will express the sheaf Ω_π of relative differentials appearing in (GRR) in terms of universal cotangent lines.

[4] That this agrees with the usual definition of cohomology groups is obvious; that it agrees with the usual definition of Chow groups follows from work of Kresch [40] (see also [16]).

Assume first that the image $\pi(\mathcal{Z})$ of the singular locus forms a divisor with normal crossings in $X_{g,n,d}$. Then there are exact sequences

$$0 \longrightarrow \Omega_\pi \longrightarrow \omega_\pi \longrightarrow i_\star \mathcal{O}_{\mathcal{Z}} \longrightarrow 0 \tag{1.13}$$

and

$$0 \longrightarrow \omega_\pi \longrightarrow L_{n+1} \longrightarrow \oplus_{i=1}^n \sigma_{i\star} \mathcal{O}_{X_{g,n,d}} \longrightarrow 0 \tag{1.14}$$

where ω_π is the relative dualizing sheaf of the family $\pi : X_{g,n+1,d} \to X_{g,n,d}$. To establish (1.13), note first that Ω_π and ω_π coincide away from $\mathcal{Z}$. Let C be a point of $\mathcal{Z}$. We can find co-ordinates (z,ϵ) near $\pi(C)$ and (z,x,y) near C where z is a (vector) co-ordinate along $\mathcal{Z}$ and the map π in these co-ordinates is

$$\pi : (z,x,y) \mapsto (z,xy)$$

Sections of ω_π near C have the form

$$f(z,x,y)\,\frac{dx \wedge dy}{d(xy)}$$

where

$$f(z,x,y) = \sum_{i,j \geq 0} f_{ij}(z) x^i y^j$$

Sections of Ω_π near C have the form

$$\alpha(z,x,y)\,dx + \beta(z,x,y)\,dy$$

where

$$\begin{aligned} \alpha(z,x,y) &= \sum_{i,j \geq 0} \alpha_{ij}(z) x^i y^j \\ \beta(z,x,y) &= \sum_{i,j \geq 0} \beta_{ij}(z) x^i y^j \end{aligned}$$

and we impose the relation $x\,dy + y\,dx = 0$. There is a natural inclusion

$$\begin{aligned} \Omega_\pi &\to \omega_\pi \\ \alpha(z,x,y)\,dx + \beta(z,x,y)\,dy &\mapsto (x\alpha(z,x,y) - y\beta(z,x,y))\,\frac{dx \wedge dy}{d(xy)} \end{aligned}$$

The cokernel consists of elements of the form

$$f_{00}(z)\,\frac{dx \wedge dy}{d(xy)}$$

The expression

$$\frac{dx \wedge dy}{d(xy)}$$

represents a locally constant section of the (orbi)bundle

$$\bigwedge^2 (L_+ \oplus L_-) \otimes L_+^{-1} \otimes L_-^{-1}$$

over $\mathcal{Z}$, so we can identify the cokernel with $i_\star \mathcal{O}_\mathcal{Z}$. This establishes (1.13); an entirely analogous argument gives (1.14).

Combining (1.13) and (1.14), we find that

$$\Omega_\pi = L_{n+1} - \sum_{i=1}^{n} \sigma_{i\star} \mathcal{O}_{X_{g,n,d}} - i_\star \mathcal{O}_\mathcal{Z} \qquad \text{in } K^0(X_{g,n+1,d}) \tag{1.15}$$

and so

$$\mathrm{Td}^\vee(\Omega_\pi) = \mathrm{Td}^\vee(L_{n+1}) \Big(\prod_{i=1}^{n} \mathrm{Td}^\vee(-\sigma_{i\star}\mathcal{O}_{X_{g,n,d}}) \Big) \mathrm{Td}^\vee(-i_\star \mathcal{O}_\mathcal{Z}) \tag{1.16}$$

We can write

$$\mathrm{Td}^\vee(\,\cdot\,) = \exp\Big(\sum_{k \geq 0} t_k \, \mathrm{ch}_k(\,\cdot\,) \Big)$$

where $t_0 = 0$ (and in fact $t_1 = \frac{1}{2}$ and $t_k = -B_k/k$ for $k \geq 2$, but we will not need this). Thus

$$\mathrm{Td}^\vee(-i_\star \mathcal{O}_\mathcal{Z}) = \exp\Big(-\sum_{k \geq 0} t_k \, \mathrm{ch}_k(i_\star \mathcal{O}_\mathcal{Z}) \Big)$$

Applying Grothendieck–Riemann–Roch again, we see that

$$\mathrm{ch}_k(i_\star \mathcal{O}_\mathcal{Z}) = i_\star(\mathrm{Td}^\vee(-L_+ - L_-))_{k-2}$$

since the (l.c.i.) virtual tangent bundle T_i is $-L_+^{-1} - L_-^{-1}$. Here $(x)_r$ denotes the component of the cohomology class x in degree $2r$. Therefore

$$\mathrm{ch}_k(i_\star \mathcal{O}_\mathcal{Z}) = i_\star \Bigg(\sum_{\substack{a+b=k-2 \\ a,b \geq 0}} \frac{\psi_+^a \psi_-^b}{(a+1)!\,(b+1)!} \Bigg)$$

and if we set

$$\alpha = -\sum_{k \geq 2} t_k \sum_{\substack{a+b=k-2 \\ a,b \geq 0}} \frac{\psi_+^a \psi_-^b}{(a+1)!\,(b+1)!}$$

then

$$\begin{aligned}\mathrm{Td}^\vee(-i_\star\mathcal{O}_{\mathcal{Z}}) &= \exp(i_\star\alpha)\\ &= 1 + (i_\star\alpha)\sum_{r\geq 1}\frac{(i_\star\alpha)^{r-1}}{r!}\\ &= 1 + i_\star\left(\frac{\exp(\alpha\cup i_\star 1)-1}{i_\star 1}\right)\end{aligned}$$

But

$$\begin{aligned}\alpha\cup i_\star 1 &= -\sum_{k\geq 2} t_k \sum_{\substack{a+b=k-2\\ a,b\geq 0}}\frac{\psi_+^{a+1}\psi_-^{b+1}}{(a+1)!\,(b+1)!}\\ &= -\sum_{k\geq 2}\frac{t_k}{k!}((\psi_+ + \psi_-)^k - \psi_+^k - \psi_-^k)\\ &= \sum_{k\geq 0} t_k(\mathrm{ch}_k(L_+) + \mathrm{ch}_k(L_-) - \mathrm{ch}_k(L_+\otimes L_-))\end{aligned}$$

and so

$$\begin{aligned}\mathrm{Td}^\vee(-i_\star\mathcal{O}_{\mathcal{Z}}) &= 1 + i_\star\left(\frac{1}{\psi_+\psi_-}\left(\frac{\mathrm{Td}^\vee(L_+)\,\mathrm{Td}^\vee(L_-)}{\mathrm{Td}^\vee(L_+\otimes L_-)} - 1\right)\right)\\ &= 1 + i_\star\left[\frac{1}{\psi_+ + \psi_-}\frac{e^{\psi_+ + \psi_-}-1}{(e^{\psi_+}-1)(e^{\psi_-}-1)}\right]_+\end{aligned}$$

Applying the inclusion-exclusion formula for the Poincaré polynomial of $\mathbb{C}[x,y]/(xy)$

$$\frac{1-uv}{(1-u)(1-v)} = \frac{1}{1-u} + \frac{1}{1-v} - 1$$

with $u = e^{\psi_+}$, $v = e^{\psi_-}$ we find that

$$\begin{aligned}\mathrm{Td}^\vee(-i_\star\mathcal{O}_{\mathcal{Z}}) &= 1 + i_\star\left[\frac{1}{\psi_+ + \psi_-}\left(\frac{1}{e^{\psi_+}-1} + \frac{1}{e^{\psi_-}-1}\right)\right]_+\\ &= 1 + i_\star\left[\frac{1}{\psi_+ + \psi_-}\left(\frac{\mathrm{Td}^\vee(L_+)}{\psi_+} + \frac{\mathrm{Td}^\vee(L_-)}{\psi_-}\right)\right]_+ \qquad (1.17)\end{aligned}$$

A similar calculation yields

$$\mathrm{Td}^\vee(-\sigma_{i\star}\mathcal{O}_{X_{g,n,d}}) = 1 - \sigma_{i\star}\left[\frac{\mathrm{Td}^\vee(L_i)}{\psi_i}\right]_+$$

Substituting into (1.16), we find

$$\begin{aligned}\mathrm{Td}^\vee(\Omega_\pi) =& \mathrm{Td}^\vee(L_{n+1}) \times \left(\prod_{i=1}^{n}\left(1-\sigma_{i\star}\left[\frac{\mathrm{Td}^\vee(L_i)}{\psi_i}\right]_+\right)\right) \\ & \times \left(1 + i_\star\left[\frac{1}{\psi_+ + \psi_-}\left(\frac{\mathrm{Td}^\vee(L_+)}{\psi_+} + \frac{\mathrm{Td}^\vee(L_-)}{\psi_-}\right)\right]_+\right)\end{aligned}$$

Since the divisors $D_i = \sigma_i(X_{g,n,d})$ and $\mathcal{Z}$ are mutually disjoint, and L_{n+1} is trivial on D_i and on $\mathcal{Z}$, this gives

$$\mathrm{Td}^\vee(\Omega_\pi) = \mathrm{Td}^\vee(L_{n+1}) - \sum_{i=1}^{n}\sigma_{i\star}\left[\frac{\mathrm{Td}^\vee(L_i)}{\psi_i}\right]_+ + i_\star\left[\frac{1}{\psi_+ + \psi_-}\left(\frac{\mathrm{Td}^\vee(L_+)}{\psi_+} + \frac{\mathrm{Td}^\vee(L_-)}{\psi_-}\right)\right]_+ \tag{1.18}$$

which implies the Proposition.

In the general case, where $\pi(\mathcal{Z})$ is not a divisor with normal crossings, this argument remains "virtually correct" in the sense that we can find an embedding [17]

$$\begin{array}{ccc} X_{g,n+1,d} & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ X_{g,n,d} & \longrightarrow & \mathcal{M} \end{array}$$

of $X_{g,n,d}$ into a non-singular space $\mathcal{M}$ with a flat family of curves $\mathcal{C} \to \mathcal{M}$ such that

- the family $\mathcal{C} \to \mathcal{M}$ restricts to the universal family over $X_{g,n,d}$
- we can extend the bundle $\mathrm{ev}^\star(E)$ over $X_{g,n+1,d}$ to a bundle over $\mathcal{C}$
- the argument above is valid for the family $\mathcal{C} \to \mathcal{M}$

We recover the Proposition by capping (1.17) for the family $\mathcal{C} \to \mathcal{M}$ with the virtual fundamental class $[X_{g,n,d}]$. □

Proposition 1.6.3 will be the main tool in the proof of our "quantum Riemann–Roch theorem".

Theorem 1.6.4.

$$\begin{aligned}&\exp\Big(-\frac{1}{24}\sum_{l>0} s_{l-1}\int_X \mathrm{ch}_l(E)c_{D-1}(T_X)\Big)(\mathrm{sdet}\sqrt{\mathbf{c}(E)})^{-\frac{1}{24}}\,\mathcal{D}_{\mathbf{s}} = \\ &\exp\Big(\sum_{m>0}\sum_{l\geq 0} s_{2m-1+l}\frac{B_{2m}}{(2m)!}(\mathrm{ch}_l(E)z^{2m-1})^{\wedge}\Big)\exp\Big(\sum_{l>0} s_{l-1}(\mathrm{ch}_l(E)/z)^{\wedge}\Big)\,\mathcal{D}_X\end{aligned} \tag{1.19}$$

Proof. Proposition A.0.2 in Appendix A shows that the right-hand side is well-defined as a formal function of $\mathbf{t}$ taking values in $\Lambda[[\hbar,\hbar^{-1}]]$, where the ground ring Λ is $\mathbb{C}[[Q]][[s_0, s_1, \ldots]]$.

It suffices to prove the infinitesimal statement

$$\begin{aligned}\frac{\partial}{\partial s_k}\mathcal{D}_{\mathbf{s}} =& \Big(\sum_{\substack{2m+r=k+1\\ m\geq 0}}\frac{B_{2m}}{(2m)!}(\mathrm{ch}_r(E)z^{2m-1})^{\wedge}\Big)\mathcal{D}_{\mathbf{s}} \\ &+\left(\begin{matrix}\frac{1}{24}\int_X c_{D-1}(X)\wedge\mathrm{ch}_{k+1}(E)+\frac{1}{48}\int_X \mathbf{e}(X)\wedge\mathrm{ch}_k(E)\\ -\frac{1}{24}\int_X \mathbf{e}(X)\wedge\mathrm{ch}_{k+1}(E)\wedge\Big(\sum_l s_{l+1}\,\mathrm{ch}_l(E)\Big)\end{matrix}\right)\mathcal{D}_{\mathbf{s}}\end{aligned} \tag{1.20}$$

Here the second exceptional term arises from the superdeterminant

$$\begin{aligned}\mathrm{sdet}\sqrt{\mathbf{c}(E)} &= \exp(\mathrm{str}(\ln\sqrt{\mathbf{c}(E)})) \\ &= \exp\Big(\frac{1}{2}\int_X \mathbf{e}(X)\wedge\Big(\sum_j s_j\,\mathrm{ch}_j(E)\Big)\Big)\end{aligned}$$

and the third exceptional term is the cocycle value

$$\begin{aligned}\mathcal{C}\Big(\frac{B_2}{2}\sum_l s_{l+1}\,\mathrm{ch}_l(E)z, \mathrm{ch}_{k+1}(E)/z\Big) &= -\frac{1}{2}\,\mathrm{str}\Big(\mathrm{ch}_{k+1}(E)\cdot\frac{1}{12}\sum_l s_{l+1}\,\mathrm{ch}_l(E)\Big) \\ &= -\frac{1}{24}\int_X \mathbf{e}(X)\wedge\mathrm{ch}_{k+1}(E)\wedge\Big(\sum_l s_{l+1}\,\mathrm{ch}_l(E)\Big)\end{aligned}$$

coming from commuting the s_k-derivative of the $1/z$-terms past the terms involving z (see Example 1.3.4.1).

Now

$$\begin{aligned}\frac{\partial\mathcal{D}_{\mathbf{s}}}{\partial s_k} =& \sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{n!}\langle\mathbf{t},\ldots,\mathbf{t};\mathrm{ch}_k(E_{g,n,d})\wedge\mathbf{c}(E_{g,n,d})\rangle_{g,n,d}\,\mathcal{D}_{\mathbf{s}} \\ &+\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{(n-1)!}\Big\langle\frac{\partial\mathbf{t}}{\partial s_k},\mathbf{t},\ldots,\mathbf{t};\mathbf{c}(E_{g,n,d})\Big\rangle_{g,n,d}\mathcal{D}_{\mathbf{s}}\end{aligned} \tag{1.21}$$

Applying Proposition 1.6.3, we see that the first summand splits into three contributions which we will call the codimension-0, codimension-1 and codimension-2 terms. We will calculate these contributions, and also the derivative contribution (the other summand in (1.21)), separately. The codimension-2 contribution will match up with the bivector field part of the quantization in (1.20) (see Example 1.3.3.1). The other contributions will combine to give the rest of the quantization and the exceptional terms in in (1.20).

Codimension-2 terms

These are

$$\sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{n!} \left\langle \mathbf{t}, \ldots, \mathbf{t}; \pi_\star i_\star \left[\left(\frac{\mathrm{ch}(E)}{\psi_+ + \psi_-} \left(\frac{\mathrm{Td}^\vee(L_+)}{\psi_+} + \frac{\mathrm{Td}^\vee(L_-)}{\psi_-} \right) \right)_{k-1} \right]_+ \mathbf{c}(E_{g,n,d}) \right\rangle_{g,n,d} \mathcal{D}_{\mathbf{s}}$$

Pulling back to $\widetilde{\mathcal{Z}}_{red} \coprod \widetilde{\mathcal{Z}}_{irr}$ and using Lemma 1.6.1, Lemma 1.6.2, and the properties of the virtual fundamental class discussed on page 57, we can write this as

$$\frac{1}{2} \sum_{g_1,g_2} \sum_{n_1,n_2} \sum_{d_1,d_2} \frac{Q^{d_1+d_2} \hbar^{g_1+g_2-1}}{n_1!\, n_2!} \sum_{r,s} \left\langle \mathbf{t}, \ldots, \mathbf{t}, \frac{\alpha_{r,s} \psi_+^r}{\sqrt{\mathbf{c}(E)}}; \mathbf{c}(E_{g_1,n_1+1,d_1}) \right\rangle_{g_1,n_1+1,d_1}$$
$$\times \left\langle \frac{\psi_-^s}{\sqrt{\mathbf{c}(E)}}, \mathbf{t}, \ldots, \mathbf{t}; \mathbf{c}(E_{g_2,n_2+1,d_2}) \right\rangle_{g_2,n_2+1,d_2}$$
$$+ \frac{1}{2} \sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{n!} \sum_{r,s} \left\langle \mathbf{t}, \ldots, \mathbf{t}, \frac{\alpha_{r,s} \psi_+^r}{\sqrt{\mathbf{c}(E)}}, \frac{\psi_-^s}{\sqrt{\mathbf{c}(E)}}; \mathbf{c}(E_{g-1,n+2,d}) \right\rangle_{g-1,n+2,d}$$

where

$$\sum_{r,s} \alpha_{r,s} \psi_+^r \psi_-^s = \left[\left(\frac{\mathrm{ch}(E)}{\psi_+ + \psi_-} \left(\frac{\mathrm{Td}^\vee(L_+)}{\psi_+} + \frac{\mathrm{Td}^\vee(L_-)}{\psi_-} \right) \right)_{k-1} \right]_+ \wedge \left(g^{\alpha\beta} \phi_\alpha \otimes \phi_\beta \right)$$
$$\in H^\star(X)[[\psi_+]] \otimes H^\star(X)[[\psi_-]]$$

and we have applied Lemmas 1.6.1 and 1.6.2. The factor of 1/2 here comes from the fact that the map

$$\widetilde{\mathcal{Z}}_{red} \coprod \widetilde{\mathcal{Z}}_{irr} \xrightarrow{\gamma_{red} \coprod \gamma_{irr}} \mathcal{Z}$$

is generically 2-to-1.

Since the twisted dilaton shift gives

$$\partial_{\alpha,k} = \frac{1}{\sqrt{\mathbf{c}(E)}} \frac{\partial}{\partial t_k^\alpha}$$

a comparison with Example 1.3.3.1 shows that we can write the codimension-2 terms as

$$\frac{\hbar}{2}\,(\partial \otimes_{A_k} \partial)\,\mathcal{D}_{\mathbf{s}} \tag{1.22}$$

where

$$A_k = \left(\operatorname{ch}(E)\frac{\operatorname{Td}^\vee(L)}{\psi}\right)_k + \frac{\operatorname{ch}_k(E)}{2}$$

We are abusing notation here: for consistency with Example 1.3.3.1 we should identify ψ with z:

$$A_k = \left(\frac{\operatorname{ch}(E)}{e^z - 1}\right)_k + \frac{\operatorname{ch}_k(E)}{2}$$

Note that A_k is a series in odd powers of z with coefficients in $H^\star(X)$, so multiplication by A_k defines an infinitesimal symplectomorphism of $\mathcal{H}$.

Codimension-1 terms

These are

$$\begin{aligned}
&-\sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{n!}\left\langle \mathbf{t},\ldots,\mathbf{t};\left(\sum_{i=1}^n \pi_\star \sigma_{i\star}\left[\operatorname{ch}(E)\frac{\operatorname{Td}^\vee(L_i)}{\psi_i}\right]_+\right)_k \mathbf{c}(E_{g,n,d})\right\rangle_{g,n,d}\mathcal{D}_{\mathbf{s}} \\
&\qquad = -\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{(n-1)!}\left\langle\left(\left[\operatorname{ch}(E)\frac{\operatorname{Td}^\vee(L)}{\psi}\right]_+\right)_k \mathbf{t}(\psi),\mathbf{t},\ldots,\mathbf{t};\mathbf{c}(E_{g,n,d})\right\rangle_{g,n,d}\mathcal{D}_{\mathbf{s}}
\end{aligned} \tag{1.23}$$

Codimension-0 terms

These are

$$\begin{aligned}
&\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{n!}\langle \mathbf{t},\ldots,\mathbf{t};(\pi_\star(\operatorname{ch}(E)\operatorname{Td}^\vee(L_{n+1})))_k \mathbf{c}(E_{g,n,d})\rangle_{g,n,d}\mathcal{D}_{\mathbf{s}} \\
&\qquad = \sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{n!}\langle \pi^\star\mathbf{t},\ldots,\pi^\star\mathbf{t},(\operatorname{ch}(E)\operatorname{Td}^\vee(L))_{k+1};\pi^\star\mathbf{c}(E_{g,n,d})\rangle_{g,n+1,d}\mathcal{D}_{\mathbf{s}}
\end{aligned}$$

Applying Lemma 1.6.2 and the comparison result (see *e.g.* [67, 54]) for universal cotangent lines

$$\pi^\star\mathbf{t}(\psi_i) = \mathbf{t}(\psi_i) - \sigma_{i\star}\left[\frac{\mathbf{t}(\psi_i)}{\psi_i}\right]_+$$

we can write this as

$$\begin{aligned}
&\sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{n!} \left\langle \mathbf{t}(\psi) - \sigma_{1\star}\left[\frac{\mathbf{t}(\psi)}{\psi}\right]_+, \dots, (\mathrm{ch}(E)\,\mathrm{Td}^\vee(L))_{k+1}; \mathbf{c}(E_{g,n+1,d}) \right\rangle_{g,n+1,d} \mathcal{D}_{\mathbf{s}} \\
=&\sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{n!} \langle \mathbf{t}, \dots, \mathbf{t}, (\mathrm{ch}(E)\,\mathrm{Td}^\vee(L))_{k+1}; \mathbf{c}(E_{g,n+1,d}) \rangle_{g,n+1,d} \mathcal{D}_{\mathbf{s}} \\
&- \sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{(n-1)!} \left\langle \mathrm{ch}_{k+1}(E) \left[\frac{\mathbf{t}(\psi)}{\psi}\right]_+, \mathbf{t}, \dots, \mathbf{t}; \mathbf{c}(E_{g,n,d}) \right\rangle_{g,n,d} \mathcal{D}_{\mathbf{s}} \\
=&\sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{(n-1)!} \langle \mathbf{t}, \dots, \mathbf{t}, (\mathrm{ch}(E)\,\mathrm{Td}^\vee(L))_{k+1}; \mathbf{c}(E_{g,n,d}) \rangle_{g,n,d} \mathcal{D}_{\mathbf{s}} \\
&- \frac{1}{2\hbar} \langle \mathbf{t}, \mathbf{t}, (\mathrm{ch}(E)\,\mathrm{Td}^\vee(L))_{k+1}; \mathbf{c}(E_{0,3,0}) \rangle_{0,3,0} - \langle (\mathrm{ch}(E)\,\mathrm{Td}^\vee(L))_{k+1}; \mathbf{c}(E_{1,1,0}) \rangle_{1,1,0} \\
&- \sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{(n-1)!} \left\langle \mathrm{ch}_{k+1}(E) \left[\frac{\mathbf{t}(\psi)}{\psi}\right]_+, \mathbf{t}, \dots, \mathbf{t}; \mathbf{c}(E_{g,n,d}) \right\rangle_{g,n,d} \mathcal{D}_{\mathbf{s}}
\end{aligned} \tag{1.24}$$

We next calculate the exceptional terms in (1.24), which arose in the reindexing since the moduli spaces $X_{0,2,0}$ and $X_{1,0,0}$ are empty and so $X_{0,3,0}$ and $X_{1,1,0}$ cannot be interpreted as universal families.

$$\begin{aligned}
-\frac{1}{2\hbar} \langle \mathbf{t}, \mathbf{t}, (\mathrm{ch}(E)\,\mathrm{Td}^\vee(L))_{k+1}; \mathbf{c}(E_{0,3,0}) \rangle_{0,3,0} &= -\frac{1}{2\hbar} \int_X t_0 \wedge t_0 \wedge \mathrm{ch}_{k+1}(E) \wedge \mathbf{c}(E) \\
&= \frac{1}{2\hbar} \Omega_{\mathbf{s}}((A_k \mathbf{q})(-z), \mathbf{q}(z))
\end{aligned} \tag{1.25}$$

where we used the facts that $[X_{0,3,0}] = [X]$ and $E_{0,3,0} = E$. Also (see *e.g.* [24])

- $X_{1,1,0} = X \times \overline{\mathcal{M}}_{1,1}$
- $[X_{1,1,0}] = \mathbf{e}(TX \otimes L_1^{-1}) \cap [X \times \overline{\mathcal{M}}_{1,1}]$
- $E_{1,1,0} = E \otimes (1 - L_1^{-1})$
- $L_1 \to X_{1,1,0}$ coincides with the pullback of the universal cotangent line over $\overline{\mathcal{M}}_{1,1}$

and

$$\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \frac{1}{24}$$

so

$$
\begin{aligned}
&-\langle(\operatorname{ch}(E)\operatorname{Td}^\vee(L))_{k+1};\mathbf{c}(E_{1,1,0})\rangle_{1,1,0} \\
&\qquad = -\int_{X\times\overline{\mathcal{M}}_{1,1}}\Big(\operatorname{ch}_{k+1}(E)-\frac{\operatorname{ch}_k(E)}{2}\psi_1\Big)\mathbf{c}(E)\,\mathbf{c}(-E\otimes L_1^{-1})\,\mathbf{e}(TX\otimes L_1^{-1})
\end{aligned}
$$

But

$$
\mathbf{c}(-E\otimes L_1^{-1})=\mathbf{c}(-E)\Big(1+\psi\sum_j s_{j+1}\operatorname{ch}_j(E)\Big)
$$

and

$$
\mathbf{e}(TX\otimes L_1^{-1})=\mathbf{e}(TX)-\psi_1 c_{D-1}(TX)
$$

so

$$
\begin{aligned}
&-\langle(\operatorname{ch}(E)\operatorname{Td}^\vee(L))_{k+1};\mathbf{c}(E_{1,1,0})\rangle_{1,1,0} \\
&\qquad = -\int_{X\times\overline{\mathcal{M}}_{1,1}}\Big(\operatorname{ch}_{k+1}(E)-\frac{\operatorname{ch}_k(E)}{2}\psi_1\Big)\Big(1+\psi\sum_j s_{j+1}\operatorname{ch}_j(E)\Big) \\
&\qquad\qquad\qquad \times(\mathbf{e}(TX)-\psi_1 c_{D-1}(TX)) \\
&\qquad = \frac{1}{48}\int_X \operatorname{ch}_k(E)\mathbf{e}(TX)-\frac{1}{24}\int_X \operatorname{ch}_{k+1}(E)\Big(\sum_j s_{j+1}\operatorname{ch}_j(E)\Big)\mathbf{e}(TX) \\
&\qquad\qquad + \frac{1}{24}\int_X \operatorname{ch}_{k+1}(E)c_{D-1}(TX)
\end{aligned}
\tag{1.26}
$$

Derivative contribution

Because of the twisted dilaton shift,

$$
\frac{\partial\mathbf{t}(z)}{\partial s_k}=-\frac{1}{2}\operatorname{ch}_k(E)(\mathbf{t}(z)-z)
$$

and so the second summand in (1.21) contributes

$$
-\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{(n-1)!}\langle\tfrac{1}{2}\operatorname{ch}_k(E)(\mathbf{t}(\psi)-\psi),\mathbf{t},\ldots,\mathbf{t};\mathbf{c}(E_{g,n,d})\rangle_{g,n,d}\mathcal{D}_{\mathbf{s}}
$$

Putting everything together

Since

$$
\Big[\Big(\operatorname{ch}(E)\frac{\operatorname{Td}^\vee(L)}{\psi}\Big)_k\Big]_+\mathbf{t}(\psi)+\operatorname{ch}_{k+1}(E)\Big[\frac{\mathbf{t}(\psi_i)}{\psi}\Big]_+=\Big[\Big(\operatorname{ch}(E)\frac{\operatorname{Td}^\vee(L)}{\psi}\Big)_k\mathbf{t}(\psi)\Big]_+
$$

we can write the sum of the codimension-0 and codimension-1 contributions as

$$\sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{(n-1)!} \langle \mathbf{t}, \ldots, \mathbf{t}, (\mathrm{ch}(E)\, \mathrm{Td}^\vee(L))_{k+1}; \mathbf{c}(E_{g,n,d}) \rangle_{g,n,d} \mathcal{D}_\mathbf{s}$$
$$- \sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{(n-1)!} \left\langle \left[\left(\mathrm{ch}(E) \frac{\mathrm{Td}^\vee(L)}{\psi} \right)_k \mathbf{t}(\psi) \right]_+, \mathbf{t}, \ldots, \mathbf{t}; \mathbf{c}(E_{g,n,d}) \right\rangle_{g,n,d} \mathcal{D}_\mathbf{s}$$
$$+ (\text{ exceptional terms (1.25) and (1.26) }) \mathcal{D}_\mathbf{s}$$

But

$$(\mathrm{ch}(E)\, \mathrm{Td}^\vee(L))_{k+1} = \left[\left(\mathrm{ch}(E) \frac{\mathrm{Td}^\vee(L)}{\psi} \right)_k \psi \right]_+$$

so we can write the sum of the codimension-0, codimension-1 and derivative contributions as

$$- \sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{(n-1)!} \left\langle \left[\left(\left(\mathrm{ch}(E) \frac{\mathrm{Td}^\vee(L)}{\psi} \right)_k + \frac{\mathrm{ch}_k(E)}{2} \right) \mathbf{q}(\psi) \right]_+, \mathbf{t}, \ldots, \mathbf{t}; \mathbf{c}(E_{g,n,d}) \right\rangle_{g,n,d} \mathcal{D}_\mathbf{s}$$
$$+ (\text{ exceptional terms (1.25) and (1.26) }) \mathcal{D}_\mathbf{s}$$

or in other words as

$$-\partial_{A_k} \mathcal{D}_\mathbf{s} + (\text{ exceptional terms (1.25) and (1.26) }) \mathcal{D}_\mathbf{s}$$

Combining this with (1.22), we find that

$$\frac{\partial \mathcal{D}_\mathbf{s}}{\partial s_k} = \left(\frac{1}{2\hbar} \Omega_\mathbf{s}((A_k \mathbf{q})(-z), \mathbf{q}(z)) - \partial_{A_k} \mathcal{D}_\mathbf{s} + \frac{\hbar}{2} (\partial \otimes_{A_k} \partial) \right) \mathcal{D}_\mathbf{s}$$
$$+ \begin{pmatrix} \frac{1}{24} \int_X c_{D-1}(X) \wedge \mathrm{ch}_{k+1}(E) + \frac{1}{48} \int_X \mathbf{e}(X) \wedge \mathrm{ch}_k(E) \\ - \frac{1}{24} \int_X \mathbf{e}(X) \wedge \mathrm{ch}_{k+1}(E) \wedge \left(\sum_l s_{l+1} \mathrm{ch}_l(E) \right) \end{pmatrix} \mathcal{D}_\mathbf{s}$$

But Example 1.3.3.1 shows that

$$\frac{1}{2\hbar} \Omega_\mathbf{s}((A_k \mathbf{q})(-z), \mathbf{q}(z)) - \partial_{A_k} \mathcal{D}_\mathbf{s} + \frac{\hbar}{2} (\partial \otimes_{A_k} \partial) = \widehat{A_k}$$

and

$$A_k(z) = \left(\frac{\mathrm{ch}(E)}{e^z - 1} \right)_k + \frac{\mathrm{ch}_k(E)}{2}$$
$$= \sum_{\substack{2m+r=k \\ r,m \geq 0}} \frac{B_{2m}}{(2m)!} \mathrm{ch}_r(E) z^{2m-1}$$

This establishes (1.20). The Theorem follows. □

Corollary 1.6.5. *The Lagrangian submanifolds* $\mathcal{L}_{\mathbf{s}}$ *are related by*

$$\mathcal{L}_{\mathbf{s}} = \exp\left(\sum_{m\geq 0}\sum_{0\leq l\leq D} s_{2m-1+l}\frac{B_{2m}}{(2m)!}\operatorname{ch}_l(E)z^{2m-1}\right)\mathcal{L}_X$$

In particular, each $\mathcal{L}_{\mathbf{s}}$ *satisfies the conclusions of Theorem 1.5.3.*

1.7 The quantum Lefschetz hyperplane principle

We specialize now to the case where $\mathbf{c}(\cdot\,) = \mathbf{e}(\cdot\,)$, the S^1-equivariant Euler class. The corresponding values of s_k satisfy

$$\lambda + x = \exp\left(\sum_{k\geq 0} s_k\frac{x^k}{k!}\right)$$

so[5]

$$s_k = \begin{cases} \ln\lambda & k=0 \\ \dfrac{(-)^{k-1}(k-1)!}{\lambda^k} & k>0 \end{cases}$$

Corollary 1.7.1. *Let* ρ_i *be the (non-equivariant) Chern roots of* E*. Then*

$$\prod_i \exp\left(-\frac{1}{24}\int_X((\lambda+\rho_i)\ln(\lambda+\rho_i)-(\lambda+\rho_i))c_{D-1}(T_X)\right)\prod_i(\operatorname{sdet}\sqrt{\lambda+\rho_i}\,)^{-\frac{1}{24}}\mathcal{D}_{\mathbf{e}} =$$

$$\prod_i \exp\left(\sum_{m>0}\frac{B_{2m}}{2m(2m-1)}\left(\widehat{\frac{z}{\lambda+\rho_i}}\right)^{2m-1}\right)\prod_i \exp\left(\left(\frac{(\lambda+\rho_i)\ln(\lambda+\rho_i)-(\lambda+\rho_i)}{z}\right)^{\wedge}\right)\mathcal{D}_X$$

Proof. The first exponent on the right-hand side of (1.19) becomes

$$\sum_i\sum_{m>0}\sum_{l\geq 0}\frac{(-)^l(2m+l-2)!}{\lambda^{2m-1+l}}\frac{B_{2m}}{(2m)!}\frac{\rho_i^l}{l!}z^{2m-1}$$

Using the binomial theorem

$$(1+x)^{1-2m} = \sum_{l\geq 0}\frac{(-)^l(2m+l-2)!}{(2m-2)!\,l!}x^l$$

we can write this as

$$\sum_i\sum_{m>0}\frac{B_{2m}}{(2m)(2m-1)}\left(1+\frac{\rho_i}{\lambda}\right)^{1-2m}\left(\frac{z}{\lambda}\right)^{2m-1}$$

[5] As a consequence, we see that we need to further extend our ground ring Λ by $\ln\lambda$.

which is

$$\sum_i \sum_{m>0} \frac{B_{2m}}{(2m)(2m-1)} \left(\frac{z}{\lambda+\rho_i}\right)^{2m-1}$$

The second exponent on the right-hand side of (1.19) is

$$\begin{aligned}
\frac{1}{z}\sum_{l>0} s_{l-1}\,\mathrm{ch}_l(E) &= \frac{1}{z}\sum_i \left(\rho_i \ln\lambda + \sum_{l\geq 2} \frac{(-)^l (l-2)!}{\lambda^{l-1}} \frac{\rho_i^l}{l!}\right) \\
&= \frac{1}{z}\sum_i \left(\rho_i \ln\lambda + \sum_{k\geq 1} \frac{(-)^{k+1}}{k(k+1)} \frac{\rho_i^{k+1}}{\lambda^k}\right) \\
&= \frac{1}{z}\sum_i \int_0^{\rho_i} \ln(\lambda + x)\, dx \\
&= \frac{1}{z}\sum_i [(\lambda+x)\ln(\lambda+x) - (\lambda+x)]_0^{\rho_i}
\end{aligned}$$

This converges in the $1/\lambda$-adic topology. We may discard the constant terms $(\lambda\ln\lambda - \lambda)/z$ as we know from Example 1.3.3.2 that the string operator $\widehat{1/z}$ annihilates $\mathcal{D}_X$. The Corollary follows. □

Corollary 1.7.2. *The Lagrangian cone $\mathcal{L}_{\mathbf{e}} \subset \mathcal{H}$ is obtained from $\mathcal{L}_X$ by multiplication (in $\mathcal{H}$) by the product over Chern roots ρ_i of*

$$\gamma_\rho(z) = \exp\left(\frac{(\lambda+\rho)\ln(\lambda+\rho) - (\lambda+\rho)}{z} + \sum_{m>0} \frac{B_{2m}}{2m(2m-1)} \left(\frac{z}{\lambda+\rho}\right)^{2m-1}\right)$$

Now

$$\ln\Gamma(x) \sim (x - \tfrac{1}{2})\ln x - x + \frac{1}{2}\ln 2\pi + \sum_{m>0} \frac{B_{2m}}{2m(2m-1)} \frac{1}{x^{2m-1}} \qquad \text{as } x\to\infty, |\arg x| < \pi$$

(see *e.g.* [1]) and so $\gamma_\rho(z)$ coincides, up to some differences in the principal term, with the asymptotic expansion of the gamma function $\Gamma((\lambda+\rho)/z)$. More precisely, it coincides with the stationary phase asymptotics of the integral

$$\frac{1}{\sqrt{2\pi z(\lambda+\rho)}} \int_0^\infty e^{\frac{-x+(\lambda+\rho)\ln x}{z}}\, dx$$

near the critical point $x = \lambda + \rho$ of the phase function.

Let us assume now that E is the direct sum of line bundles, so that the Chern roots ρ_i of E lie in $H^2(X;\mathbb{Z})$. Consider the J-function

$$J_X(t,z) = \sum_d J_d(t,z) Q^d$$

and introduce the following hypergeometric modification of J_X:

$$I_E(t,z) = \sum_d J_d(t,z) Q^d \prod_i \frac{\prod_{k=-\infty}^{\langle \rho_i, d\rangle} (\lambda + \rho_i + kz)}{\prod_{k=-\infty}^{0} (\lambda + \rho_i + kz)}$$

Due to our choice of topology on Λ, this gives a well-defined element of $\mathcal{H}$ for each $t \in H^\star(X)$.

Theorem 1.7.3. *The family*

$$t \mapsto I_E(t,-z) \qquad t \in H^\star(X,\Lambda)$$

of vectors in $(\mathcal{H}_{\mathbf{e},E}, \Omega_{\mathbf{e},E})$ *lies on the Lagrangian submanifold* $\mathcal{L}^{\text{nat}}_{\mathbf{e},E}$ *defined by the differential of the twisted genus-*0 *descendent potential.*

Proof. This is equivalent to the assertion that the family

$$t \mapsto \sqrt{\mathbf{e}(E)}\, I_E(t,-z) \qquad t \in H^\star(X,\Lambda)$$

lies on $\mathcal{L}_{\mathbf{e}} \subset (\mathcal{H}, \Omega)$. Thus we need to show that

$$\sqrt{\mathbf{e}(E)}\, I_E(t,-z) \in \Big(\prod_i \gamma_{\rho_i}(z)\Big) \mathcal{L}_X \qquad \forall t \in H^\star(X;\Lambda)$$

or in other words that

$$\Big(\prod_i \gamma_{\rho_i}(-z)\Big) \sqrt{\mathbf{e}(E)}\, I_E(t,-z) \in \mathcal{L}_X$$

But

$$\Big(\prod_i \gamma_{\rho_i}(z)\Big) \sqrt{\mathbf{e}(E)}\, I_E(t,z)$$

is equal to (the asymptotic expansion of)

$$\sqrt{\mathbf{e}(E)} \sum_d J_d(t,z) Q^d \prod_i \frac{1}{\sqrt{2\pi z(\lambda+\rho_i)}} \int_0^\infty e^{\frac{-x_i + (\lambda+\rho_i)\ln x_i}{z}} dx_i \frac{\prod_{k=-\infty}^{\langle \rho_i, d\rangle} (\lambda + \rho_i + kz)}{\prod_{k=-\infty}^{0} (\lambda + \rho_i + kz)}$$

$$= \sum_d J_d(t,z) Q^d \prod_i \frac{1}{\sqrt{2\pi z}} \int_0^\infty e^{\frac{-x_i + (\lambda+\rho_i+\langle \rho_i, d\rangle)\ln x_i}{z}} dx_i \qquad \text{(integrating by parts)}$$

$$= \sum_d \prod_i \frac{1}{\sqrt{2\pi z}} \int_0^\infty dx_i\, e^{\frac{-x_i + (\lambda+\rho_i)\ln x_i}{z}} x_i^{\langle \rho_i, d\rangle / z} Q^d J_d(t,z)$$

Using the string equation (see Example 1.3.3.2) and the divisor equation (see Example 1.3.3.3) we find that

$$\prod_i e^{\frac{(\lambda+\rho_i)\ln x_i}{z}} \sum_d x_i^{\langle \rho_i, d\rangle/z} Q^d J_d(t,z) = J_X\Big(t + \sum_i (\lambda+\rho_i)\ln x_i, z\Big)$$

and so

$$\Big(\prod_i \gamma_{\rho_i}(z)\Big)\sqrt{\mathbf{e}(E)}\, I_E(t,z) = \Big(\prod_i \frac{1}{\sqrt{2\pi z}}\int_0^\infty dx_i\Big) e^{-\sum x_i/z} J_X\Big(t + \sum_i (\lambda+\rho_i)\ln x_i, z\Big)$$

We need to show that this belongs to the cone determined by the family

$$t \mapsto J_X(t,z) \qquad t \in H^\star(X,\Lambda)$$

In fact, we will show

Claim. *For each $t \in H^\star(X)$ there exists $t^* \in H^\star(X)$ such that the element*

$$\Big(\prod_i \gamma_{\rho_i}(z)\Big)\sqrt{\mathbf{e}(E)}\, I_E(t,z)$$

differs from

$$\lambda^{(\dim E)/2} J_X(t^*, z)$$

by a linear combination of first derivatives of J_X at t^* with coefficients in $z\Lambda[\![z]\!]$ which converge in the sense of Section 1.3.2.

This will follow from the fact that J_X is the generator for the quantum $\mathcal{D}$-module [28] of X. In other words, J_X satisfies the system of partial differential equations

$$z\frac{\partial}{\partial t^\alpha}\frac{\partial}{\partial t^\beta} J_X(t,z) = A_{\alpha\beta}{}^\gamma(t)\frac{\partial}{\partial t^\gamma} J_X(t,z) \tag{1.27}$$

where $A_{\alpha\beta}{}^\gamma$ are the structure constants of the quantum cohomology algebra

$$\phi_\alpha \bullet \phi_\beta = A_{\alpha\beta}{}^\gamma \phi_\gamma$$

We know that

$$\begin{aligned}\Big(\prod_i \gamma_{\rho_i}(z)\Big)\sqrt{\mathbf{e}(E)}\, I_E(t,z) &= \Big(\prod_i \frac{1}{\sqrt{2\pi z}}\int_0^\infty dx_i\Big) e^{-\sum x_i/z} J_X\Big(t + \sum_i (\lambda+\rho_i)\ln x_i, z\Big) \\ &= \prod_i \Big(\frac{1}{\sqrt{2\pi z}}\int_0^\infty dx_i\, e^{-x_i/z + \ln x_i(\lambda\partial_1 + \partial_{\rho_i})}\Big) J_X(t,z)\end{aligned}$$

where ∂_v denotes the derivative in the direction of $v \in H^\star(X)$. Since $z\partial_1 J_X = J_X$, this is

$$\prod_i \left(\frac{1}{\sqrt{2\pi z}} \int_0^\infty dx_i \, e^{\frac{-x_i + \ln x_i (\lambda + z\partial_{\rho_i})}{z}} \right) J_X(t,z) \tag{1.28}$$

We can evaluate (1.28) using the relations (1.27) in the $\mathcal{D}$-module generated by $J_X(t,z)$. These relations imply that

$$(z\partial_{v_1}) \dots (z\partial_{v_n}) \, J_X(t,z) = (z\partial_{v_1 \bullet \dots \bullet v_n}) \, J_X(t,z) + o(z) \tag{1.29}$$

where $o(z)$ denotes a linear combination of $z\partial_{\phi_\alpha} J_X(t,z)$ with coefficients in $z\Lambda[\![z]\!]$, convergent in the above-mentioned sense. We take the asymptotic expansion of the oscillating integral appearing in (1.28) and apply the relations (1.29):

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi z}} \int_0^\infty dx \, e^{(-x + \ln x (\lambda + z\partial_\rho))/z} J_X(t,z) \\
&\quad \sim \sqrt{\lambda + z\partial_\rho} \, \left(e^{\frac{(\lambda + z\partial_\rho)\ln(\lambda + z\partial_\rho) - (\lambda + z\partial_\rho)}{z} + \sum_{m>0} \frac{B_{2m}}{2m(2m-1)} \left(\frac{z}{\lambda + z\partial_\rho} \right)^{2m-1}} \right) J_X(t,z) \\
&\quad = \sqrt{\lambda + z\partial_\rho} \, \left(e^{\frac{(\lambda + z\partial_\rho)\ln(\lambda + z\partial_\rho) - (\lambda + z\partial_\rho)}{z}} \right) \left(J_X(t,z) + \frac{o(z)}{z} \right) \\
&\quad = \sqrt{\lambda + z\partial_\rho} \, \left(e^{\partial_{[(\lambda + \rho\bullet)\ln(\lambda + \rho\bullet) - (\lambda + \rho\bullet)]1}} \right) \left(J_X(t,z) + \frac{o(z)}{z} \right)
\end{aligned}$$

But

$$\sqrt{\lambda + z\partial_\rho} = \sqrt{\lambda} \left(1 + \frac{1}{2\lambda}(z\partial_\rho) - \frac{1}{8\lambda^2}(z\partial_\rho)^2 + \dots \right)$$

so

$$\frac{1}{\sqrt{2\pi z}} \int_0^\infty dx \, e^{\frac{-x + \ln x (\lambda + z\partial_\rho)}{z}} J_X(t,z) = \sqrt{\lambda} \left(e^{\partial_{[(\lambda + \rho\bullet)\ln(\lambda + \rho\bullet) - (\lambda + \rho\bullet)]1}} \right) \left(J_X(t,z) + \frac{o(z)}{z} \right)$$

Applying this to (1.28), we find that

$$\prod_i \left(\frac{1}{\sqrt{2\pi z}} \int_0^\infty dx_i \, e^{\frac{-x_i + \ln x_i (\lambda + z\partial_{\rho_i})}{z}} \right) J_X(t,z)$$

is equal to

$$\lambda^{(\dim E)/2} J_X(t^*, z) + C^\alpha(t^*, z) z\partial_{\phi_\alpha} J_X(t^*, z)$$

where

$$t^* = t + \sum_i [(\lambda + \rho_i\bullet)\ln(\lambda + \rho_i\bullet) - (\lambda + \rho_i\bullet)]1$$

and the coefficients $C^\alpha(t^*, z)$ are appropriately convergent elements of $\Lambda[\![z]\!]$. This proves the Claim. Since the cone $\mathcal{L}_X$ is ruled by $zT_{\mathfrak{f}}\mathcal{L}_X$, the Theorem follows. □

Let L_t denote the tangent space to $\mathcal{L}^{\text{nat}}_{\mathbf{e},E}$ at the point $I_E(t,-z)$. Since

$$I_E(t,z) \equiv J_X(t,z) \bmod Q$$

the argument of section 1.5.1 shows that the family

$$t \mapsto I_E(t,-z) \qquad t \in H^\star(X,\Lambda)$$

is transverse to the ruling of $\mathcal{L}^{\text{nat}}_{\mathbf{e},E}$ by zL_t. Thus zL_t meets the slice $-z + z\mathcal{H}_-$ at a unique point.

Corollary 1.7.4. *The intersection of zL_t with $-z + z\mathcal{H}_-$ coincides with the value*

$$J_{\mathbf{e},E}(\tau(t),-z) \in -z + \tau(t) + \mathcal{H}_-$$

where $J_{\mathbf{e},E}$ is the $(\mathbf{e},E)$-twisted J-function (see section 1.6.1). In other words

$$J_{\mathbf{e},E}(\tau,z) = I_E(t,z) + C^\alpha(t,z)\, z\, \partial_{\phi_\alpha} I_E(t,z) \tag{1.30}$$

where the $C^\alpha(t,z)$ are appropriately convergent elements of $\Lambda[\![z]\!]$ and $\tau(t)$ is determined by the asymptotics $z + \tau + O(z^{-1})$ of the right-hand side of (1.30).

Remarks:

(i) This procedure for computing $J_{\mathbf{e},E}$ from I_E is reminiscent of Birkhoff factorization in the theory of loop groups. Indeed, the procedure applied to the first derivatives of I_E rather than I_E actually is an example of Birkhoff factorization.

(ii) The Corollary gives a geometrical description of the "mirror map" $t \mapsto \tau$: the J-function obtained as the intersection $L_t \cap (-z + z\mathcal{H}_-)$ comes naturally parameterized by t which may have little common with the projections $\tau - z$ of the intersection points along $\mathcal{H}_-$.

1.7.1 Mirror theorems

Let us assume now that E is a direct sum of convex line bundles (a line bundle F over X is convex if $H^1(C, f^\star F) = 0$ for all genus-0 stable maps $f : C \to X$). Let $j : Y \hookrightarrow X$ be the inclusion into X of a complete intersection Y cut out by a generic global section of

E. We will deduce the relationship between Gromov–Witten invariants of X and of Y by taking the non-equivariant limit $\lambda \to 0$ in Corollary 1.7.4. Although the proof of Theorem 1.7.3 fails when $\lambda = 0$, the statement of Corollary 1.7.4 survives: both I_E and $J_{\mathbf{e},E}$ have non-equivariant limits, and the relationship between them is that described by Corollary 1.7.4.

We can write

$$J_{\mathbf{e},E}(t,z) = z + t + \sum_{n,d} \frac{Q^d}{n!} (\mathrm{ev}_{n+1})_\star \left[\mathrm{ev}_1^\star t \wedge \ldots \wedge \mathrm{ev}_n^\star t \wedge \frac{\mathbf{e}(E'_{0,n+1,d})}{z - \psi_{n+1}} \right]$$

where $(\mathrm{ev}_{n+1})_\star$ denotes the cohomological pushforward along $\mathrm{ev}_{n+1} : X_{g,n+1,d} \to X$ and $E'_{0,n+1,d}$ is the kernel of the evaluation map

$$E_{0,n+1,d} = H^0(C, f^\star E) \to E$$

at the $(n+1)$st marked point. In the non-equivariant limit, $J_{\mathbf{e},E}(t,z)$ degenerates to

$$J_{X,Y}(t,z) = z + t + \sum_{n,d} \frac{Q^d}{n!} (\mathrm{ev}_{n+1})_\star \left[\mathrm{ev}_1^\star t \wedge \ldots \wedge \mathrm{ev}_n^\star t \wedge \frac{\mathbf{e}(E'_{0,n+1,d})}{z - \psi_{n+1}} \right]$$

where $\mathbf{e}$ denotes now the non-equivariant Euler class. The function $J_{X,Y}$ encodes Gromov–Witten invariants of Y via

$$\mathbf{e}(E) J_{X,Y}(u,z) =_{H_2(Y) \to H_2(X)} j_\star J_Y(j^\star u, z) \tag{1.31}$$

since $[Y_{0,n+1,d}] = \mathbf{e}(E_{0,n+1,d}) \cap [X_{0,n+1,d}]$ (see [36]). The long subscript here indicates that corresponding homomorphism between Novikov rings should be applied to the right-hand side of the equation.

The non-equivariant limit of $I_E(t,z)$ is

$$I_{X,Y}(t,z) = \sum_d J_d(t,z) Q^d \prod_i \prod_{k=1}^{\langle \rho_i, d\rangle} (\rho_i + kz)$$

where, as before, $\{\rho_i\}$ are the Chern roots of E.

Corollary 1.7.5. *The series $I_{X,Y}(t,-z)$ and $J_{X,Y}(\tau,-z)$ determine the same cone. In particular, $J_{X,Y}(\tau,-z)$ is determined from $I_{X,Y}(t,-z)$ by the "Birkhoff factorization" procedure followed by the mirror map $t \mapsto \tau$ as described in Corollary 1.7.4.*

Now, restricting $I_{X,Y}$ and $J_{X,Y}$ to the small parameter space $H^{\leq 2}(X;\Lambda)$ and assuming that $c_1(E) \leq c_1(TX)$ we can derive the quantum Lefschetz theorems of [35, 4, 43, 21, 9].

Proposition 1.7.6. *If $c_1(E) \leq c_1(X)$ then, for $t \in H^{\leq 2}(X;\Lambda)$*

$$I_{X,Y}(t,z) = zF(t) + \sum_i G^i(t)\phi_i + O(z^{-1})$$

for some scalar-valued functions $F(t)$ and $G^i(t)$ with $F(t)$ invertible. The $\{\phi_i\}$ here are a basis for $H^{\leq 2}(X)$.

Proof. Since $I_{X,Y}(t,z) \equiv J_X(t,z) \bmod Q$,

$$I_{X,Y}(t) = z + t + \sum_{d>0} Q^d J_d(t,z) \prod_i \prod_{k=1}^{\langle \rho_i, d\rangle} (\rho_i + kz) + O(z^{-1})$$

We need to work out the highest power of z occurring in $J_d(t,z)$. But, for $d > 0$,

$$J_d(t,z) = \sum_{n,k} \frac{1}{n!} \left\langle \overbrace{t,\ldots,t}^{n}, \frac{\phi_\alpha \psi^k}{z^{k+1}} \right\rangle_{0,n+1,d} \phi^\alpha \tag{1.32}$$

and the highest power of z occurs here with the lowest power of ψ. In other words, we should take the degree of t equal to 2 and the degree of ϕ_α equal to $D = \dim X$. The maximum power of z occurring is $-(k+1)$ where

$$\begin{aligned} n + D + k &= \dim_{\mathbb{C}} X_{0,n+1,d} \\ &= n + D - 2 + \langle c_1(TX), d\rangle \end{aligned}$$

Thus the highest power of z occurring in

$$J_d(t,z) \prod_i \prod_{k=1}^{\langle \rho_i, d\rangle} (\rho_i + kz)$$

is

$$1 + \langle c_1(E), d\rangle - \langle c_1(TX), d\rangle$$

which is at most 1. For this to equal 1, we need ϕ_α in (1.32) to be a volume form, in which case ϕ^α has degree zero. Similarly, for z^0 to occur we need $\deg \phi_\alpha \geq 2D - 2$, in which case ϕ^α has degree at most 2. Thus

$$I_{X,Y}(t,z) = zF(t) + \sum_i G^i(t)\phi_i + O(z^{-1})$$

where $F(t)$ and $G^i(t)$ are scalar-valued functions such that

$$F(t) \equiv 1 \bmod Q$$
$$G^i(t) \equiv t^i \bmod Q$$

□

We see from the proof that if $c_1(E) - c_1(TX)$ is sufficiently negative then $F(t) = 1$ and $G^i(t) = t^i$.

Corollary 1.7.7. *When $c_1(E) \leq c_1(TX)$, the restriction of $J_{X,Y}(\tau, z)$ to the small parameter space $\tau \in H^2(X;\Lambda)$ is given by*

$$J_{X,Y}(\tau, z) = \frac{I_{X,Y}(t,z)}{F(t)}$$

where

$$\tau = \sum_i \frac{G^i(t)}{F(t)}\phi_i$$

The J-function of $X = \mathbb{C}P^{n-1}$ restricted to the small parameter space $t_0 + tP$, where P is the hyperplane class generating the algebra $H^*(X;\Lambda) = \Lambda[P]/(P^n)$, takes the form

$$J_X = z\, e^{(t_0+Pt)/z} \sum_{d\geq 0} \frac{Q^d e^{dt}}{\prod_{k=1}^d (P+kz)^n}$$

For a hypersurface Y of degree l in $\mathbb{C}P^{n-1}$ we then have

$$I_{X,Y} = ze^{(t_0+Pt)/z} \sum_{d\geq 0} Q^d e^{dt} \frac{\prod_{k=1}^{ld}(lP+kz)}{\prod_{k=1}^d (P+kz)^n}$$

Corollary 1.7.8. *On the small parameter space*

(i) *for $l < n-1$,*

$$J_{X,Y}(t_0, t, z) = I_{X,Y}(t_0, t, z)$$

(ii) *for $l = n-1$,*

$$J_{X,Y}(\tau_0, t, z) = I_{X,Y}(t_0, t, z)$$

where $\tau_0 = t_0 + l!\, Qe^t$.

(iii) *for* $l = n$,

$$J_{X,Y}(t_0, \tau, z) = I_{X,Y}(t_0, t, z)/F(t)$$

where $\tau = G(t)/F(t)$ *and the series* F *and* G *are found from the expansion* $I_{X,Y} = \exp(t_0/z)(zF + GP + O(z^{-1}))$.

Taking $n = l = 5$ and applying the relation (1.31) we recover the quintic mirror formula of Candelas *et al.* [11].

1.8 Quantum Serre duality

Consider again the general situation where $E \to X$ is a holomorphic vector bundle with Chern roots ρ_i and

$$\mathbf{c}(\,\cdot\,) = \exp\Big(\sum_k s_k \operatorname{ch}_k(\,\cdot\,)\Big)$$

is a multiplicative characteristic class. Put

$$\mathbf{c}^*(\,\cdot\,) = \exp\Big(\sum_k (-)^{k+1} s_k \operatorname{ch}_k(\,\cdot\,)\Big)$$

so that in particular

$$\mathbf{c}^*(E^*) = \frac{1}{\mathbf{c}(E)}$$

Despite the fact that there is no obvious relationship between $\mathbf{c}^*((E^*)_{g,n,d})$ and $\mathbf{c}(E_{g,n,d})$, the twisted descendent potentials $\mathcal{D}_{\mathbf{c},E}$ and $\mathcal{D}_{\mathbf{c}^*,E^*}$ are closely related.

Corollary 1.8.1. *We have*

$$\mathcal{D}_{\mathbf{c}^*,E^*} = (\operatorname{sdet}(\mathbf{c}(E)))^{-1/24}\mathcal{D}_{\mathbf{c},E}$$

More explicitly,

$$\mathcal{D}_{\mathbf{c}^*,E^*}(\mathbf{t}^*) = (\operatorname{sdet}(\mathbf{c}(E)))^{-\frac{1}{24}}\mathcal{D}_{\mathbf{c},E}(\mathbf{t})$$

where $\mathbf{t}^*(z) = \mathbf{c}(E)\mathbf{t}(z) + (1 - \mathbf{c}(E))z$.

Proof. Replacing $\operatorname{ch}_l(E)$ with $(-1)^l \operatorname{ch}_l(E)$, and s_k with $(-1)^{k+1} s_k$ in Theorem 1.6.4 preserves all terms except the super-determinant. □

Corollary 1.8.2. *Consider the dual bundle E^* equipped with the dual S^1-action, and the S^1-equivariant inverse Euler class $\mathbf{e}^{-1}$. Put*

$$\mathbf{t}^*(z) = z + (-1)^{\dim E/2}\mathbf{e}(E)(\mathbf{t}(z) - z)$$

and introduce the change $\pm : Q^d \mapsto Q^d(-1)^{\langle c_1(E),d\rangle}$ in the Novikov ring. With this notation

$$\mathcal{D}_{\mathbf{e}^{-1},E^*}(\mathbf{t}^*, Q) = \operatorname{sdet}((-1)^{(\dim E)/2}\mathbf{e}(E))^{-\frac{1}{24}}\mathcal{D}_{\mathbf{e},E}(\mathbf{t}, \pm Q)$$

Proof. We have

$$\mathbf{e}^{-1}(E^*) = \prod_i(-\lambda - \rho_i)^{-1}$$

Since

$$(-\lambda + x)^{-1} = \exp\Big(-\ln(-\lambda) + \sum_k \frac{x^k}{k\lambda^k}\Big)$$

we find that

$$\mathbf{e}^{-1}(\,\cdot\,) = \exp\Big(\sum s_k^* \operatorname{ch}_k(\,\cdot\,)\Big)$$

where

$$s_k^* = \begin{cases} -\ln(-\lambda) & k = 0 \\ \dfrac{(k-1)!}{\lambda^k} & k > 0 \end{cases}$$

For $k > 0$ we have $s_k^* = (-1)^{k+1}s_k$ as in Corollary 1.8.1. However, $s_0^* = -s_0 - \pi\sqrt{-1}$. Examining Theorem 1.6.4, we see that s_0 occurs on the right-hand side of (1.19) only in the form $\exp\,(s_0\rho/z)^\wedge$ where $\rho = \operatorname{ch}_1(E)$. Example 1.3.3.3 therefore implies that the action of the s_0-flow can be absorbed by the change

$$Q^d \mapsto Q^d \exp(s_0\langle\rho, d\rangle) \tag{1.33}$$

together with multiplication of $\mathcal{D}_X$ by the factor $\exp(s_0(\dim E)/48)$ coming from the superdeterminant. In our case, where we need to move along the s_0-flow for time $-\pi\sqrt{-1}$, (1.33) becomes the change $Q^d \mapsto \pm Q$. $\square$

Chapter 2

Quantum extraordinary cohomology

2.1 Introduction

Given the spectacular progress in enumerative geometry associated with the study of quantum cohomology, it is natural to ask whether one can obtain more detailed enumerative information by studying the extraordinary cohomology of moduli spaces of stable maps. The goal of this chapter is to define quantum extraordinary cohomology — a collection of invariants of a Kähler manifold X which encodes information about the extraordinary cohomology of the spaces $X_{g,n,d}$ — and to understand the relationship between this and the usual quantum cohomology of X. The main result of this chapter, Theorem 2.4.1, expresses the extraordinary descendent potential for complex cobordism (and hence that for any other complex-oriented cohomology theory) in terms of the cohomological descendent potential.

As we have seen in chapter 0, this determines all tangent-twisted Gromov—Witten invariants of X — Gromov–Witten invariants of X twisted by characteristic classes of the virtual tangent bundles $\mathcal{T}^{vir}_{g,n,d}$ — in terms of untwisted Gromov–Witten invariants. The relationship is formulated in terms of an extension of the quantization formalism. In particular, it implies that each genus-0 extraordinary descendent potential of X can be encoded by a semi-infinite ruled cone in the corresponding extraordinary cohomology groups of X with coefficients in certain Laurent series in $1/z$. As in chapter 1, our main technical tool will be the Grothendieck–Riemann–Roch theorem, which we apply to various calculations on the universal family over the moduli space of stable maps.

The material of this chapter represents joint work with Givental. The chapter is organized as follows. In section 2.2 we recall various facts about complex-oriented cohomology theories and fix the notation involved. In section 2.3 we define quantum extraordinary cohomology and extend the quantization formalism to this setting. In section 2.4, we formulate the main result of this chapter, Theorem 2.4.1, and various corollaries of it. The proof of Theorem 2.4.1 is contained in section 2.5.

2.2 Complex-oriented cohomology theories

In this section we collect various standard results about complex-oriented cohomology theories which we will need below. Good references for this material include [2] and Appendix 4 of [63]. A complex-oriented cohomology theory is a multiplicative cohomology theory $E^\star$ together with a choice of element $u \in \widetilde{E}^2(\mathbb{C}P^\infty)$ such that if $j : \mathbb{C}P^1 \to \mathbb{C}P^\infty$ is the inclusion map then $j^\star u$ is the standard generator for $\widetilde{E}^2(\mathbb{C}P^1)$. The element u is called the orientation. We write $\Omega_E^\star = E^\star(pt)$. For any space X, the map $X \to pt$ makes $E^\star(X)$ into a module over $\Omega_E^\star$.

Given a complex-oriented cohomology theory (E, u) we can construct Chern classes in the usual way; the first Chern class of the universal line bundle over $\mathbb{C}P^\infty$ is u, and

$$E^\star(\mathbb{C}P^\infty) \cong \Omega_E^\star[[u]]$$

The operation of tensor product of complex line bundles equips $E^\star(\mathbb{C}P^\infty)$ with the structure $F(u, v) \in \Omega_E^\star[[u, v]]$ of a formal group over $\Omega_E^\star$. The inversion $u \mapsto u^\star$ in this formal group is induced by inversion of complex line bundles.

Example 2.2.0.1 Take $E^\star(X) = H^\star(X; \mathbb{C})$ and u to be the usual first Chern class of the universal line bundle ξ over $\mathbb{C}P^\infty$. Then $F(u, v) = u + v$, and $u^\star = -u$. ⋄

Example 2.2.0.2 Take $E^\star(X) = K^\star(X; \mathbb{Z}) \otimes \mathbb{C}$ and let $u = 1 - \xi^{-1} \in \widetilde{E}^0(\mathbb{C}P^\infty)$. (The orientation can be regarded as lying in $\widetilde{E}^2(\mathbb{C}P^\infty)$ via Bott periodicity.) Then $F(u,v) = u+v-uv$, and $u^\star = -u-u^2-u^3-\dots$ ◇

We will always assume that the ground ring $\Omega_E^\star$ contains $\mathbb{C}$, so there is a logarithm $g_E \in \Omega_E^\star[[u]]$ such that

$$g_E(u) = u + \sum_{i>0} \beta_i u^{i+1} \qquad \beta_i \in \Omega_E^{-2i}$$

and

$$g_E(F(u,v)) = g_E(u) + g_E(v)$$

This logarithm is unique, and it determines the complex-oriented cohomology theory. We write $u_E(z) \in \Omega_E^\star[[z]]$ for the power series inverse to g_E.

The Chern–Dold character [10]

$$\mathrm{ch}_E : E^\star(\,\cdot\,) \to H^\star(\,\cdot\,;\Omega_E^\star)$$

is the unique multiplicative natural transformation from $E^\star(\,\cdot\,)$ to $H^\star(\,\cdot\,;\Omega_E^\star)$ which is the identity map on $\Omega_E^\star$. If z is the standard orientation of $H^\star(\,\cdot\,;\Omega_E^\star)$ and u_E is the orientation of $E^\star(\,\cdot\,)$ then $\mathrm{ch}_E(u_E) = u_E(z)$. Given a proper l.c.i. map of quasi-projective schemes $f : X \to Y$, Baum, Fulton and MacPherson have constructed [5] a push-forward $f_\star : E^\star(X) \to E^\star(Y)$. Their construction is functorial, and satisfies

$$\begin{array}{ccc} E^\star(X) & \xrightarrow{\mathrm{ch}_E(\,\cdot\,)\,\mathrm{Td}_E(T_f)} & H^\star(X,\Omega_E^\star) \\ f_\star \downarrow & \square & \downarrow f_\star \\ E^\star(Y) & \xrightarrow{\mathrm{ch}_E(\,\cdot\,)} & H^\star(Y,\Omega_E^\star) \end{array} \tag{RR}$$

where $T_f \in K^0(X)$ is the l.c.i. virtual tangent bundle of f and $\mathrm{Td}_E(\,\cdot\,)$ is the multiplicative characteristic class (with values in $H^\star(\,\cdot\,;\Omega_E^\star)$) which on a line bundle L with (cohomological) first Chern class ρ takes the value

$$\mathrm{Td}_E(L) = \frac{\rho}{u_E(\rho)}$$

In order to work with many complex-oriented cohomology theories at once, we consider complex cobordism $MU^\star$ equipped with the standard orientation u [2, page 38]. $MU^\star(pt)$ is a polynomial algebra [55, 64] on generators $p_1, p_2, \dots$ of degree $-2, -4, \dots$, where p_i is Poincaré-dual (see page 22) to $[\mathbb{C}P^i \to pt]$. Since we consider only complex-oriented cohomology theories with ground rings that contain $\mathbb{C}$, we tensor with $\mathbb{C}$ throughout. This gives $\Omega^\star_{MU} = \mathbb{C}[p_1, p_2, \ldots]$. Complex cobordism is universal among complex-oriented cohomology theories: given a complex-oriented cohomology theory (E, u_E) there is a unique multiplicative natural transformation $\theta_E : MU \to E$ such that $\theta_E(u) = u_E$. For any space X, $MU^\star(X)$ defines a sheaf on $\mathrm{Spec}\,\Omega^\star_{MU}$. The natural transformation $\theta_E : MU \to E$ gives a map

$$\tilde{\theta}_E : \mathrm{Spec}\,\Omega^\star_E \to \mathrm{Spec}\,\Omega^\star_{MU}$$

and the pullback of the sheaf $MU^\star(X)$ by $\tilde{\theta}_E$ is $E^\star(X)$.

Since $\Omega^\star_{MU} \cong \mathbb{C}[p_1, p_2, \ldots]$, the p_i give co-ordinates on $\mathrm{Spec}\,\Omega^\star_{MU}$. Using Miščenko's formula for the logarithm in complex cobordism

$$g(u) = u + \sum_{n>0} \frac{p_n}{n+1} u^n$$

we see that if we define $s_1, s_2, \ldots$ by

$$\exp\left(\sum_{k>0} s_k \frac{x^k}{k!}\right) = \frac{x}{u(x)}$$

then $\Omega^\star_{MU} \cong \mathbb{C}[s_1, s_2, \ldots]$. We take x to have degree 2 in the above formula, so $\deg s_i = -2i$. The s_i give another co-ordinate system on $\Omega^\star_{MU}$, which we make extensive use of below; we write $\mathbf{s} = (s_1, s_2, \ldots)$ throughout.

2.3 Quantum extraordinary cohomology

In view of the Riemann–Roch formula (RR), we define the genus-g extraordinary descendent potential $\mathcal{F}^g_E$ of X to be

$$\mathcal{F}^g_E(t_0, t_1, \ldots) = \sum_{\substack{d \in H_2(X;\mathbb{Z}) \\ n \geq 0}} \frac{Q^d}{n!} \int_{[X_{g,n,d}]} \bigwedge_{i=1}^{i=n} \Big(\sum_{k_i \geq 0} \mathrm{ch}_E(\mathrm{ev}_i^\star\, t_{k_i}) \wedge u_E(\psi_i)^{k_i} \Big) \wedge \mathrm{Td}_E(\mathcal{T}^{vir}_{g,n,d}) \quad (2.1)$$

Here $t_0, t_1, \ldots \in E^\star(X)$ are extraordinary cohomology classes on X and $\mathcal{T}^{vir}_{g,n,d}$ is the virtual tangent bundle to $X_{g,n,d}$ (see pages 20–21). We regard $\mathcal{F}^g_E$ as a formal function of

$$\mathbf{t}(u) = t_0 + t_1 u + t_2 u^2 + \ldots \in E^\star(X)[[u]]$$

which takes values in $\Omega_E^\star[[Q]]$. The total extraordinary descendent potential

$$\mathcal{D}_E = \exp\Big(\sum_{g\geq 0} \hbar^{g-1}\mathcal{F}_E^g\Big)$$

is a generating function for E-valued Gromov–Witten invariants of all genera. The argument of Lemma 1.3.1 shows that this is well-defined as a formal function of $\mathbf{t}$ which takes values in $\Omega_E^\star[[Q]][[\hbar,\hbar^{-1}]]$. The $\Omega_E^\star$-module $E^\star(X)[[u]]$ defines a sheaf on $\operatorname{Spec}\Omega_E^\star$, and $\mathcal{D}_E$ is a function on a formal neighbourhood of the zero section of this sheaf. The pullback of the total cobordism potential $\mathcal{D}_{MU}$ by the map $\tilde{\theta}_E : \operatorname{Spec}\Omega_E^\star \to \operatorname{Spec}\Omega_{MU}^\star$ coincides with $\mathcal{D}_E$, so we can think of $\mathcal{D}_{MU}$ as a family of functions (depending on $s_1, s_2, \ldots \in \Omega_{MU}^\star$) which encodes the extraordinary descendent potentials for all complex-oriented cohomology theories.

2.3.1 Aside: quantum K-theory

Note that if we take the complex-oriented cohomology theory E to be complex K-theory then our E-valued Gromov-Witten invariants do not coincide with the K-theoretic Gromov-Witen invariants of Givental and Lee [29, 41, 42]. In essence, this is because we deal with the K-theory of the moduli spaces $X_{g,n,d}$ as topological spaces, whereas Givental and Lee consider the orbifold K-theory of $X_{g,n,d}$. For example, to define the K-theoretic correlator

$$\chi(X_{g,n,d}; \operatorname{ev}_1^\star \alpha_1 \otimes L_1^{i_1} \otimes \ldots \otimes \operatorname{ev}_n^\star \alpha_n \otimes L_n^{i_n}) \tag{2.2}$$

where $\alpha_i \in K^0(X)$, they take the orbibundle push-forward of

$$\operatorname{ev}_1^\star \alpha_1 \otimes L_1^{i_1} \otimes \ldots \otimes \operatorname{ev}_n^\star \alpha_n \otimes L_n^{i_n}$$

from $X_{g,n,d}$ to a point. This can be computed using the Kawasaki–Riemann–Roch formula [34]. We take (2.2) to be

$$\int_{[X_{g,n,d}]} \operatorname{ch}(\operatorname{ev}_1^\star \alpha_1 \otimes L_1^{i_1} \otimes \ldots \otimes \operatorname{ev}_n^\star \alpha_n \otimes L_n^{i_n}) \operatorname{Td}(\mathcal{T}_{g,n,d}^{vir}) \tag{2.3}$$

where ch is the usual Chern character and Td is the usual Todd class. This corresponds to taking only the principal term in Kawasaki–Riemann–Roch. In this sense, our K-theoretic correlators give an approximation to the quantum K-theory of [29, 41].

2.3.2 The quantization formalism

The main result of this chapter, Theorem 2.4.1, determines the total cobordism potential $\mathcal{D}_{MU}$ in terms of the usual (cohomological) total descendent potential $\mathcal{D}_X$. The Theorem is formulated in terms of an extension of Givental's quantization formalism to the cobordism-valued setting. In order to make this extension, we need:

(a) to find a symplectic space $\mathcal{U}$ over a ground ring which contains $\Omega^\star_{MU}$ — we can regard this as a family of symplectic spaces $\mathcal{U}_\mathbf{s}$ depending on $\mathbf{s}$ — and a polarization $\mathcal{U} = \mathcal{U}_+ \oplus \mathcal{U}_-$ which identifies $\mathcal{U}$ with $T^\star\mathcal{U}_+$.

(b) to equip $\mathcal{U}$ with the structure of (some completion of) an algebra of Laurent polynomials.

(c) to regard $\mathcal{D}_{MU}$ as function on $\mathcal{U}$, or in other words as a family of elements of the Fock spaces corresponding to $\mathcal{U}_\mathbf{s}$.

(d) to identify the Fock spaces corresponding to different $\mathcal{U}_\mathbf{s}$, so that $\mathcal{D}_{MU}$ gives a family $\mathcal{D}_\mathbf{s}$ of functions in a single Fock space. We can then study how $\mathcal{D}_\mathbf{s}$ varies with $\mathbf{s}$.

Note that (a) and (b) are essential ingredients of the formalism — without (a) there is no quantization and without (b) there is no loop group. In order to achieve both (a) and (b), we will need to restrict attention to a formal neighbourhood of $\mathbf{s} = (0, 0, \ldots)$ in $\operatorname{Spec} \Omega^\star_{MU}$. This corresponds to working in a formal neighbourhood of (usual) cohomology.

We work over the ground ring

$$\widetilde{\Omega}^\star_{MU} = \mathbb{C}[[Q]] \otimes \mathbb{C}[[s_1, s_2, \ldots]]$$

and regard $\mathcal{F}^g_E$ (respectively $\mathcal{D}_E$) as a formal function of

$$\mathbf{t} = t_0 + t_1 u + \ldots \in MU^\star(X; \widetilde{\Omega}^\star_{MU})[[u]]$$

which takes values in $\widetilde{\Omega}^\star_{MU}$ (respectively $\widetilde{\Omega}^\star_{MU}[[\hbar, \hbar^{-1}]]$). We equip $\widetilde{\Omega}^\star_{MU}$ with the topology coming from the norm

$$\|Q^d s_{i_1}^{j_1} \ldots s_{i_n}^{j_n}\| = 2^{-\int_d \omega - i_1 j_1 - \ldots - i_n j_n}$$

where ω is the symplectic form on X. Consider the supervector space

$$\begin{aligned}\mathcal{U} &= \Big\{\sum_{n\in\mathbb{Z}} \alpha_n u^n : \alpha_n \in MU^\star(X;\widetilde{\Omega}^\star_{MU}), \alpha_n \to 0 \text{ as } n \to \infty\Big\} \\ &\subset MU^\star(X;\widetilde{\Omega}^\star_{MU})[\![u,u^{-1}]\!]\end{aligned}$$

where the degree of u is 2, equipped with the even symplectic form

$$\Omega(f_1,f_2) = \frac{1}{2\pi i}\oint (f_1(u^\star), f_2(u))\, dg(u) \tag{2.4}$$

Here $(\cdot\,,\cdot)$ denotes the Poincaré pairing in cobordism and the integral denotes the residue[1] at $u = 0$. Note that since $u^\star$ is a power series in u of degree 2, $f_1(u^\star) \in \mathcal{U}$ whenever $f_1(u) \in \mathcal{U}$. The symplectic form Ω takes values in $\widetilde{\Omega}^\star_{MU}$, and so we can regard it as a family of symplectic structures depending on $\mathbf{s}$.

Darboux co-ordinates and the polarization

Using the Chern–Dold character, we can write the symplectic form Ω in cohomological terms. Let $\mathrm{Td}_{\mathbf{s}}$ be the multiplicative characteristic class with values in $H(\cdot\,,\widetilde{\Omega}^\star_{MU})$ defined by

$$\mathrm{Td}_{\mathbf{s}}\,(\cdot) = \exp\Big(\sum_{k>0} s_k\, \mathrm{ch}_k(\cdot)\Big)$$

It is the composition of Td_{MU} with the inclusion $\Omega^\star_{MU} \to \widetilde{\Omega}^\star_{MU}$. Define an $\widetilde{\Omega}^\star_{MU}$-valued inner product on $MU^\star(X;\widetilde{\Omega}^\star_{MU})$ by

$$(\alpha,\beta)_{\mathbf{s}} = \int_X \mathrm{ch}_{\mathbf{s}}(\alpha) \wedge \mathrm{ch}_{\mathbf{s}}(\beta) \wedge \mathrm{Td}_{\mathbf{s}}\,(TX)$$

where $\mathrm{ch}_{\mathbf{s}} : MU^\star(\cdot\,;\widetilde{\Omega}^\star_{MU}) \to H^\star(\cdot\,,\widetilde{\Omega}^\star_{MU})$ is the map induced by the Chern–Dold character ch_{MU}. Then

$$\Omega(f_1,f_2) = \frac{1}{2\pi i}\oint (f_1(u(-z)), f_2(u(z)))_{\mathbf{s}}\, dz$$

This makes it easy to write down Darboux co-ordinates on $(\mathcal{U},\Omega)$.

Pick a basis $\{\phi_\alpha : \alpha = 1,\ldots,N\}$ for $MU^\star(X;\widetilde{\Omega}^\star_{MU})$ over $\widetilde{\Omega}^\star_{MU}$ and let $g^{\mathbf{s}}_{\alpha\beta} = (\phi_\alpha,\phi_\beta)_{\mathbf{s}}$. Write $g_{\mathbf{s}}^{\alpha\beta}$ for the entries of the matrix inverse to that with entries $g^{\mathbf{s}}_{\alpha\beta}$. Define Laurent series $v_k(u)$, $k = 0,1,2,\ldots$ by

$$\frac{1}{u(-x-y)} = \sum_{k\geq 0} (u(x))^k v_k(u(y))$$

[1]Note that the differential dg occurring in (2.4) is the invariant differential on the formal group $\Omega^\star_{MU}[\![u]\!]$.

where we expand the left-hand side in the region where $|x| < |y|$.

Claim.

$$\mathbf{f} = q_k^\alpha(\mathbf{f})\phi_\alpha u^k + p_l^\beta(\mathbf{f}) g_{\mathbf{s}}^{\beta\epsilon}\phi_\epsilon v_l(u) \qquad \mathbf{f} \in \mathcal{U} \tag{2.5}$$

gives a Darboux co-ordinate system on $\mathcal{U}$.

Proof. Expressions of the form (2.5) certainly lie in $\mathcal{U}$. We have

$$\begin{aligned}
&\sum_{k,l\geq 0} (u(x))^k v_l(u(y)) \Omega(g_{\mathbf{s}}^{\beta\epsilon}\phi_\epsilon v_k(u), \phi_\alpha u^l) \\
&\qquad = \Omega\Big(g_{\mathbf{s}}^{\beta\epsilon}\phi_\epsilon \sum_{k\geq 0}(u(x))^k v_k(u), \phi_\alpha \sum_{l\geq 0} u^l v_l(u(y))\Big) \\
&\qquad = g_{\mathbf{s}}^{\beta\epsilon}(\phi_\epsilon, \phi_\alpha)_{\mathbf{s}} \frac{1}{2\pi i} \oint \Big(\sum_{k\geq 0}(u(x))^k v_k(u(-z))\Big)\Big(\sum_{l\geq 0}(u(z))^l v_l(u(y))\Big)\, dz \\
&\qquad = \frac{\delta_\alpha^\beta}{2\pi i} \oint \frac{1}{u(-x+z)}\frac{1}{u(-z-y)}\, dz
\end{aligned}$$

Here $|x| < |z| < |y|$, so the only pole inside the contour of integration is the simple pole of $1/u(-x+z)$ at $z = x$. Thus

$$\begin{aligned}
\sum_{k,l\geq 0} (u(x))^k v_l(u(y)) \Omega(g_{\mathbf{s}}^{\beta\epsilon}\phi_\epsilon v_k(u), \phi_\alpha u^l) &= \frac{\delta_\alpha^\beta}{u(-x-y)} \\
&= \delta_\alpha^\beta \sum_{m\geq 0} (u(x))^m v_m(y)
\end{aligned}$$

and so

$$\Omega\Big(\frac{\partial}{\partial p_l^\beta}, \frac{\partial}{\partial q_k^\alpha}\Big) = \delta_{\alpha\beta}\delta_{kl} \qquad \text{for all } \alpha, \beta, k, l$$

Also,

$$\Omega\Big(\phi_\alpha \sum_{k\geq 0} v_k(u(x)) u^k, \phi_\beta \sum_{l\geq 0} u^l v_l(u(y))\Big) = g_{\alpha\beta}^{\mathbf{s}} \frac{1}{2\pi i} \oint \frac{1}{u(-x+z)}\frac{1}{u(-z-y)}\, dz$$

where $|z| < |x|$ and $|z| < |y|$. This is zero, as there is no pole inside the contour of integration, and so

$$\Omega\Big(\frac{\partial}{\partial q_k^\alpha}, \frac{\partial}{\partial q_l^\beta}\Big) = 0 \qquad \text{for all } \alpha, \beta, k, l$$

Similarly,

$$\Omega\Big(g_{\mathbf{s}}^{\alpha\epsilon}\phi_\epsilon\sum_{k\geq 0}(u(x))^k v_k(u), g_{\mathbf{s}}^{\beta\epsilon'}\phi_{\epsilon'}\sum_{l\geq 0} v_l(u)(u(y))^l\Big) = \frac{g_{\mathbf{s}}^{\alpha\beta}}{2\pi i}\oint \frac{1}{u(-x+z)}\frac{1}{u(-y-z)}\,dz$$

where $|x| < |z|$ and $|y| < |z|$. The contributions from the (simple) poles at $z = x$ and $z = -y$ cancel, so

$$\Omega\Big(\frac{\partial}{\partial p_k^\alpha}, \frac{\partial}{\partial p_l^\beta}\Big) = 0 \qquad \text{for all } \alpha, \beta, k, l$$

Since

$$(u(z))^k \equiv z^k \bmod \mathbf{s} \qquad \text{and} \qquad v_k(u(z)) \equiv (-z)^{-1-k} \bmod \mathbf{s}$$

any element $f \in \mathcal{U}$ such that

$$\Omega\Big(\frac{\partial}{\partial p_k^\alpha}, f\Big) = \Omega\Big(\frac{\partial}{\partial q_k^\alpha}, f\Big) = 0 \qquad \text{for all } k, \alpha$$

is in fact zero, so every element in $\mathcal{U}$ has the form (2.5). Thus $\{p_k^\alpha, q_l^\beta\}$ forms a Darboux co-ordinate system on $\mathcal{U}$. □

The polarization of $(\mathcal{U}, \Omega)$ by the Lagrangian subspaces

$$\mathcal{U}_+ = \Big\{\sum_{n\geq 0}\alpha_n u^n : \alpha_n \in MU^\star(X; \widetilde{\Omega}^\star_{MU}), \alpha_n \to 0 \text{ as } n \to \infty\Big\}$$
$$\mathcal{U}_- = \Big\{\sum_{n\geq 0}\alpha_n v_n(u) : \alpha_n \in MU^\star(X; \widetilde{\Omega}^\star_{MU})\Big\}$$

gives a symplectic identification of $(\mathcal{U}, \Omega)$ with the cotangent bundle $T^\star\mathcal{U}_+$. We can regard this as a formal family of polarizations, depending on $\mathbf{s}$, of the formal family of symplectic spaces $(\mathcal{U}_{\mathbf{s}}, \Omega_{\mathbf{s}})$. Thus we have achieved (a) and (b).

The dilaton shift

Recall that the cobordism-valued potentials $\mathcal{D}_{MU}$ and $\mathcal{F}^0_{MU}$ are formal functions of

$$\mathbf{t}(u) = t_0 + t_1 u + \ldots \in MU^\star(X; \widetilde{\Omega}^\star_{MU})[[u]]$$

We regard them as formal functions of

$$\mathbf{q}(u) = q_0 + q_1 u + \ldots \in \mathcal{U}_+$$

via the dilaton shift

$$\mathbf{q}(u) = \mathbf{t}(u) + u^\star \tag{2.6}$$

(*cf* section 1.3.3). This achieves (c).

Identification of Fock spaces

It remains to deal with (d). We will first identify the symplectic spaces corresponding to different values of $\mathbf{s}$ via the Chern–Dold character. This will not induce an identification of the Fock spaces $\mathfrak{Fock}_{\mathbf{s}}$: the representation of the Lie algebra of infinitesimal symplectomorphisms on $\mathfrak{Fock}_{\mathbf{s}}$ is built from a representation of the Heisenberg algebra which is determined by the polarization corresponding to $\mathbf{s}$. Since the Chern–Dold character identifies the symplectic spaces corresponding to different $\mathbf{s}$ but not the polarizations, there is more work to do. We discuss this further below, after introducing the cohomological version $(\mathcal{H}, \Omega_0)$ of the symplectic space which is the target of the Chern–Dold character.

Define an $\widetilde{\Omega}^\star_{MU}$-valued inner product on $H^\star(X; \widetilde{\Omega}^\star_{MU})$ by

$$(\alpha, \beta)_0 = \int_X \alpha \wedge \beta$$

and set

$$\begin{aligned}\mathcal{H} &= \Big\{ \sum_{n \in \mathbb{Z}} a_n z^n : a_n \in H^\star(X; \widetilde{\Omega}^\star_{MU}), a_n \to 0 \text{ as } n \to \infty \Big\} \\ &\subset H^\star(X; \widetilde{\Omega}^\star_{MU})[\![z, z^{-1}]\!]\end{aligned}$$

Define

$$\Omega_0(f_1, f_2) = \frac{1}{2\pi i} \oint (f_1(-z), f_2(z))_0 \, dz$$

where, as before, the contour of integration winds once anticlockwise about the origin. The polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ by Lagrangian subspaces

$$\begin{aligned}\mathcal{H}_+ &= \Big\{ \sum_{n \geq 0} a_n z^n : a_n \in H^\star(X; \widetilde{\Omega}^\star_{MU}), a_n \to 0 \text{ as } n \to \infty \Big\} \\ \mathcal{H}_- &= \Big\{ \sum_{n < 0} a_n z^n : a_n \in H^\star(X; \widetilde{\Omega}^\star_{MU}) \Big\}\end{aligned}$$

gives a symplectic identification of $(\mathcal{H}, \Omega_0)$ with $T^\star \mathcal{H}_+$. We pick an $\widetilde{\Omega}^\star_{MU}$-basis $\{\phi_\alpha : \alpha = 1, \ldots, N\}$ for $H^\star(X; \widetilde{\Omega}^\star_{MU})$ and use Darboux co-ordinates (1.1) on $\mathcal{H}$, constructed exactly as in Example 1.3.1.1.

Let the Fock space $\mathfrak{Fock}$ consist of formal functions of

$$\mathbf{t}_0(z) = t_0 + t_1 z + \ldots \in MU^\star(X; \widetilde{\Omega}^\star_{MU})[[z]]$$

which take values in $\widetilde{\Omega}^\star_{MU}[[\hbar, \hbar^{-1}]]$. As before, we regard this as a space of formal functions of

$$\mathbf{q}_0(z) = q_0 + q_1 z + \ldots \in \mathcal{H}_+$$

(near the point $\mathbf{q}_0(z) = -z$) via the dilaton shift

$$\mathbf{q}_0(z) = \mathbf{t}_0(z) - z \tag{2.7}$$

Quantizations of quadratic Darboux monomials act on $\mathfrak{Fock}$ as described in section 1.3.3.

The quantum Chern–Dold character $\mathrm{qch} : \mathcal{U} \to \mathcal{H}$, defined by

$$\mathrm{qch}\Big(\sum_{n \in \mathbb{Z}} \alpha_n u^n\Big) = \sqrt{\mathrm{Td}_{\mathbf{s}}\,(TX)} \sum_{n \in \mathbb{Z}} \mathrm{ch}_{\mathbf{s}}(\alpha_n)(u(z))^n \tag{2.8}$$

is a symplectomorphism from $\mathcal{U}$ to $\mathcal{H}$. It maps $\mathcal{U}_+$ isomorphically to $\mathcal{H}_+$, and we regard $\mathcal{D}_{MU}$ as a function on $\mathcal{H}_+$ via this isomorphism. This gives a formal family $\mathcal{D}_{\mathbf{s}}$, depending on $\mathbf{s}$, of formal functions on $\mathcal{H}_+$. Despite the fact that $\mathcal{D}_{\mathbf{s}}$ is a formal function of $\mathbf{q}_0$ near the point

$$\begin{aligned} \mathbf{q}_0 &= \sqrt{\mathrm{Td}_{\mathbf{s}}\,(TX)}\, u(-z) \\ &\equiv -z \qquad \mod s_1, s_2, \ldots \end{aligned}$$

and so can be considered as a formal function of $\mathbf{t}$ and $\mathbf{s}$ taking values in $\widetilde{\Omega}^\star_{MU}[[\hbar, \hbar^{-1}]]$, we should not regard it as an element of the Fock space $\mathfrak{Fock}$ for the following reason. Given a symplectic vector space V, a polarization $V = V_+ \oplus V_-$ induces a representation of the corresponding Heisenberg algebra $\mathrm{Heis}(V)$ on the space of formal functions on V_+. Explicitly, if $\{p^a, q^b\}$ is a Darboux co-ordinate system adapted to the polarization (so V_+ is given by $p^1 = p^2 = \ldots = 0$ and V_- is given by $q^1 = q^2 = \ldots = 0$) and we regard $\mathrm{Heis}(V)$ as consisting of affine-linear functions on V under the Poisson bracket then the affine-linear function

$$\alpha + \sum_i \beta_i q^i + \sum_j \gamma_j p^j$$

acts as

$$f \mapsto \alpha f + \frac{1}{\sqrt{\hbar}} \sum_i \beta_i q^i f + \sqrt{\hbar} \sum_j \gamma_j \partial_j f$$

where ∂_j is differentiation in the direction of q_j. The projective representation of the Lie algebra of infinitesimal symplectomorphisms that we use — our quantization procedure — is constructed from such a representation of a Heisenberg algebra. By the Stone–von Neumann theorem, this representation of the Heisenberg algebra is projectively unique. Symplectic transformations act as automorphisms of the Heisenberg algebra, and the projective representation of the Lie algebra of infinitesimal symplectomorphisms that this induces is our quantization procedure. The quantum Chern–Dold character identifies the family of Heisenberg algebras corresponding to the family of symplectic spaces $(\mathcal{U}_{\mathbf{s}}, \Omega_{\mathbf{s}})$ with the Heisenberg algebra of $(\mathcal{H}, \Omega_0)$. It does not, however, identify the family of polarizations of $\mathcal{U}_{\mathbf{s}}$ with $\mathcal{H}_+ \oplus \mathcal{H}_-$. We should therefore regard $\mathcal{D}_{\mathbf{s}}$ as living in the Fock space $\mathfrak{Fock}_{\mathbf{s}}$ corresponding to the representation of the Heisenberg algebra of $(\mathcal{H}, \Omega_0)$ given by the polarization

$$\mathcal{H} = \mathcal{H}_+ \oplus \operatorname{span}\{v_0(u(z)), v_1(u(z)), \ldots\}$$

We need to identify $\mathfrak{Fock}_{\mathbf{s}}$ with $\mathfrak{Fock}$. To do this, it suffices to identify them as representations of the Heisenberg algebra. We return to our model situation: a symplectic vector space V equipped with a polarization $V = V_+ \oplus V_-$ with Darboux co-ordinates $\{p^a, q^b\}$ adapted to the polarization. Suppose that $V = V_+ \oplus \overline{V_-}$ is another polarization, and $\{p^a, \bar{q}^b\}$ is a Darboux co-ordinate system adapted to $V_+ \oplus \overline{V_-}$. We have

$$\bar{q}^b = q^b + \sum_a A^{ba} p^a$$

for some symmetric matrix A^{ba}. If Fock is the representation of $\operatorname{Heis}(V)$ corresponding to the polarization $V = V_+ \oplus V_-$ and $\overline{\text{Fock}}$ is the representation corresponding to $V_+ \oplus \overline{V_-}$ then the affine-linear function q^i acts on Fock as

$$f \mapsto \frac{1}{\sqrt{\hbar}} q^i f$$

and on $\overline{\text{Fock}}$ as

$$f \mapsto \frac{1}{\sqrt{\hbar}} q^i f - \sqrt{\hbar} \sum_j A^{ij} \partial_j f$$

Thus

$$\begin{array}{ccc} \overline{\text{Fock}} & \rightarrow & \text{Fock} \\ f & \mapsto & e^{(\hbar/2)\sum_{m,n} A^{mn} \partial_m \partial_n} f \end{array} \tag{2.9}$$

is a map[2] of Heis(V)-modules. In other words,

$$\begin{aligned} p^a(e^{(\hbar/2)\sum_{m,n} A^{mn}\partial_m\partial_n} f) &= \sqrt{\hbar}\,\partial_a e^{(\hbar/2)\sum_{m,n} A^{mn}\partial_m\partial_n} f \\ &= e^{(\hbar/2)\sum_{m,n} A^{mn}\partial_m\partial_n} \sqrt{\hbar}\,\partial_a f \\ &= e^{(\hbar/2)\sum_{m,n} A^{mn}\partial_m\partial_n} p^a(f) \end{aligned}$$

and

$$\begin{aligned} q^a(e^{(\hbar/2)\sum_{m,n} A^{mn}\partial_m\partial_n} f) &= \frac{1}{\sqrt{\hbar}} q^a e^{(\hbar/2)\sum_{m,n} A^{mn}\partial_m\partial_n} f \\ &= e^{(\hbar/2)\sum_{m,n} A^{mn}\partial_m\partial_n}\left(\frac{1}{\sqrt{\hbar}} q^a - \sqrt{\hbar}\sum_b A^{ab}\partial_b\right) f \\ &= e^{(\hbar/2)\sum_{m,n} A^{mn}\partial_m\partial_n} \bar{q}^a(f) \end{aligned}$$

Put another way, the quantization of the symplectic transformation $e^{\frac{1}{2}\sum_{m,n} A^{mn} p_m p_n}$ which maps $V_+ \oplus \overline{V_-}$ to $V_+ \oplus V_-$ intertwines the representations $\overline{\text{Fock}}$ and Fock.

To apply this to our situation, we need to find appropriate Darboux co-ordinate systems adapted to the polarizations $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and to $\mathcal{H} = \mathcal{H}_+ \oplus \text{span}\{v_0(u(z)), v_1(u(z)), \ldots\}$. For a Darboux co-ordinate system adapted to $\mathcal{H}_+ \oplus \mathcal{H}_-$ we use (1.1). For co-ordinates adapted to $\mathcal{H}_+ \oplus \text{span}\{v_0(u(z)), v_1(u(z)), \ldots\}$, take

$$\mathbf{f} = \sum_{r\geq 0} \bar{q}_r^\alpha(\mathbf{f})\phi_\alpha z^r + \sum_{s\geq 0} \bar{p}_s^\beta(\mathbf{f}) g^{\beta\epsilon}\phi_\epsilon w_s(z) \qquad \mathbf{f} \in \mathcal{H} \tag{2.10}$$

where the Laurent series $w_s(z)$ are defined by

$$\frac{1}{u(-x-z)} = \sum_{s\geq 0} x^s w_s(z) \qquad (|x| < |z|)$$

An argument parallel to that on pages 89 and 90 shows that $\{\bar{p}_r^\alpha, \bar{q}_s^\beta\}$ is a Darboux co-ordinate system on $\mathcal{U}$. We have

$$\begin{aligned} \bar{p}_r^\alpha(\,\cdot\,) &= \Omega(\,\cdot\,, \phi_\alpha z^r) \\ &= p_r^\alpha(\,\cdot\,) \end{aligned}$$

and

$$\begin{aligned} \bar{q}_s^\beta(\,\cdot\,) &= \Omega(g^{\beta\epsilon}\phi_\epsilon w_s(z), \,\cdot\,) \\ &= \sum_{r\geq 0} A^{\beta,s;\alpha,r} p_r^\alpha(\,\cdot\,) + \sum_{r\geq 0} B^{\beta,s;\alpha,r} q_r^\alpha(\,\cdot\,) \end{aligned}$$

[2] We assume here that the map (2.9) is well-defined. In the situation which we consider below it will be, due to the presence of the auxiliary variables $\mathbf{s}$.

where

$$A^{\beta,s;\alpha,r} = \bar{q}_s^\beta(g^{\alpha\epsilon}\phi_\epsilon(-z)^{-1-r})$$
$$B^{\beta,s;\alpha,r} = \bar{q}_s^\beta(\phi_\alpha z^r)$$

Equation (2.10) shows that $B^{\beta,s;\alpha,r} = \delta^{\alpha\beta}\delta^{rs}$, so

$$\bar{p}_r^\alpha = p_r^\alpha \qquad \text{and} \qquad \bar{q}_s^\beta = q_s^\beta + \sum_{r\geq 0} A^{\beta,s;\alpha,r} p_r^\alpha$$

as in our model situation. We therefore consider the formal family

$$\mathcal{G}_{\mathbf{s}} = \exp\left(\frac{\hbar}{2}\sum_{r,s} A^{\alpha,r;\beta,s}\partial_{\alpha,r}\partial_{\beta,s}\right)\mathcal{D}_{\mathbf{s}}$$

of elements of the Fock space $\mathfrak{Fock}$. Proposition A.0.3 shows that $\mathcal{G}_{\mathbf{s}}$ is well-defined as a formal function of $\mathbf{t}$ and $\mathbf{s}$ which takes values in $\widetilde{\Omega}^\star_{MU}[[\hbar,\hbar^{-1}]]$.

Note that

$$\begin{aligned}
\sum_{r,s} A^{\beta,s;\alpha,r}x^r y^s &= \Omega(g^{\beta\epsilon}\phi_\epsilon w_s(z)y^s, g^{\alpha\epsilon'}\phi_{\epsilon'}(-z)^{-1-r}x^r) \\
&= g^{\beta\epsilon}g^{\alpha\epsilon'}\Omega\left(\frac{\phi_\epsilon}{u(-y-z)}, \frac{\phi_{\epsilon'}}{-x-z}\right) \qquad (|x|,|y|<|z|) \\
&= \frac{g^{\alpha\beta}}{2\pi i}\oint \frac{1}{u(-y+z)}\frac{1}{-x-z}\,dz \qquad (|x|,|y|<|z|) \\
&= g^{\alpha\beta}\left(\frac{1}{-x-y} - \frac{1}{u(-x-y)}\right) \\
&= -\left[\frac{g^{\alpha\beta}}{u(-x-y)}\right]_+
\end{aligned} \tag{2.11}$$

2.4 Computing the extraordinary descendent potential

We are now in a position to state the main result of this chapter, which describes the relationship between $\mathcal{G}_{\mathbf{s}}$ and the cohomological descendent potential $\mathcal{D}_X$.

Theorem 2.4.1. *Let $E = TX - 1$. Then*

$$\begin{aligned}
&\exp\left(-\frac{1}{24}\sum_{l>0} s_{l-1}\int_X \mathrm{ch}_l(E)c_{D-1}(T_X)\right)(\mathrm{sdet}\sqrt{\mathrm{Td}_{\mathbf{s}}(E)})^{-\frac{1}{24}}\mathcal{G}_{\mathbf{s}} = \\
&\exp\left(\sum_{m>0}\sum_{l\geq 0} s_{2m-1+l}\frac{B_{2m}}{(2m)!}(\mathrm{ch}_l(E)z^{2m-1})^\wedge\right)\exp\left(\sum_{l>0} s_{l-1}(\mathrm{ch}_l(E)/z)^\wedge\right)\mathcal{D}_X
\end{aligned} \tag{2.12}$$

Comparing with Theorem 1.6.4 we see that $\mathcal{G}_{\mathbf{s}}$ coincides with the descendent potential of X twisted by the class $\mathrm{Td}_{\mathbf{s}}$ and the bundle E. This is perhaps surprising — the integrals involved in (2.1) appear significantly more complicated than those for the twisted theory, since they contain contributions not only from the bundle TX but also from variations of complex structure on the domain curve (see the discussion on pages 20 and 21). Remarkably, this extra complication is entirely absorbed by the modified dilaton shift (2.6, 2.7) and the change of polarization $\mathcal{D}_{\mathbf{s}} \rightsquigarrow \mathcal{G}_{\mathbf{s}}$.

The graph of the differential of the genus-0 cobordism potential $\mathcal{F}^0_{MU}$ in $(\mathcal{U}, \Omega) \cong T^\star\mathcal{U}_+$ gives a family $\mathcal{L}_{\mathbf{s}}$ of Lagrangian submanifolds of $(\mathcal{U}_{\mathbf{s}}, \Omega_{\mathbf{s}})$. Since the genus-0 part of $\mathcal{D}_{\mathbf{s}}$ is the generating function of $\mathrm{qch}(\mathcal{L}_{\mathbf{s}})$ with respect to the polarization

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathrm{span}\{v_0(u(z)), v_1(u(z)), \ldots\}$$

and since $\mathcal{G}_{\mathbf{s}}$ differs from $\mathcal{D}_{\mathbf{s}}$ by the quantization of the transformation which maps this polarization to the standard one

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$$

the genus-0 part of $\mathcal{G}_{\mathbf{s}}$ is the generating function of $\mathrm{qch}(\mathcal{L}_{\mathbf{s}})$ with respect to the standard polarization.

Corollary 2.4.2. $\mathrm{qch}(\mathcal{L}_{\mathbf{s}})$ *coincides with the Lagrangian cone for* $(E, \mathrm{Td}_{\mathbf{s}})$*-twisted Gromov–Witten theory. In other words*

$$\mathrm{qch}(\mathcal{L}_{\mathbf{s}}) = \exp\left(\sum_{m\geq 0}\sum_{0\leq l\leq D} s_{2m-1+l}\frac{B_{2m}}{(2m)!}\,\mathrm{ch}_l(E)z^{2m-1}\right)\mathcal{L}_X$$

In particular, this implies

Corollary 2.4.3. *The submanifolds* $\mathcal{L}_{\mathbf{s}} \subset (\mathcal{U}_{\mathbf{s}}, \Omega_{\mathbf{s}})$ *satisfy the conclusions of Theorem 1.5.3: they are ruled Lagrangian cones.*

In the case $X = pt$, $\mathcal{L}_X$ is invariant under multiplication by

$$\exp\left(\sum_{m\geq 0}\sum_{0\leq l\leq D} s_{2m-1+l}\frac{B_{2m}}{(2m)!}\,\mathrm{ch}_l(E)z^{2m-1}\right)$$

Corollary 2.4.4. *When* $X = pt$, $\mathrm{qch}(\mathcal{L}_{\mathbf{s}}) = \mathcal{L}_X$.

2.5 The proof of Theorem 2.4.1

2.5.1 Outline of the proof

By Proposition A.0.2 and Proposition A.0.3, both sides of (2.12) are well-defined. Since $\mathcal{G}_{\mathbf{0}} = \mathcal{D}_X$, it therefore suffices to prove the infinitesimal version

$$\begin{aligned}\frac{\partial}{\partial s_k}\mathcal{G}_{\mathbf{s}} =& \Bigg(\sum_{\substack{2m+r=k+1\\ m\geq 0}} \frac{B_{2m}}{(2m)!}\left(\operatorname{ch}_r(E)z^{2m-1}\right)^{\wedge}\Bigg)\mathcal{G}_{\mathbf{s}} \\ &+ \left(\begin{array}{l}\frac{1}{24}\int_X c_{D-1}(X)\wedge \operatorname{ch}_{k+1}(E) + \frac{1}{48}\int_X \mathbf{e}(X)\wedge \operatorname{ch}_k(E) \\ \quad -\frac{1}{24}\int_X \mathbf{e}(X)\wedge \operatorname{ch}_{k+1}(E)\wedge \Big(\sum_l s_{l+1}\operatorname{ch}_l(E)\Big)\end{array}\right)\mathcal{G}_{\mathbf{s}}\end{aligned} \tag{2.13}$$

(see page 65). If we can prove this for the case in which $\pi(\mathcal{Z})$ is a divisor with normal crossings in $X_{g,n,d}$ then the arguments of the latter part of Proposition 1.6.3 (see page 64) will deal with the general case. Thus we assume that $\pi(\mathcal{Z})$ is a divisor with normal crossings.

Lacking a more intelligent approach, we will compute the left-hand side of (2.13) and then observe that it is equal to the right-hand side. As a first step, we calculate $\partial\mathcal{D}_{\mathbf{s}}/\partial s_k$. The discussion of section 2.3 shows that $\mathcal{D}_{\mathbf{s}}$ depends on $\mathbf{t}_0$ as

$$\mathcal{D}_{\mathbf{s}}(\mathbf{t}_0) = \exp\Bigg(\sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{n!}\langle T_{\mathbf{s}}(\psi),\ldots,T_{\mathbf{s}}(\psi); \operatorname{Td}_{\mathbf{s}}(\mathcal{T}^{vir})\rangle_{g,n,d}\Bigg)$$

where

$$\begin{aligned} T_{\mathbf{s}}(z) &= \frac{1}{\sqrt{\operatorname{Td}_{\mathbf{s}}(TX)}}(\mathbf{t}_0(z)-z) - u(-z) \\ u(z) &= \frac{z}{\operatorname{Td}_{\mathbf{s}}(L)} \\ &= z\exp\Bigg(-\sum_{k>0} s_k \frac{z^k}{k!}\Bigg)\end{aligned}$$

and

$$\operatorname{Td}_{\mathbf{s}}(\,\cdot\,) = \exp\Bigg(\sum_{k>0} s_k \operatorname{ch}_k(\,\cdot\,)\Bigg)$$

Thus

$$\begin{aligned}\mathcal{D}_{\mathbf{s}}^{-1}\frac{\partial \mathcal{D}_{\mathbf{s}}}{\partial s_k} =&\sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{(n-1)!}\left\langle \frac{\partial T_{\mathbf{s}}(\psi)}{\partial s_k}, T_{\mathbf{s}}(\psi), \ldots, T_{\mathbf{s}}(\psi); \mathrm{Td}_{\mathbf{s}}\left(\mathcal{T}^{vir}\right)\right\rangle_{g,n,d} \\ &+ \sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{n!} \langle T_{\mathbf{s}}(\psi), \ldots, T_{\mathbf{s}}(\psi); \mathrm{ch}_k(\mathcal{T}^{vir})\,\mathrm{Td}_{\mathbf{s}}\left(\mathcal{T}^{vir}\right)\rangle_{g,n,d}\end{aligned} \tag{2.14}$$

Call the first sum in (2.14) the derivative term and the second sum the main term. We will calculate the derivative term in section 2.5.2 and the main term in section 2.5.4. Along the way, we will need an expression for the virtual tangent bundle $\mathcal{T}^{vir}$ in terms of bundles pulled back from the target space X (which we handle much as in Chapter 1) and universal cotangent lines; we compute this in section 2.5.3. We collect the results of our computations in section 2.5.5, obtaining the rather complicated-looking expression (2.49) for $\partial \mathcal{D}_{\mathbf{s}}/\partial s_k$. We want to express $\partial \mathcal{G}_{\mathbf{s}}/\partial s_k$ in terms of quantized infinitesimal symplectomorphisms acting on $\mathcal{G}_{\mathbf{s}}$, and so in section 2.5.6 we rewrite (2.49) in terms of the action of the Heisenberg algebra. The results of this (equation (2.50) below) still look rather complicated, largely because (2.50) is written in the wrong co-ordinates: it is expressed in terms of the action of p_k^α and q_l^β, whereas the Heisenberg algebra acts naturally on $\mathcal{D}_{\mathbf{s}}$ via p_k^α and $\bar{q}_l^\beta$. Once we rewrite (2.50) in terms of this latter action, it becomes easy to see that $\partial \mathcal{G}_{\mathbf{s}}/\partial s_k$ has the desired form.

It is worth noting that all the geometric ingredients in the proof of Theorem 2.4.1 already occur in the proof of Theorem 1.6.4 — Theorem 2.4.1 is also just a consequence of the Grothendieck–Riemann–Roch theorem applied to the universal family of stable maps. The only difference is that the computations in this case are somewhat more involved.

2.5.2 The derivative term

We have

$$\frac{\partial T_{\mathbf{s}}}{\partial s_k} = -\frac{\mathrm{ch}_k(TX)}{2\sqrt{\mathrm{Td}_{\mathbf{s}}\,(TX)}}(\mathbf{t}_0 - z) + \frac{(-z)^{k+1}}{k!}\exp\left(-\sum_{k>0} s_k \frac{(-z)^k}{k!}\right)$$

and so the derivative term in (2.14) is

$$
\begin{aligned}
&-\sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{(n-1)!} \left\langle \frac{\mathrm{ch}_k(TX)}{2\sqrt{\mathrm{Td}_{\mathbf{s}}(TX)}} (\mathbf{t}_0(\psi) - \psi), T_{\mathbf{s}}(\psi), \ldots, T_{\mathbf{s}}(\psi); \mathrm{Td}_{\mathbf{s}}(\mathcal{T}^{vir}) \right\rangle_{g,n,d} \\
&\qquad + \sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{(n-1)!} \left\langle \frac{(-\psi)^{k+1}}{k!} \mathrm{Td}_{\mathbf{s}}(-L^{-1}), T_{\mathbf{s}}(\psi), \ldots, T_{\mathbf{s}}(\psi); \mathrm{Td}_{\mathbf{s}}(\mathcal{T}^{vir}) \right\rangle_{g,n,d}
\end{aligned}
\tag{2.15}
$$

2.5.3 Calculating $\mathcal{T}^{vir}$

From the discussion on page 21 we know that

$$
\begin{aligned}
\mathcal{T}^{vir} &= \pi_\star \mathrm{ev}^\star(TX) - \mathrm{Aut}(C) + \mathrm{Def}(C) \\
&= \pi_\star \mathrm{ev}^\star(TX) - H^0(\Omega_\pi^\vee(-D)) + H^1(\Omega_\pi^\vee(-D))
\end{aligned}
$$

where $D = D_1 + \ldots + D_n$ is the divisor given by the marked points. Applying Serre duality,

$$
\begin{aligned}
\mathcal{T}^{vir} &= \pi_\star \mathrm{ev}^\star(TX) - \pi_\star(\Omega_\pi^\vee(-D)) \\
&= \pi_\star \mathrm{ev}^\star(TX) + (\pi_\star(\Omega_\pi(D) \otimes \omega_\pi))^\vee \\
&= \pi_\star \mathrm{ev}^\star(TX) + (\pi_\star(\Omega_\pi \otimes L_{n+1}))^\vee
\end{aligned}
$$

Using (1.13), we find

$$
\begin{aligned}
\mathcal{T}^{vir} &= \pi_\star \mathrm{ev}^\star(TX) + (\pi_\star(\omega_\pi \otimes L_{n+1}))^\vee - (\pi_\star(i_\star(\mathcal{O}_{\mathcal{Z}}) \otimes L_{n+1}))^\vee \\
&= \pi_\star \mathrm{ev}^\star(TX) - \pi_\star(L_{n+1}^{-1}) - (\pi_\star(i_\star(\mathcal{O}_{\mathcal{Z}}) \otimes L_{n+1}))^\vee
\end{aligned}
$$

by Serre duality again. Since L_{n+1} is trivial on $\mathcal{Z}$, this implies that

$$
\mathcal{T}^{vir} = \pi_\star \mathrm{ev}^\star(TX) - \pi_\star(L_{n+1}^{-1}) - (\pi_\star i_\star(\mathcal{O}_{\mathcal{Z}}))^\vee
$$

(An immediate consequence of this, which we will not need, is that the logarithmic virtual tangent bundle of $X_{g,n,d}$ with respect to the virtual divisor $\pi(\mathcal{Z})$ is $\pi_\star \mathrm{ev}^\star(TX) - \pi_\star(L_{n+1}^{-1})$.) Since

$$
E = TX - 1
$$

if we set

$$
\mathcal{T}^{cs} = -\pi_\star(L_{n+1}^{-1} - 1) - (\pi_\star i_\star \mathcal{O}_{\mathcal{Z}})^\vee
$$

then

$$
\mathcal{T}^{vir} = \pi_\star \mathrm{ev}^\star(E) + \mathcal{T}^{cs} \tag{2.16}
$$

2.5.4 The main term

This is

$$\sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{n!} \langle T_{\mathbf{s}}(\psi), \ldots, T_{\mathbf{s}}(\psi); (\mathrm{ch}_k(\pi_\star \operatorname{ev}^\star E) + \mathrm{ch}_k(\mathcal{T}^{cs}))\, \mathrm{Td}_{\mathbf{s}}\, (\mathcal{T}^{vir})\rangle_{g,n,d} \tag{2.17}$$

We call the terms involving $\pi_\star \operatorname{ev}^\star E$ the target space terms, and those involving $\mathcal{T}^{cs}$ the complex structure terms. This is because, as we saw in section 2.5.3, the terms involving $\mathcal{T}^{cs}$ roughly speaking arise from deformations of the complex structure on the domain of the stable map.

The complex structure terms

We have

$$\begin{aligned} \mathrm{ch}_k(\mathcal{T}^{cs}) &= -\,\mathrm{ch}_k(\pi_\star(L_{n+1}^{-1} - 1)) + (-)^{k+1}\,\mathrm{ch}_k(\pi_\star i_\star \mathcal{O}_{\mathcal{Z}}) \\ &= -\pi_\star[\mathrm{ch}(\underbrace{L_{n+1}^{-1} - 1}_{\text{log term}} + \underbrace{(-)^k i_\star \mathcal{O}_{\mathcal{Z}}}_{\text{nodal term}}) \cdot \mathrm{Td}^\vee\,(\Omega_\pi)]_{k+1} \end{aligned} \tag{2.18}$$

by Grothendieck–Riemann–Roch again, where $[\cdot]_r$ denotes the degree-$2r$ component of a cohomology class.

Using (1.18), we see that the log term in (2.18) is

$$-\pi_\star[(e^{-\psi} - 1)(\mathrm{Td}^\vee\,(L_{n+1}) + \boxed{\text{codim-1}} + \boxed{\text{codim-2}})]_{k+1}$$

where $\boxed{\text{codim-1}}$ and $\boxed{\text{codim-2}}$ are as in Proposition 1.6.3. The $\boxed{\text{codim-1}}$ terms are supported on the divisors $D_i = \sigma_i(X_{g,n,d})$ and the $\boxed{\text{codim-2}}$ terms are supported on the singular locus $\mathcal{Z}$. But $e^{-\psi} - 1$ vanishes on D_i and on $\mathcal{Z}$, since it is divisible by ψ, so the log term is

$$-\pi_\star\left[(e^{-\psi} - 1)\frac{\psi}{e^{\psi} - 1}\right]_{k+1} = -\pi_\star\left[\frac{(-\psi)^{k+1}}{k!}\right] \tag{2.19}$$

We calculate the nodal term in (2.18) using Grothendieck–Riemann–Roch again. It is

$$\begin{aligned}(-)^{k+1}&\pi_\star[\mathrm{ch}(i_\star\mathcal{O}_\mathcal{Z})(\mathrm{Td}^\vee(L_{n+1})+\boxed{\text{codim-1}}+\boxed{\text{codim-2}})]_{k+1}\\ &=(-)^{k+1}\pi_\star\left[(i_\star\mathrm{Td}^\vee(-L_+-L_-))\left(\mathrm{Td}^\vee(L_{n+1})+\boxed{\text{codim-1}}+\left(\frac{1}{\mathrm{Td}^\vee(i_\star\mathcal{O}_\mathcal{Z})}-1\right)\right)\right]_{k+1}\\ &=(-)^{k+1}\pi_\star i_\star\left[\mathrm{Td}^\vee(-L_+-L_-)\left(1+\frac{1}{\mathrm{Td}^\vee(i^\star i_\star\mathcal{O}_\mathcal{Z})}-1\right)\right]_{k-1}\end{aligned} \tag{2.20}$$

where we used the facts that the conormal bundle to $\mathcal{Z}$ in $X_{g,n+1,d}$ is $L_+\oplus L_-$, that L_{n+1} is trivial when restricted to $\mathcal{Z}$ and that $\mathcal{Z}$ misses the divisors D_i. Since

$$i^\star i_\star\mathcal{O}_\mathcal{Z}=(1-L_+)(1-L_-)$$

the nodal term (2.20) is

$$\begin{aligned}(-)^{k+1}\pi_\star i_\star\left[\mathrm{Td}^\vee(-L_+-L_-)\left(\frac{\mathrm{Td}^\vee(L_+)\,\mathrm{Td}^\vee(L_-)}{\mathrm{Td}^\vee(L_+\otimes L_-)}\right)\right]_{k-1}&=(-)^{k+1}\pi_\star i_\star\left[\frac{e^{\psi_++\psi_-}-1}{\psi_++\psi_-}\right]_{k-1}\\ &=\pi_\star i_\star\left[\frac{(-\psi_+-\psi_-)^{k-1}}{k!}\right]_+\end{aligned} \tag{2.21}$$

where the $[\cdot]_+$ ensures that the formula is correct for $k=0$. Combining (2.18), (2.19) and (2.21) we find that

$$\mathrm{ch}_k(\mathcal{T}^{cs})=\pi_\star\left[-\frac{(-\psi)^{k+1}}{k!}+i_\star\frac{(-\psi_+-\psi_-)^{k-1}}{k!}\right]_+ \tag{2.22}$$

Thus the complex structure terms in (2.17) are

$$\begin{aligned}\sum_{g,n,d}&\frac{Q^d\hbar^{g-1}}{n!}\left\langle T_\mathbf{s}(\psi),\ldots,T_\mathbf{s}(\psi);\pi_\star\left[-\frac{(-\psi)^{k+1}}{k!}+i_\star\frac{(-\psi_+-\psi_-)^{k-1}}{k!}\right]_+\mathrm{Td}_\mathbf{s}(\mathcal{T}^{vir})\right\rangle_{g,n,d}\\ &=\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{n!}\left\langle\pi^\star(T_\mathbf{s}(\psi)),\ldots,\pi^\star(T_\mathbf{s}(\psi)),\left(-\frac{(-\psi)^{k+1}}{k!}\right);\mathrm{Td}_\mathbf{s}(\pi^\star\mathcal{T}^{vir})\right\rangle_{g,n+1,d}\\ &\quad+\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{n!}\left\langle\pi^\star(T_\mathbf{s}(\psi)),\ldots,\pi^\star(T_\mathbf{s}(\psi)),i_\star\left[\frac{(-\psi_+-\psi_-)^{k-1}}{k!}\right]_+;\mathrm{Td}_\mathbf{s}(\pi^\star\mathcal{T}^{vir})\right\rangle_{g,n+1,d}\end{aligned} \tag{2.23}$$

The comparison result for universal cotangent lines (see *e.g.* [67, 54])

$$\pi^\star\psi_i=\psi_i-\sigma_{i\star}\mathcal{O}_{X_{g,n,d}}$$

implies that

$$\pi^\star T_{\mathbf{s}}(\psi_i) = T_{\mathbf{s}}(\psi_i) - \sigma_{i\star}\left[\frac{T_{\mathbf{s}}(\psi_i)}{\psi_i}\right]_+ \tag{2.24}$$

Also,

$$\begin{aligned} \pi^\star \mathcal{T}^{vir} &= \mathcal{T}^{vir} - \Omega_\pi^\vee \\ &= \mathcal{T}^{vir} - L_{n+1}^{-1} + \sum_{i=1}^n (\sigma_{i\star}\mathcal{O}_{X_{g,n,d}})^\vee + (i_\star \mathcal{O}_{\mathcal{Z}})^\vee \end{aligned} \tag{2.25}$$

where we used (1.13) and (1.14), and so

$$\mathrm{Td}_{\mathbf{s}}(\pi^\star\mathcal{T}^{vir}) = \mathrm{Td}_{\mathbf{s}}(\mathcal{T}^{vir})\,\mathrm{Td}_{\mathbf{s}}(-L_{n+1}^{-1})\,\mathrm{Td}_{\mathbf{s}}\left(\sum_{i=1}^n (\sigma_{i\star}\mathcal{O}_{X_{g,n,d}})^\vee\right)\mathrm{Td}_{\mathbf{s}}((i_\star\mathcal{O}_{\mathcal{Z}})^\vee)$$

Thus the complex structure terms (2.23) become

$$\begin{aligned} &\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{n!}\left\langle T_{\mathbf{s}}(\psi) - \sigma_{1\star}\left[\frac{T_{\mathbf{s}}(\psi)}{\psi}\right]_+, \ldots, \left(-\frac{(-\psi)^{k+1}}{k!}\right); \mathrm{Td}_{\mathbf{s}}(\pi^\star\mathcal{T}^{vir})\right\rangle_{g,n+1,d} \\ &+\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{n!}\left\langle T_{\mathbf{s}}(\psi) - \sigma_{1\star}\left[\frac{T_{\mathbf{s}}(\psi)}{\psi}\right]_+, \ldots, i_\star\left[\frac{(-\psi_+ - \psi_-)^{k-1}}{k!}\right]_+; \mathrm{Td}_{\mathbf{s}}(\pi^\star\mathcal{T}^{vir})\right\rangle_{g,n+1,d} \end{aligned} \tag{2.26}$$

Since ψ vanishes on the divisors D_i and on the singular locus $\mathcal{Z}$, the first sum in (2.26) is

$$\begin{aligned} &-\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{n!}\left\langle T_{\mathbf{s}}(\psi), \ldots, T_{\mathbf{s}}(\psi), \frac{(-\psi)^{k+1}}{k!}; \mathrm{Td}_{\mathbf{s}}(\mathcal{T}^{vir})\,\mathrm{Td}_{\mathbf{s}}(-L_{n+1}^{-1})\right\rangle_{g,n+1,d} \\ &\qquad = -\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{(n-1)!}\left\langle T_{\mathbf{s}}(\psi), \ldots, T_{\mathbf{s}}(\psi), \frac{(-\psi)^{k+1}}{k!}\,\mathrm{Td}_{\mathbf{s}}(-L^{-1}); \mathrm{Td}_{\mathbf{s}}(\mathcal{T}^{vir})\right\rangle_{g,n,d} \\ &\qquad + \frac{1}{2\hbar}\left\langle T_{\mathbf{s}}, T_{\mathbf{s}}, \frac{(-\psi)^{k+1}}{k!}\,\mathrm{Td}_{\mathbf{s}}(-L^{-1}); \mathrm{Td}_{\mathbf{s}}(\mathcal{T}^{vir})\right\rangle_{0,3,0} \\ &\qquad + \left\langle \frac{(-\psi)^{k+1}}{k!}\,\mathrm{Td}_{\mathbf{s}}(-L^{-1}); \mathrm{Td}_{\mathbf{s}}(\mathcal{T}^{vir})\right\rangle_{1,1,0} \end{aligned} \tag{2.27}$$

But ψ^2 vanishes on both $X_{0,3,0}$ and $X_{1,1,0}$, so the exceptional terms in (2.27) vanish. The first sum in (2.26) is therefore

$$-\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{(n-1)!}\left\langle T_{\mathbf{s}}(\psi), \ldots, T_{\mathbf{s}}(\psi), \frac{(-\psi)^{k+1}}{k!}\,\mathrm{Td}_{\mathbf{s}}(-L^{-1}); \mathrm{Td}_{\mathbf{s}}(\mathcal{T}^{vir})\right\rangle_{g,n,d} \tag{2.28}$$

We compute the second sum in (2.26) by pulling back to the singular locus $\mathcal{Z}$. The divisors D_i miss this locus and L_{n+1} is trivial there, so the second sum in (2.26) becomes

$$\sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{n!} \int_{i_\star[\mathcal{Z}]} T_{\mathbf{s}}(\psi_1) \wedge \ldots \wedge \left[\frac{(-\psi_+ - \psi_-)^{k-1}}{k!}\right]_+ \wedge \operatorname{Td}_{\mathbf{s}}(i^\star \mathcal{T}^{vir}) \operatorname{Td}_{\mathbf{s}}((i^\star i_\star \mathcal{O}_{\mathcal{Z}})^\vee) \tag{2.29}$$

where we use the notation of section 1.6.2. The integration takes place over the singular locus $\mathcal{Z}$ in $X_{g,n+1,d}$. Since the normal bundle to $\mathcal{Z}$ in $X_{g,n+1,d}$ is $L_+^{-1} \oplus L_-^{-1}$, we have

$$\begin{aligned} \operatorname{Td}_{\mathbf{s}}(i^\star \mathcal{T}^{vir}) \operatorname{Td}_{\mathbf{s}}((i^\star i_\star \mathcal{O}_{\mathcal{Z}})^\vee) &= \operatorname{Td}_{\mathbf{s}}(\mathcal{T}^{vir}_{\mathcal{Z}}) \operatorname{Td}_{\mathbf{s}}(L_+^{-1} + L_-^{-1}) \operatorname{Td}_{\mathbf{s}}((1 - L_+^{-1})(1 - L_-^{-1})) \\ &= \operatorname{Td}_{\mathbf{s}}(\mathcal{T}^{vir}_{\mathcal{Z}}) \operatorname{Td}_{\mathbf{s}}(L_+^{-1} \otimes L_-^{-1}) \end{aligned}$$

Much as we did when processing the codimension-2 contributions in the proof of the quantum Riemann–Roch theorem (see page 66), we compute (2.29) by pulling back along

$$\widetilde{\mathcal{Z}}_{red} \coprod \widetilde{\mathcal{Z}}_{irr} \xrightarrow{\gamma_{red} \coprod \gamma_{irr}} \mathcal{Z}$$

We know that

$$\widetilde{\mathcal{Z}}_{red} = \coprod_{\substack{g=g_+ + g_- \\ n=n_+ + n_- \\ d=d_+ + d_-}} X_{g_+, n_+ + \bullet, d_+} \times_X X_{0,1+\bullet+\circ,0} \times_X X_{g_-, n_- + \circ, d_-}$$

and, in the notation of Lemma 1.6.1,

$$\gamma_{red}^\star \mathcal{T}^{vir}_{\mathcal{Z}} = p_+^\star \mathcal{T}^{vir}_{X_{g_+, n_+ + \bullet, d_+}} + p_-^\star \mathcal{T}^{vir}_{X_{g_-, n_- + \circ, d_-}} - \operatorname{ev}_\Delta^\star TX$$

Also,

$$\widetilde{\mathcal{Z}}_{irr} = X_{g-1, n+\bullet+\circ} \times_{X \times X} X_{0,1+\bullet+\circ,0}$$

and

$$\gamma_{irr}^\star \mathcal{T}^{vir}_{\mathcal{Z}} = \mathcal{T}^{vir}_{X_{g-1,n+\bullet+\circ,d}} - \operatorname{ev}_\Delta^\star TX$$

Thus we can write (2.29) as

$$\begin{aligned} \frac{1}{2} \sum_{g_1, g_2} \sum_{n_1, n_2} \sum_{d_1, d_2} \frac{Q^{d_1+d_2} \hbar^{g_1+g_2-1}}{n_1!\, n_2!} \sum_{r,s} a_{r,s} g^{\alpha\beta} \left\langle T_{\mathbf{s}}, \ldots, T_{\mathbf{s}}, \frac{\phi_\alpha \psi_+^r}{\sqrt{\operatorname{Td}_{\mathbf{s}}(TX)}}; \operatorname{Td}_{\mathbf{s}}(\mathcal{T}^{vir}) \right\rangle_{g_1, n_1+1, d_1} \\ \times \left\langle \frac{\phi_\beta \psi_-^s}{\sqrt{\operatorname{Td}_{\mathbf{s}}(TX)}}, T_{\mathbf{s}}, \ldots, T_{\mathbf{s}}; \operatorname{Td}_{\mathbf{s}}(\mathcal{T}^{vir}) \right\rangle_{g_2, n_2+1, d_2} \\ + \frac{1}{2} \sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{n!} \sum_{r,s} a_{r,s} g^{\alpha\beta} \left\langle T_{\mathbf{s}}, \ldots, T_{\mathbf{s}}, \frac{\phi_\alpha \psi_+^r}{\sqrt{\operatorname{Td}_{\mathbf{s}}(TX)}}, \frac{\phi_\beta \psi_-^s}{\sqrt{\operatorname{Td}_{\mathbf{s}}(TX)}}; \operatorname{Td}_{\mathbf{s}}(\mathcal{T}^{vir}) \right\rangle_{g-1, n+2, d} \end{aligned} \tag{2.30}$$

where

$$\sum_{r,s} a_{r,s}\psi_+^r\psi_-^s = \frac{(-\psi_+ - \psi_-)^{k-1}}{k!}\operatorname{Td}_{\mathbf{s}}\left(L_+^{-1}\otimes L_-^{-1}\right) \in \widetilde{\Omega}^\star_{MU}[\![\psi_+,\psi_-]\!]$$

If we write

$$T_{\mathbf{s}}(z) = \sum_k (T_{\mathbf{s}})_k^\alpha \phi_\alpha z^k$$

then the affine-linear function p_k^α acts on $\mathfrak{Fock}$ as

$$\frac{\sqrt{\hbar}}{\sqrt{\operatorname{Td}_{\mathbf{s}}(TX)}}\frac{\partial}{\partial(T_{\mathbf{s}})_k^\alpha}$$

and so (2.30) is

$$-\mathcal{D}_{\mathbf{s}}^{-1}\Big(\frac{1}{2}\sum_{r,s} A_k^{\alpha,r;\beta,s}p_r^\alpha p_s^\beta\Big)\mathcal{D}_{\mathbf{s}}$$

where

$$\sum_{r,s} A_k^{\alpha,r;\beta,s}\psi_+^r\psi_-^s = -g^{\alpha\beta}\frac{(-\psi_+ - \psi_-)^{k-1}}{k!}\operatorname{Td}_{\mathbf{s}}\left(L_+^{-1}\otimes L_-^{-1}\right) \tag{2.31}$$

The complex structure terms (2.23) are therefore

$$\begin{aligned}
-\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{(n-1)!}\Big\langle T_{\mathbf{s}}(\psi),\ldots,T_{\mathbf{s}}(\psi),\frac{(-\psi)^{k+1}}{k!}\operatorname{Td}_{\mathbf{s}}\left(-L^{-1}\right);\operatorname{Td}_{\mathbf{s}}\left(\mathcal{T}^{vir}\right)\Big\rangle_{g,n,d} \\
-\mathcal{D}_{\mathbf{s}}^{-1}\Big(\frac{1}{2}\sum_{r,s} A_k^{\alpha,r;\beta,s}p_r^\alpha p_s^\beta\Big)\mathcal{D}_{\mathbf{s}}
\end{aligned} \tag{2.32}$$

Note that

$$\begin{aligned}
\sum_{r,s} A_k^{\alpha,r;\beta,s}\psi_+^r\psi_-^s &= -g^{\alpha\beta}\frac{\partial}{\partial s_k}\left(\frac{\operatorname{Td}_{\mathbf{s}}\left(L_+^{-1}\otimes L_-^{-1}\right)-1}{-\psi_+ - \psi_-}\right) \\
&= -\frac{\partial}{\partial s_k}\left[\frac{g^{\alpha\beta}}{u(-\psi_+ - \psi_-)}\right]_+ \\
&= \frac{\partial}{\partial s_k}\sum_{r,s} A^{\alpha,r;\beta,s}\psi_+^r\psi_-^s
\end{aligned}$$

The target space terms

It remains to calculate the target space terms

$$\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{n!}\langle T_{\mathbf{s}}(\psi),\ldots,T_{\mathbf{s}}(\psi);\pi_\star([\operatorname{ev}^\star(\operatorname{ch}(E))\cdot\operatorname{Td}^\vee(\Omega_\pi)]_{k+1})\operatorname{Td}_{\mathbf{s}}\left(\mathcal{T}^{vir}\right)\rangle_{g,n,d} \tag{2.33}$$

from (2.17). By (1.18), we have

$$\mathrm{Td}^{\vee}(\Omega_\pi) = \boxed{\text{codim-0}} + \boxed{\text{codim-1}} + \boxed{\text{codim-2}}$$

where, as before,

$$\boxed{\text{codim-0}} = \mathrm{Td}^{\vee} L_{n+1}$$

$$\boxed{\text{codim-1}} = -\sum_{i=1}^{n} \sigma_{i\star}\left[\frac{\mathrm{Td}^{\vee}(L_i)}{\psi_i}\right]_+$$

$$\boxed{\text{codim-2}} = i_\star\left[\frac{1}{\psi_+ + \psi_-}\left(\frac{\mathrm{Td}^{\vee}(L_+)}{\psi_+} + \frac{\mathrm{Td}^{\vee}(L_-)}{\psi_-}\right)\right]_+$$

Thus (2.33) splits into codimension-0, codimension-1 and codimension-2 terms.

The codimension-1 terms in (2.33)

These are

$$-\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{n!}\left\langle T_{\mathbf{s}}(\psi),\dots,T_{\mathbf{s}}(\psi);\pi_\star\left[\mathrm{ev}^\star(\mathrm{ch}(E))\sum_{i=1}^{n}\sigma_{i\star}\left[\frac{\mathrm{Td}^{\vee}(L_i)}{\psi_i}\right]_+\right]_{k+1}\mathrm{Td}_{\mathbf{s}}(\mathcal{T}^{vir})\right\rangle_{g,n,d}$$

$$= -\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{(n-1)!}\left\langle\left[\left[\mathrm{ch}(E)\frac{\mathrm{Td}^{\vee}(L)}{\psi}\right]_k\right]_+ T_{\mathbf{s}}(\psi),T_{\mathbf{s}}(\psi),\dots,T_{\mathbf{s}}(\psi);\mathrm{Td}_{\mathbf{s}}(\mathcal{T}^{vir})\right\rangle_{g,n,d} \tag{2.34}$$

The codimension-2 terms in (2.33)

These are

$$\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{n!}\langle T_{\mathbf{s}}(\psi),\dots,T_{\mathbf{s}}(\psi);\pi_\star[\mathrm{ev}^\star(\mathrm{ch}(E))\cdot\boxed{\text{codim-2}}]_{k+1}\,\mathrm{Td}_{\mathbf{s}}(\mathcal{T}^{vir})\rangle_{g,n,d}$$

$$= \sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{n!}\langle \pi^\star(T_{\mathbf{s}}(\psi)),\dots,\pi^\star(T_{\mathbf{s}}(\psi)),[\mathrm{ev}^\star(\mathrm{ch}(E))\cdot\boxed{\text{codim-2}}]_{k+1};\mathrm{Td}_{\mathbf{s}}(\pi^\star\mathcal{T}^{vir})\rangle_{g,n+1,d}$$

which is

$$\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{n!}\int_{i_\star[\mathcal{Z}]}T_{\mathbf{s}}(\psi_1)\wedge\dots\wedge\left[\left[\frac{\mathrm{ch}(E)}{\psi_+ + \psi_-}\left(\frac{\mathrm{Td}^{\vee}(L_+)}{\psi_+}+\frac{\mathrm{Td}^{\vee}(L_-)}{\psi_-}\right)\right]_{k-1}\right]_+\wedge\mathrm{Td}_{\mathbf{s}}(i^\star\pi^\star\mathcal{T}^{vir}) \tag{2.35}$$

where the integration takes place over the singular locus $\mathcal{Z} \subset X_{g,n+1,d}$. Now

$$i^\star \pi^\star \mathcal{T}^{vir} = \mathcal{T}^{vir}_{\mathcal{Z}} + L_+^{-1} \otimes L_-^{-1}$$

so, pulling back to $\widetilde{\mathcal{Z}}_{red} \coprod \widetilde{\mathcal{Z}}_{irr}$ as before, we find that the codimension-2 terms (2.35) can be written as

$$\begin{aligned}
&\frac{1}{2} \sum_{g_1,g_2} \sum_{n_1,n_2} \sum_{d_1,d_2} \frac{Q^{d_1+d_2} \hbar^{g_1+g_2-1}}{n_1!\, n_2!} \sum_{r,s} b^{\alpha\beta}_{r,s} \left\langle T_{\mathbf{s}}, \ldots, T_{\mathbf{s}}, \frac{\phi_\alpha \psi_+^r}{\sqrt{\mathrm{Td}_{\mathbf{s}}\,(TX)}}; \mathrm{Td}_{\mathbf{s}}\,(\mathcal{T}^{vir}) \right\rangle_{g_1,n_1+1,d_1} \\
&\qquad\qquad \times \left\langle \frac{\phi_\beta \psi_-^s}{\sqrt{\mathrm{Td}_{\mathbf{s}}\,(TX)}}, T_{\mathbf{s}}, \ldots, T_{\mathbf{s}}; \mathrm{Td}_{\mathbf{s}}\,(\mathcal{T}^{vir}) \right\rangle_{g_2,n_2+1,d_2} \\
&+ \frac{1}{2} \sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{n!} \sum_{r,s} b^{\alpha\beta}_{r,s} \left\langle T_{\mathbf{s}}, \ldots, T_{\mathbf{s}}, \frac{\phi_\alpha \psi_+^r}{\sqrt{\mathrm{Td}_{\mathbf{s}}\,(TX)}}, \frac{\phi_\beta \psi_-^s}{\sqrt{\mathrm{Td}_{\mathbf{s}}\,(TX)}}; \mathrm{Td}_{\mathbf{s}}\,(\mathcal{T}^{vir}) \right\rangle_{g-1,n+2,d}
\end{aligned} \tag{2.36}$$

where

$$\sum_{r,s} b^{\alpha\beta}_{r,s} \psi_+^r \psi_-^s = \left(\left[\frac{1}{\psi_+ + \psi_-} \left[\mathrm{ch}(E) \frac{\mathrm{Td}^\vee\,(L_+)}{\psi_+} + \mathrm{ch}(E) \frac{\mathrm{Td}^\vee\,(L_-)}{\psi_-} \right]_k \right]_+ \mathrm{Td}_{\mathbf{s}}\,(L_+^{-1} \otimes L_-^{-1}) \right)^{\alpha\beta}$$

The right-hand side here means that we take the element of $\mathrm{End}(H^\star(X))[\![\psi_+, \psi_-]\!]$ given by multiplication by

$$\left[\frac{1}{\psi_+ + \psi_-} \left[\mathrm{ch}(E) \frac{\mathrm{Td}^\vee\,(L_+)}{\psi_+} + \mathrm{ch}(E) \frac{\mathrm{Td}^\vee\,(L_-)}{\psi_-} \right]_k \right]_+ \mathrm{Td}_{\mathbf{s}}\,(L_+^{-1} \otimes L_-^{-1})$$

write it as a matrix-valued power series, with entries

$$\left(\left[\frac{1}{\psi_+ + \psi_-} \left[\mathrm{ch}(E) \frac{\mathrm{Td}^\vee\,(L_+)}{\psi_+} + \mathrm{ch}(E) \frac{\mathrm{Td}^\vee\,(L_-)}{\psi_-} \right]_k \right]_+ \mathrm{Td}_{\mathbf{s}}\,(L_+^{-1} \otimes L_-^{-1}) \right)^\alpha_\beta$$

with respect to the basis $\{\phi_\alpha\}$, and raise the index using the metric. It will turn out to be convenient to write

$$b^{\alpha\beta}_{r,s} = B_k^{\alpha,r;\beta,s} + C_k^{\alpha,r;\beta,s}$$

where

$$\begin{aligned}
&\sum_{r,s} B_k^{\alpha,r;\beta,s} \psi_+^r \psi_-^s = \left(\left[\frac{1}{\psi_+ + \psi_-} \left[\mathrm{ch}(E) \frac{\mathrm{Td}^\vee\,(L_+)}{\psi_+} + \mathrm{ch}(E) \frac{\mathrm{Td}^\vee\,(L_-)}{\psi_-} \right]_k \right]_+ \right)^{\alpha\beta} \\
&\sum_{r,s} C_k^{\alpha,r;\beta,s} \psi_+^r \psi_-^s = \\
&\qquad \left(\left[\frac{1}{\psi_+ + \psi_-} \left[\mathrm{ch}(E) \frac{\mathrm{Td}^\vee\,(L_+)}{\psi_+} + \mathrm{ch}(E) \frac{\mathrm{Td}^\vee\,(L_-)}{\psi_-} \right]_k \right]_+ (\mathrm{Td}_{\mathbf{s}}\,(L_+^{-1} \otimes L_-^{-1}) - 1) \right)^{\alpha\beta}
\end{aligned} \tag{2.37}$$

Thus the codimension-2 terms in (2.33) are

$$\mathcal{D}_{\mathbf{s}}^{-1}\Big(\frac{1}{2}\sum_{r,s} B_k^{\alpha,r;\beta,s} p_r^\alpha p_s^\beta\Big)\mathcal{D}_{\mathbf{s}} + \mathcal{D}_{\mathbf{s}}^{-1}\Big(\frac{1}{2}\sum_{r,s} C_k^{\alpha,r;\beta,s} p_r^\alpha p_s^\beta\Big)\mathcal{D}_{\mathbf{s}} \tag{2.38}$$

The codimension-0 terms in (2.33)

These are

$$\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{n!}\langle T_{\mathbf{s}}(\psi),\ldots,T_{\mathbf{s}}(\psi);\pi_\star([\mathrm{ev}^\star(\mathrm{ch}(E))\cdot \mathrm{Td}^\vee(L_{n+1})]_{k+1})\,\mathrm{Td}_{\mathbf{s}}(\mathcal{T}^{vir})\rangle_{g,n,d}$$
$$= \sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{n!}\langle \pi^\star(T_{\mathbf{s}}(\psi)),\ldots,\pi^\star(T_{\mathbf{s}}(\psi)),[\mathrm{ch}(E)\,\mathrm{Td}^\vee(L)]_{k+1};\mathrm{Td}_{\mathbf{s}}(\pi^\star\mathcal{T}^{vir})\rangle_{g,n+1,d}$$

Using (2.24) and the fact that L_{n+1} is trivial on the divisors $D_i=\sigma_i(X_{g,n,d})$, we can write this as

$$\begin{aligned}&-\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{(n-1)!}\left\langle\left[\frac{\mathrm{ch}_{k+1}(E)T_{\mathbf{s}}(\psi)}{\psi}\right]_+, T_{\mathbf{s}}(\psi),\ldots,T_{\mathbf{s}}(\psi);\mathrm{Td}_{\mathbf{s}}(\mathcal{T}^{vir})\right\rangle_{g,n,d}\\ &\quad+\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{n!}\langle T_{\mathbf{s}}(\psi),\ldots,T_{\mathbf{s}}(\psi),[\mathrm{ch}(E)\,\mathrm{Td}^\vee(L)]_{k+1};\mathrm{Td}_{\mathbf{s}}(\pi^\star\mathcal{T}^{vir})\rangle_{g,n+1,d}\end{aligned} \tag{2.39}$$

We concentrate on the second sum in (2.39). Applying (2.25) we find that

$$\mathrm{Td}_{\mathbf{s}}(\pi^\star\mathcal{T}^{vir}) = \mathrm{Td}_{\mathbf{s}}(\mathcal{T}^{vir})\,\mathrm{Td}_{\mathbf{s}}(-L_{n+1}^{-1})\,\mathrm{Td}_{\mathbf{s}}\Big(\sum_{i=1}^n(\sigma_{i\star}\mathcal{O}_{X_{g,n,d}})^\vee\Big)\,\mathrm{Td}_{\mathbf{s}}((i_\star\mathcal{O}_{\mathcal{Z}})^\vee) \tag{2.40}$$

Grothendieck–Riemann–Roch calculations parallel to that on pages 62–63 yield

$$\mathrm{Td}_{\mathbf{s}}\Big(\sum_{i=1}^n(\sigma_{i\star}\mathcal{O}_{X_{g,n,d}})^\vee\Big) = 1-\sum_{i=1}^n\sigma_{i\star}\left(\frac{1}{\psi_i}\left(\frac{1}{\mathrm{Td}_{\mathbf{s}}(L_i^{-1})}-1\right)\right)$$

and

$$\mathrm{Td}_{\mathbf{s}}((i_\star\mathcal{O}_{\mathcal{Z}})^\vee) = 1+i_\star\left[\frac{1}{\psi_+\psi_-}\left(\frac{\mathrm{Td}_{\mathbf{s}}(L_+^{-1}\otimes L_-^{-1})}{\mathrm{Td}_{\mathbf{s}}(L_+^{-1})\,\mathrm{Td}_{\mathbf{s}}(L_-^{-1})}-1\right)\right]$$

so

$$
\begin{aligned}
&\mathrm{Td}_{\mathbf{s}}\,(-L_{n+1}^{-1})\,\mathrm{Td}_{\mathbf{s}}\left(\sum_{i=1}^{n}(\sigma_{i\star}\mathcal{O}_{X_{g,n,d}})^{\vee}q\right)\mathrm{Td}_{\mathbf{s}}\,((i_\star\mathcal{O}_{\mathcal{Z}})^{\vee}) \\
&\quad = (1+\mathrm{Td}_{\mathbf{s}}\,(-L_{n+1}^{-1})-1)\times\left(1-\sum_{i=1}^{n}\sigma_{i\star}\left[\frac{1}{\psi_i}\left(\frac{1}{\mathrm{Td}_{\mathbf{s}}\,(L_i^{-1})}-1\right)\right]\right) \\
&\qquad\qquad \times\left(1+i_\star\left[\frac{1}{\psi_+\psi_-}\left(\frac{\mathrm{Td}_{\mathbf{s}}\,(L_+^{-1}\otimes L_-^{-1})}{\mathrm{Td}_{\mathbf{s}}\,(L_+^{-1})\,\mathrm{Td}_{\mathbf{s}}\,(L_-^{-1})}-1\right)\right]\right) \\
&\quad = \mathrm{Td}_{\mathbf{s}}\,(-L_{n+1}^{-1})-\sum_{i=1}^{n}\sigma_{i\star}\left[\frac{1}{\psi_i}\left(\frac{1}{\mathrm{Td}_{\mathbf{s}}\,(L_i^{-1})}-1\right)\right] \\
&\qquad\qquad + i_\star\left[\frac{1}{\psi_+\psi_-}\left(\frac{\mathrm{Td}_{\mathbf{s}}\,(L_+^{-1}\otimes L_-^{-1})}{\mathrm{Td}_{\mathbf{s}}\,(L_+^{-1})\,\mathrm{Td}_{\mathbf{s}}\,(L_-^{-1})}-1\right)\right]
\end{aligned} \tag{2.41}
$$

Here we used the fact that $\mathrm{Td}_{\mathbf{s}}\,(-L_{n+1}^{-1})-1$, which is divisible by ψ_{n+1}, vanishes on the divisors D_i and on $\mathcal{Z}$. Combining (2.40) and (2.41) we see that we can divide the second sum in (2.39) into three parts, which correspond to the three summands in (2.41). We call the part of (2.39) corresponding to the first summand in (2.41) the *smooth contribution*, the part of (2.39) corresponding to the second summand in (2.41) the *divisor contribution* and the part of (2.39) corresponding to the third summand in (2.41) the *nodal contribution*. We evaluate these parts separately.

The divisor contribution to (2.39) is

$$
\begin{aligned}
-\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{n!}\Big\langle & T_{\mathbf{s}}(\psi),\ldots,T_{\mathbf{s}}(\psi),[\mathrm{ch}(E)\,\mathrm{Td}^{\vee}\,(L)]_{k+1}; \\
&\mathrm{Td}_{\mathbf{s}}\,(\mathcal{T}^{vir})\sum_{i=1}^{n}\sigma_{i\star}\left[\frac{1}{\psi_i}\left(\frac{1}{\mathrm{Td}_{\mathbf{s}}\,(L_i^{-1})}-1\right)\right]\Big\rangle_{g,n+1,d}
\end{aligned} \tag{2.42}
$$

We evaluate this by pulling back along the maps σ_i. Since the normal bundle to the divisor D_i in $X_{g,n+1,d}$ is L_i^{-1},

$$\mathrm{Td}_{\mathbf{s}}\,(\sigma_i^\star\mathcal{T}^{vir})=\mathrm{Td}_{\mathbf{s}}\,(\mathcal{T}^{vir})\,\mathrm{Td}_{\mathbf{s}}\,(L_i^{-1})$$

and since ψ_i and ψ_{n+1} vanish on D_i, (2.42) becomes

$$-\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{(n-1)!}\Big\langle\frac{\mathrm{ch}_{k+1}(E)}{\psi}\Big(\frac{1}{\mathrm{Td}_{\mathbf{s}}\,(L^{-1})}-1\Big)T_{\mathbf{s}}(0),T_{\mathbf{s}}(\psi),\ldots,T_{\mathbf{s}}(\psi);$$
$$\mathrm{Td}_{\mathbf{s}}\,(\mathcal{T}^{vir})\,\mathrm{Td}_{\mathbf{s}}\,(L_1^{-1})\Big\rangle_{g,n,d}$$
$$=-\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{(n-1)!}\Big\langle\frac{\mathrm{ch}_{k+1}(E)}{\psi}(1-\mathrm{Td}_{\mathbf{s}}\,(L^{-1}))T_{\mathbf{s}}(0),T_{\mathbf{s}}(\psi),\ldots,T_{\mathbf{s}}(\psi);\mathrm{Td}_{\mathbf{s}}\,(\mathcal{T}^{vir})\Big\rangle_{g,n,d} \tag{2.43}$$

The nodal contribution to (2.39) is

$$\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{n!}\Big\langle T_{\mathbf{s}}(\psi),\ldots,T_{\mathbf{s}}(\psi),[\mathrm{ch}(E)\,\mathrm{Td}^\vee\,(L)]_{k+1};$$
$$\mathrm{Td}_{\mathbf{s}}\,(\mathcal{T}^{vir})i_\star\Big[\frac{1}{\psi_+\psi_-}\Big(\frac{\mathrm{Td}_{\mathbf{s}}\,(L_+^{-1}\otimes L_-^{-1})}{\mathrm{Td}_{\mathbf{s}}\,(L_+^{-1})\,\mathrm{Td}_{\mathbf{s}}\,(L_-^{-1})}-1\Big)\Big]\Big\rangle_{g,n+1,d}$$

Since the normal bundle to $\mathcal{Z}$ in $X_{g,n+1,d}$ is $L_+^{-1}\oplus L_-^{-1}$, and since L_{n+1} is trivial on $\mathcal{Z}$, this is

$$\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{n!}\int_{i_\star[\mathcal{Z}]}T_{\mathbf{s}}(\psi_1)\wedge\ldots\wedge T_{\mathbf{s}}(\psi_n)\wedge\Big[\frac{\mathrm{ch}_{k+1}(E)}{\psi_+\psi_-}\Big(\frac{\mathrm{Td}_{\mathbf{s}}\,(L_+^{-1}\otimes L_-^{-1})}{\mathrm{Td}_{\mathbf{s}}\,(L_+^{-1})\,\mathrm{Td}_{\mathbf{s}}\,(L_-^{-1})}-1\Big)\Big]$$
$$\wedge\,\mathrm{Td}_{\mathbf{s}}\,(\mathcal{T}_{\mathcal{Z}}^{vir})\wedge\mathrm{Td}_{\mathbf{s}}\,(L_+^{-1}+L_-^{-1})$$
$$=\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{n!}\int_{i_\star[\mathcal{Z}]}T_{\mathbf{s}}(\psi_1)\wedge\ldots\wedge T_{\mathbf{s}}(\psi_n)\wedge\mathrm{Td}_{\mathbf{s}}\,(\mathcal{T}_{\mathcal{Z}}^{vir})$$
$$\wedge\Big[\frac{\mathrm{ch}_{k+1}(E)}{\psi_+\psi_-}(\mathrm{Td}_{\mathbf{s}}\,(L_+^{-1}\otimes L_-^{-1})-\mathrm{Td}_{\mathbf{s}}\,(L_+^{-1})\,\mathrm{Td}_{\mathbf{s}}\,(L_-^{-1}))\Big]$$

Processing this as before, we find that the nodal contribution to (2.39) is

$$\mathcal{D}_{\mathbf{s}}^{-1}\Big(\frac{1}{2}\sum_{r,s}D_k^{\alpha,r;\beta,s}p_r^\alpha p_s^\beta\Big)\mathcal{D}_{\mathbf{s}} \tag{2.44}$$

where

$$\sum_{r,s}D_k^{\alpha,r;\beta,s}\psi_+^r\psi_-^s=\Big(\frac{\mathrm{ch}_{k+1}(E)}{\psi_+\psi_-}(\mathrm{Td}_{\mathbf{s}}\,(L_+^{-1}\otimes L_-^{-1})-\mathrm{Td}_{\mathbf{s}}\,(L_+^{-1})\,\mathrm{Td}_{\mathbf{s}}\,(L_-^{-1}))\Big)^{\alpha\beta} \tag{2.45}$$

The smooth contribution to (2.39) is

$$\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{n!}\langle T_{\mathbf{s}}(\psi),\ldots,T_{\mathbf{s}}(\psi),[\mathrm{ch}(E)\,\mathrm{Td}^\vee\,(L)]_{k+1};\mathrm{Td}_{\mathbf{s}}\,(\mathcal{T}^{vir})\,\mathrm{Td}_{\mathbf{s}}\,(-L_{n+1}^{-1})\rangle_{g,n+1,d}$$

Renumbering, this is

$$\begin{aligned}\sum_{g,n,d} \frac{Q^d \hbar^{g-1}}{(n-1)!} \langle [\mathrm{ch}(E)\,\mathrm{Td}^\vee(L)]_{k+1}\,\mathrm{Td}_{\mathbf{s}}(-L^{-1}), T_{\mathbf{s}}(\psi), \ldots, T_{\mathbf{s}}(\psi); \mathrm{Td}_{\mathbf{s}}(\mathcal{T}^{vir}) \rangle_{g,n,d} \\ -\frac{1}{2\hbar} \langle [\mathrm{ch}(E)\,\mathrm{Td}^\vee(L)]_{k+1}\,\mathrm{Td}_{\mathbf{s}}(-L^{-1}), T_{\mathbf{s}}(\psi), T_{\mathbf{s}}(\psi); \mathrm{Td}_{\mathbf{s}}(\mathcal{T}^{vir}) \rangle_{0,3,0} \\ - \langle [\mathrm{ch}(E)\,\mathrm{Td}^\vee(L)]_{k+1}\,\mathrm{Td}_{\mathbf{s}}(-L^{-1}); \mathrm{Td}_{\mathbf{s}}(\mathcal{T}^{vir}) \rangle_{1,1,0}\end{aligned} \tag{2.46}$$

Using the facts that

- $X_{0,3,0} = X$
- $[X_{0,3,0}]$ is the fundamental class of X
- All universal cotangent lines over $X_{0,3,0}$ are trivial
- $\mathcal{T}^{vir}_{X_{0,3,0}} = TX$

we can evaluate the first exceptional term in (2.46):

$$\begin{aligned}-\frac{1}{2\hbar} \langle [\mathrm{ch}(E)\,\mathrm{Td}^\vee(L)]_{k+1}\,\mathrm{Td}_{\mathbf{s}}(-L^{-1}), T_{\mathbf{s}}(\psi), T_{\mathbf{s}}(\psi); \mathrm{Td}_{\mathbf{s}}(\mathcal{T}^{vir}) \rangle_{0,3,0} \\ = -\frac{1}{2\hbar} \int_X \mathrm{ch}_{k+1}(E) \wedge (T_{\mathbf{s}})_0 \wedge (T_{\mathbf{s}})_0 \wedge \mathrm{Td}_{\mathbf{s}}(TX) \\ = -\frac{1}{2\hbar} \int_X \mathrm{ch}_{k+1}(E) \wedge q_0 \wedge q_0 \\ = -\frac{1}{2\hbar} (\mathrm{ch}_{k+1}(E) q_0, q_0)\end{aligned} \tag{2.47}$$

To evaluate the second exceptional term in (2.46) we need to compute $\mathrm{Td}_{\mathbf{s}}(\mathcal{T}^{vir}_{X_{1,1,0}})$. This is

$$\exp\Big(\sum_{l>0} s_l\, \mathrm{ch}_l(\mathcal{T}^{vir}_{X_{1,1,0}}) \Big)$$

which, using (2.16) and (2.22), is

$$\exp\Big(\sum_{l>0} s_l\, \mathrm{ch}_l(E_{1,1,0}) \Big) \exp\Big(\sum_{l>0} s_l \pi_\star \Big[-\frac{(-\psi)^{l+1}}{l!} + i_\star \frac{(-\psi_+ - \psi_-)^{l-1}}{l!} \Big] \Big)$$

Applying the discussion on page 68 yields

$$\begin{aligned}\mathrm{Td}_{\mathbf{s}}\,(\mathcal{T}^{vir}_{X_{1,1,0}}) &= \exp\Big(\sum_{l>0} s_l\psi_1\,\mathrm{ch}_{l-1}(E)\Big)\exp(-s_1\pi_\star(\psi_2)^2 + s_1\pi_\star i_\star 1)\\ &= 1+\psi_1\sum_{l>0} s_l\,\mathrm{ch}_{l-1}(E) - s_1\pi_\star(\psi_2)^2 + s_1\pi_\star i_\star 1\end{aligned}$$

Similarly

$$\mathrm{Td}_{\mathbf{s}}\,(-L_1^{-1}) = 1 + s_1\psi_1$$

and so, applying the discussion on page 68 once again,

$$\begin{aligned}&-\langle[\mathrm{ch}(E)\,\mathrm{Td}^\vee\,(L)]_{k+1}\,\mathrm{Td}_{\mathbf{s}}\,(-L^{-1});\mathrm{Td}_{\mathbf{s}}\,(\mathcal{T}^{vir})\rangle_{1,1,0}\\ &\quad= -\int_{X\times\overline{\mathcal{M}}_{1,1}}\left(\mathrm{ch}_{k+1}(E)-\frac{\mathrm{ch}_k(E)}{2}\psi_1\right)(1+s_1\psi_1)(\mathbf{e}(TX)+\psi_1 c_{D-1}(TX))\\ &\qquad\qquad\times\Big(1+\psi_1\sum_{l>0}s_l\,\mathrm{ch}_{l-1}(E)-s_1\pi_\star(\psi_2)^2+s_1\pi_\star i_\star 1\Big)\\ &\quad=\frac{1}{48}\int_X \mathrm{ch}_k(E)\mathbf{e}(TX)-\frac{s_1}{24}\int_X\mathrm{ch}_{k+1}(E)\mathbf{e}(TX)\\ &\qquad+\frac{1}{24}\int_X\mathrm{ch}_{k+1}(E)c_{D-1}(TX)-\frac{1}{24}\int_X\mathrm{ch}_{k+1}(E)\Big(\sum_{l>0}s_l\,\mathrm{ch}_{l-1}(E)\Big)\mathbf{e}(TX)\\ &\qquad+s_1\int_X\mathrm{ch}_{k+1}(E)\mathbf{e}(TX)\int_{\overline{\mathcal{M}}_{1,1}}\pi_\star(\psi_2^2)-s_1\int_X\mathrm{ch}_{k+1}(E)\mathbf{e}(TX)\int_{\overline{\mathcal{M}}_{1,1}}\pi_\star i_\star 1\end{aligned}$$

Now

$$\begin{aligned}\int_{\overline{\mathcal{M}}_{1,1}}\pi_\star(\psi_2^2) &= \int_{\overline{\mathcal{M}}_{1,2}}\psi_2^2\\ &=\frac{1}{24}\qquad\text{(string equation)}\end{aligned}$$

and

$$\begin{aligned}\int_{\overline{\mathcal{M}}_{1,1}}\pi_\star i_\star 1 &= \int_{i_\star[\mathcal{Z}]} 1\\ &=\frac{1}{2}\int_{\tilde{\mathcal{Z}}}1\\ &=\frac{1}{2}\end{aligned}$$

so

$$
\begin{aligned}
-\langle [\mathrm{ch}(E)\,\mathrm{Td}^\vee(L)]_{k+1}\,\mathrm{Td}_{\mathbf{s}}(-L^{-1});\mathrm{Td}_{\mathbf{s}}(\mathcal{T}^{vir})\rangle_{1,1,0}&\\
=&\frac{1}{48}\int_X \mathrm{ch}_k(E)\mathbf{e}(TX)+\frac{1}{24}\int_X \mathrm{ch}_{k+1}(E)c_{D-1}(TX)\\
&-\frac{1}{24}\int_X \mathrm{ch}_{k+1}(E)\Big(\sum_{l>0} s_l\,\mathrm{ch}_{l-1}(E)\Big)\mathbf{e}(TX)-\frac{s_1}{2}\int_X \mathrm{ch}_{k+1}(E)\mathbf{e}(TX)
\end{aligned}
$$

Combining this with (2.47), (2.46), (2.44), (2.43), (2.39), (2.38), (2.34), and (2.33) we see that the target space terms in (2.17) are

$$
\begin{aligned}
&-\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{(n-1)!}\Big\langle\Big[\frac{\mathrm{ch}_{k+1}(E)T_{\mathbf{s}}(\psi)}{\psi}\Big]_+,T_{\mathbf{s}}(\psi),\ldots,T_{\mathbf{s}}(\psi);\mathrm{Td}_{\mathbf{s}}(\mathcal{T}^{vir})\Big\rangle_{g,n,d}\\
&+\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{(n-1)!}\langle[\mathrm{ch}(E)\,\mathrm{Td}^\vee(L)]_{k+1}\,\mathrm{Td}_{\mathbf{s}}(-L^{-1}),T_{\mathbf{s}}(\psi),\ldots,T_{\mathbf{s}}(\psi);\mathrm{Td}_{\mathbf{s}}(\mathcal{T}^{vir})\rangle_{g,n,d}\\
&-\frac{1}{2\hbar}(\mathrm{ch}_{k+1}(E)q_0,q_0)\\
&+\frac{1}{48}\int_X \mathrm{ch}_k(E)\mathbf{e}(TX)-\frac{1}{24}\int_X\Big(\mathrm{ch}_{k+1}(E)\sum_{l>0}s_l\,\mathrm{ch}_{l-1}(E)\Big)\mathbf{e}(TX)\\
&+\frac{1}{24}\int_X \mathrm{ch}_{k+1}(E)c_{D-1}(TX)-\frac{s_1}{2}\int_X \mathrm{ch}_{k+1}(E)\mathbf{e}(TX) \qquad (2.48)\\
&-\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{(n-1)!}\Big\langle\frac{\mathrm{ch}_{k+1}(E)}{\psi}(1-\mathrm{Td}_{\mathbf{s}}(L^{-1}))T_{\mathbf{s}}(0),T_{\mathbf{s}}(\psi),\ldots,T_{\mathbf{s}}(\psi);\mathrm{Td}_{\mathbf{s}}(\mathcal{T}^{vir})\Big\rangle_{g,n,d}\\
&-\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{(n-1)!}\Big\langle\Big[\Big[\mathrm{ch}(E)\frac{\mathrm{Td}^\vee(L)}{\psi}\Big]_k\Big]_+T_{\mathbf{s}}(\psi),T_{\mathbf{s}}(\psi),\ldots,T_{\mathbf{s}}(\psi);\mathrm{Td}_{\mathbf{s}}(\mathcal{T}^{vir})\Big\rangle_{g,n,d}\\
&+\mathcal{D}_{\mathbf{s}}^{-1}\Big(\frac{1}{2}\sum_{r,s}B_k^{\alpha,r;\beta,s}p_r^\alpha p_s^\beta\Big)\mathcal{D}_{\mathbf{s}}+\mathcal{D}_{\mathbf{s}}^{-1}\Big(\frac{1}{2}\sum_{r,s}C_k^{\alpha,r;\beta,s}p_r^\alpha p_s^\beta\Big)\mathcal{D}_{\mathbf{s}}\\
&+\mathcal{D}_{\mathbf{s}}^{-1}\Big(\frac{1}{2}\sum_{r,s}D_k^{\alpha,r;\beta,s}p_r^\alpha p_s^\beta\Big)\mathcal{D}_{\mathbf{s}}
\end{aligned}
$$

2.5.5 Collecting everything

Our expression (2.14) for $\mathcal{D}_{\mathbf{s}}^{-1}(\partial\mathcal{D}_{\mathbf{s}}/\partial s_k)$ is the sum of (2.15), (2.32) and (2.48). The second term in (2.15) cancels with the first term in (2.32), so

$$\begin{aligned}
\mathcal{D}_{\mathbf{s}}^{-1}\frac{\partial\mathcal{D}_{\mathbf{s}}}{\partial s_k} = &-\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{(n-1)!}\left\langle\frac{\mathrm{ch}_k(TX)}{2\sqrt{\mathrm{Td}_{\mathbf{s}}\,(TX)}}(\mathbf{t}_0(\psi)-\psi),T_{\mathbf{s}}(\psi),\ldots,T_{\mathbf{s}}(\psi);\mathrm{Td}_{\mathbf{s}}\,(\mathcal{T}^{vir})\right\rangle_{g,n,d}\\
&-\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{(n-1)!}\left\langle\left[\frac{\mathrm{ch}_{k+1}(E)T_{\mathbf{s}}(\psi)}{\psi}\right]_+,T_{\mathbf{s}}(\psi),\ldots,T_{\mathbf{s}}(\psi);\mathrm{Td}_{\mathbf{s}}\,(\mathcal{T}^{vir})\right\rangle_{g,n,d}\\
&+\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{(n-1)!}\langle[\mathrm{ch}(E)\,\mathrm{Td}^\vee\,(L)]_{k+1}\,\mathrm{Td}_{\mathbf{s}}\,(-L^{-1}),T_{\mathbf{s}}(\psi),\ldots,T_{\mathbf{s}}(\psi);\mathrm{Td}_{\mathbf{s}}\,(\mathcal{T}^{vir})\rangle_{g,n,d}\\
&-\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{(n-1)!}\left\langle\left[\left[\mathrm{ch}(E)\frac{\mathrm{Td}^\vee\,(L)}{\psi}\right]_k\right]_+T_{\mathbf{s}}(\psi),T_{\mathbf{s}}(\psi),\ldots,T_{\mathbf{s}}(\psi);\mathrm{Td}_{\mathbf{s}}\,(\mathcal{T}^{vir})\right\rangle_{g,n,d}\\
&-\mathcal{D}_{\mathbf{s}}^{-1}\Big(\frac{1}{2}\sum_{r,s}A_k^{\alpha,r;\beta,s}p_r^\alpha p_s^\beta\Big)\mathcal{D}_{\mathbf{s}}+\mathcal{D}_{\mathbf{s}}^{-1}\Big(\frac{1}{2}\sum_{r,s}B_k^{\alpha,r;\beta,s}p_r^\alpha p_s^\beta\Big)\mathcal{D}_{\mathbf{s}}\\
&+\mathcal{D}_{\mathbf{s}}^{-1}\Big(\frac{1}{2}\sum_{r,s}C_k^{\alpha,r;\beta,s}p_r^\alpha p_s^\beta\Big)\mathcal{D}_{\mathbf{s}}+\mathcal{D}_{\mathbf{s}}^{-1}\Big(\frac{1}{2}\sum_{r,s}D_k^{\alpha,r;\beta,s}p_r^\alpha p_s^\beta\Big)\mathcal{D}_{\mathbf{s}}\\
&-\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{(n-1)!}\left\langle\frac{\mathrm{ch}_{k+1}(E)}{\psi}(1-\mathrm{Td}_{\mathbf{s}}\,(L^{-1}))T_{\mathbf{s}}(0),T_{\mathbf{s}}(\psi),\ldots,T_{\mathbf{s}}(\psi);\mathrm{Td}_{\mathbf{s}}\,(\mathcal{T}^{vir})\right\rangle_{g,n,d}\\
&-\frac{1}{2\hbar}(\mathrm{ch}_{k+1}(E)q_0,q_0)+\frac{1}{48}\int_X\mathrm{ch}_k(E)\mathbf{e}(TX)+\frac{1}{24}\int_X\mathrm{ch}_{k+1}(E)c_{D-1}(TX)\\
&-\frac{1}{24}\int_X\Big(\mathrm{ch}_{k+1}(E)\sum_{l>0}s_l\,\mathrm{ch}_{l-1}(E)\Big)\mathbf{e}(TX)-\frac{s_1}{2}\int_X\mathrm{ch}_{k+1}(E)\mathbf{e}(TX)
\end{aligned}$$

The first four terms together insert

$$-\frac{\mathrm{ch}_k(TX)}{2\sqrt{\mathrm{Td}_{\mathbf{s}}\,(TX)}}(\mathbf{t}_0(\psi)-\psi)-\left[\frac{\mathrm{ch}_{k+1}(E)T_{\mathbf{s}}(\psi)}{\psi}\right]_++[\mathrm{ch}(E)\,\mathrm{Td}^\vee\,(L)]_{k+1}\,\mathrm{Td}_{\mathbf{s}}\,(-L^{-1})$$
$$-\left[\left[\mathrm{ch}(E)\frac{\mathrm{Td}^\vee\,(L)}{\psi}\right]_k\right]_+T_{\mathbf{s}}(\psi)$$

at the first marked point. This is

$$-\frac{\mathrm{ch}_k(TX)}{2\sqrt{\mathrm{Td}_{\mathbf{s}}(TX)}}(\mathbf{t}_0(\psi)-\psi)-\left[\left[\mathrm{ch}(E)\frac{\mathrm{Td}^{\vee}(L)}{\psi}\right]_k T_{\mathbf{s}}(\psi)\right]_+ + \left[\left[\mathrm{ch}(E)\frac{\mathrm{Td}^{\vee}(L)}{\psi}\right]_k \psi\,\mathrm{Td}_{\mathbf{s}}(-L^{-1})\right]_+$$

$$= -\frac{\mathrm{ch}_k(TX)}{2\sqrt{\mathrm{Td}_{\mathbf{s}}(TX)}}(\mathbf{t}_0(\psi)-\psi)-\left[\left[\mathrm{ch}(E)\frac{\mathrm{Td}^{\vee}(L)}{\psi}\right]_k (T_{\mathbf{s}}(\psi)+u(-\psi))\right]_+$$

$$= -\frac{\mathrm{ch}_k(TX)}{2\sqrt{\mathrm{Td}_{\mathbf{s}}(TX)}}(\mathbf{t}_0(\psi)-\psi)-\left[\left[\mathrm{ch}(E)\frac{\mathrm{Td}^{\vee}(L)}{\psi}\right]_k \frac{\mathbf{q}_0(\psi)}{\sqrt{\mathrm{Td}_{\mathbf{s}}(TX)}}\right]_+$$

or in other words

$$-\left[\Delta_k(\psi)\frac{\mathbf{q}_0(\psi)}{\sqrt{\mathrm{Td}_{\mathbf{s}}(TX)}}\right]_+$$

where

$$\Delta_k(\psi)=\left[\mathrm{ch}(E)\frac{\mathrm{Td}^{\vee}(L)}{\psi}\right]_k+\frac{\mathrm{ch}_k(E)}{2}$$

Thus

$$\begin{aligned}
\mathcal{D}_{\mathbf{s}}^{-1}\frac{\partial \mathcal{D}_{\mathbf{s}}}{\partial s_k} =& -\frac{1}{2\hbar}(\mathrm{ch}_{k+1}(E)q_0,q_0)\\
&-\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{(n-1)!}\left\langle\left[\Delta_k(\psi)\frac{\mathbf{q}_0(\psi)}{\sqrt{\mathrm{Td}_{\mathbf{s}}(TX)}}\right]_+, T_{\mathbf{s}}(\psi),\dots,T_{\mathbf{s}}(\psi);\mathrm{Td}_{\mathbf{s}}(\mathcal{T}^{vir})\right\rangle_{g,n,d}\\
&-\sum_{g,n,d}\frac{Q^d\hbar^{g-1}}{(n-1)!}\left\langle\frac{\mathrm{ch}_{k+1}(E)}{\psi}(1-\mathrm{Td}_{\mathbf{s}}(L^{-1}))T_{\mathbf{s}}(0),T_{\mathbf{s}}(\psi),\dots,T_{\mathbf{s}}(\psi);\mathrm{Td}_{\mathbf{s}}(\mathcal{T}^{vir})\right\rangle_{g,n,d}\\
&-\mathcal{D}_{\mathbf{s}}^{-1}\Big(\frac{1}{2}\sum_{r,s}A_k^{\alpha,r;\beta,s}p_r^{\alpha}p_s^{\beta}\Big)\mathcal{D}_{\mathbf{s}}+\mathcal{D}_{\mathbf{s}}^{-1}\Big(\frac{1}{2}\sum_{r,s}B_k^{\alpha,r;\beta,s}p_r^{\alpha}p_s^{\beta}\Big)\mathcal{D}_{\mathbf{s}}\\
&+\mathcal{D}_{\mathbf{s}}^{-1}\Big(\frac{1}{2}\sum_{r,s}C_k^{\alpha,r;\beta,s}p_r^{\alpha}p_s^{\beta}\Big)\mathcal{D}_{\mathbf{s}}+\mathcal{D}_{\mathbf{s}}^{-1}\Big(\frac{1}{2}\sum_{r,s}D_k^{\alpha,r;\beta,s}p_r^{\alpha}p_s^{\beta}\Big)\mathcal{D}_{\mathbf{s}}\\
&+\frac{1}{48}\int_X \mathrm{ch}_k(E)\mathbf{e}(TX)-\frac{1}{24}\int_X\Big(\mathrm{ch}_{k+1}(E)\sum_{l>0}s_l\,\mathrm{ch}_{l-1}(E)\Big)\mathbf{e}(TX)\\
&+\frac{1}{24}\int_X\mathrm{ch}_{k+1}(E)c_{D-1}(TX)-\frac{s_1}{2}\int_X\mathrm{ch}_{k+1}(E)\mathbf{e}(TX)
\end{aligned}\tag{2.49}$$

2.5.6 Hunting the quantized operators

Using the metric to lower the index on the matrix of multiplication by $\mathrm{ch}_{k+1}(E)$, we can write the first term in (2.49) as

$$-\mathcal{D}_{\mathbf{s}}^{-1}\left(\frac{1}{2\hbar}(\mathrm{ch}_{k+1}(E))_{\alpha\beta}q_0^\alpha q_0^\beta\right)\mathcal{D}_{\mathbf{s}}$$

since the affine-linear function q_0^α acts on $\mathfrak{Fock}$ as multiplication by

$$\frac{q_0^\alpha}{\sqrt{\hbar}}$$

If we set

$$\Delta_k(\psi) = \sum_{m\geq 0}\Delta_{k,2m-1}\psi^{2m-1} \qquad \Delta_{k,2m-1}\in H^\star(X;\widetilde{\Omega}^\star_{MU})$$

and define p_{-1}^ϵ to be zero for all ϵ then the second term in (2.49) is

$$-\mathcal{D}_{\mathbf{s}}^{-1}\Big(\sum_{m,n}(\Delta_{k,2m-1})_\alpha^\epsilon q_n^\alpha p_{n+2m-1}^\epsilon\Big)\mathcal{D}_{\mathbf{s}}$$

The third term in (2.49) is

$$-\mathcal{D}_{\mathbf{s}}^{-1}\Big(\sum_l c_l(\mathrm{ch}_{k+1}(E))_\beta^\alpha q_0^\beta p_l^\alpha\Big)\mathcal{D}_{\mathbf{s}}$$

where

$$\begin{aligned}\sum_l c_l\psi^l &= \frac{1-\mathrm{Td}_{\mathbf{s}}\left(L^{-1}\right)}{\psi}\\ &= \left[\frac{1}{u(-\psi)}\right]_+\end{aligned}$$

so

$$\begin{aligned}\frac{\partial\mathcal{D}_{\mathbf{s}}}{\partial s_k} =&-(\tfrac{1}{2}\left(\mathrm{ch}_{k+1}(E)\right)_{\alpha\beta}q_0^\alpha q_0^\beta)\mathcal{D}_{\mathbf{s}} - \Big(\sum_l c_l(\mathrm{ch}_{k+1}(E))_\beta^\alpha q_0^\beta p_l^\alpha\Big)\mathcal{D}_{\mathbf{s}}\\ &-\Big(\sum_{m,n}(\Delta_{k,2m-1})_\alpha^\epsilon q_n^\alpha p_{n+2m-1}^\epsilon\Big)\mathcal{D}_{\mathbf{s}}\\ &-\Big(\frac{1}{2}\sum_{r,s}A_k^{\alpha,r;\beta,s}p_r^\alpha p_s^\beta\Big)\mathcal{D}_{\mathbf{s}} + \Big(\frac{1}{2}\sum_{r,s}B_k^{\alpha,r;\beta,s}p_r^\alpha p_s^\beta\Big)\mathcal{D}_{\mathbf{s}}\\ &+\Big(\frac{1}{2}\sum_{r,s}C_k^{\alpha,r;\beta,s}p_r^\alpha p_s^\beta\Big)\mathcal{D}_{\mathbf{s}} + \Big(\frac{1}{2}\sum_{r,s}D_k^{\alpha,r;\beta,s}p_r^\alpha p_s^\beta\Big)\mathcal{D}_{\mathbf{s}}\\ &+\left(\begin{array}{c}\frac{1}{48}\int_X \mathrm{ch}_k(E)\mathbf{e}(TX) + \frac{1}{24}\int_X \mathrm{ch}_{k+1}(E)c_{D-1}(TX)\\ -\frac{1}{24}\int_X\Big(\mathrm{ch}_{k+1}(E)\sum_{l>0}s_l\,\mathrm{ch}_{l-1}(E)\Big)\mathbf{e}(TX) - \frac{s_1}{2}\int_X\mathrm{ch}_{k+1}(E)\mathbf{e}(TX)\end{array}\right)\mathcal{D}_{\mathbf{s}}\end{aligned} \tag{2.50}$$

Making the substitution

$$q_r^\alpha = \bar{q}_r^\alpha - \sum_s A^{\alpha,r;\beta,s} p_s^\beta$$

we find that

$$\begin{aligned}-\frac{1}{2}\left(\mathrm{ch}_{k+1}(E)\right)_{\alpha\beta} q_0^\alpha q_0^\beta =& -\frac{1}{2}\left(\mathrm{ch}_{k+1}(E)\right)_{\alpha\beta} \bar{q}_0^\alpha \bar{q}_0^\beta + \left(\mathrm{ch}_{k+1}(E)\right)_{\alpha\beta} \sum_l A^{\alpha,0;\epsilon,l} \bar{q}_0^\beta p_l^\epsilon \\ & -\frac{1}{2}\left(\mathrm{ch}_{k+1}(E)\right)_{\alpha\beta} \sum_{r,s} A^{\alpha,0;\mu,r} A^{\beta,0;\nu,s} p_r^\mu p_s^\nu\end{aligned}$$

Equation (2.11) gives

$$\begin{aligned}\sum_l A^{\alpha,0;\epsilon,l} \psi^l &= \left[\frac{g^{\alpha\epsilon}}{u(-\psi)}\right]_+ \\ &= g^{\alpha\epsilon} \sum_l c_l \psi^l\end{aligned}$$

so

$$A^{\alpha,0;\epsilon,l} = g^{\alpha\epsilon} c_l$$

Thus

$$\begin{aligned}-\frac{1}{2}\left(\mathrm{ch}_{k+1}(E)\right)_{\alpha\beta} q_0^\alpha q_0^\beta =& -\frac{1}{2}\left(\mathrm{ch}_{k+1}(E)\right)_{\alpha\beta} \bar{q}_0^\alpha \bar{q}_0^\beta + \sum_l \left(\mathrm{ch}_{k+1}(E)\right)^\epsilon{}_\beta c_l \bar{q}_0^\beta p_l^\epsilon \\ & -\frac{1}{2} \sum_{r,s} \left(\mathrm{ch}_{k+1}(E)\right)^{\mu\nu} c_r c_s p_r^\mu p_s^\nu\end{aligned}$$

Also,

$$\begin{aligned}-\sum_l c_l \left(\mathrm{ch}_{k+1}(E)\right)^\alpha_\beta q_0^\beta p_l^\alpha &= -\sum_l c_l \left(\mathrm{ch}_{k+1}(E)\right)^\alpha_\beta \bar{q}_0^\beta p_l^\alpha + \sum_{r,s} c_r \left(\mathrm{ch}_{k+1}(E)\right)^\alpha_\beta A^{\beta,0;\nu,s} p_s^\nu p_r^\alpha \\ &= -\sum_l c_l \left(\mathrm{ch}_{k+1}(E)\right)^\alpha_\beta \bar{q}_0^\beta p_l^\alpha + \sum_{r,s} \left(\mathrm{ch}_{k+1}(E)\right)^{\mu\nu} c_r c_s p_r^\mu p_s^\nu\end{aligned}$$

so the first two terms in (2.50) together give

$$-\left(\tfrac{1}{2}\left(\mathrm{ch}_{k+1}(E)\right)_{\alpha\beta} \bar{q}_0^\alpha \bar{q}_0^\beta\right)\mathcal{D}_{\mathbf{s}} + \left(\frac{1}{2}\sum_{r,s} \left(\mathrm{ch}_{k+1}(E)\right)^{\mu\nu} c_r c_s p_r^\mu p_s^\nu\right)\mathcal{D}_{\mathbf{s}} \tag{2.51}$$

The third term in (2.50) is

$$-\Big(\sum_{m,n} (\Delta_{k,2m-1})^\epsilon{}_\alpha \bar{q}_n^\alpha p_{n+2m-1}^\epsilon\Big)\mathcal{D}_{\mathbf{s}} + \Big(\sum_{l,m,n} (\Delta_{k,2m-1})^\epsilon{}_\alpha A^{\alpha,n;\beta,l} p_l^\beta p_{n+2m-1}^\epsilon\Big)\mathcal{D}_{\mathbf{s}} \tag{2.52}$$

Using the symmetry of $A^{\alpha,n;\beta,l}$ and the fact that multiplication by $\Delta_{k,2m-1}$ is self-adjoint, we can write the second term in (2.52) as

$$\Big(\frac{1}{2}\sum_{l,m,n}(\Delta_{k,2m-1})^{\epsilon}_{\alpha}A^{\alpha,n;\beta,l}(p^{\beta}_{l}p^{\epsilon}_{n+2m-1}+p^{\beta}_{l+2m-1}p^{\epsilon}_{n})\Big)\mathcal{D}_{\mathbf{s}}$$

But

$$\begin{aligned}\frac{1}{2}\sum_{l,m,n}&(\Delta_{k,2m-1})^{\epsilon}_{\alpha}A^{\alpha,n;\beta,l}(\psi^{l}_{+}[\psi^{n+2m-1}_{-}]_{+}+[\psi^{l+2m-1}_{+}]_{+}\psi^{n}_{-})\\
&=\frac{1}{2}\Big(\sum_{m}(\Delta_{k,2m-1})^{\epsilon}_{\alpha}(\psi^{2m-1}_{+}+\psi^{2m-1}_{-})\Big)\Big(\sum_{l,n\geq 0}A^{\alpha,n;\beta,l}\psi^{l}_{+}\psi^{n}_{-}\Big)\\
&\qquad-\frac{1}{2}\sum_{l}(\Delta_{k,-1})^{\epsilon}_{\alpha}A^{\alpha,0;\beta,l}(\psi^{l}_{+}\psi^{-1}_{-}+\psi^{-1}_{+}\psi^{l}_{-})\\
&=\frac{1}{2}\left(\Delta_k(\psi_+)+\Delta_k(\psi_-)\right)^{\epsilon\beta}\left[\frac{1}{u(-\psi_+-\psi_-)}\right]_{+}\\
&\qquad-\frac{(\mathrm{ch}_{k+1}(E))^{\epsilon\beta}}{2}\left(\frac{1}{\psi_-}\left[\frac{1}{u(-\psi_+)}\right]_{+}+\frac{1}{\psi_+}\left[\frac{1}{u(-\psi_-)}\right]_{+}\right)\end{aligned}\tag{2.53}$$

Comparing this with (2.37), we see that we can write it in terms of the $C^{\epsilon,r;\beta,s}_{k}$. The right-hand side of equation (2.53) is

$$\begin{aligned}-\frac{1}{2}&\left(\Delta_k(\psi_+)+\Delta_k(\psi_-)\right)^{\epsilon\beta}\frac{\mathrm{Td}_{\mathbf{s}}\left(L^{-1}_{+}\otimes L^{-1}_{-}\right)-1}{\psi_++\psi_-}+\frac{(\mathrm{ch}_{k+1}(E))^{\epsilon\beta}}{2\psi_+\psi_-}(\mathrm{Td}_{\mathbf{s}}\left(L^{-1}_{+}\right)-1+\mathrm{Td}_{\mathbf{s}}\left(L^{-1}_{-}\right)-1)\\
&=-\frac{1}{2}\left[\frac{1}{\psi_++\psi_-}(\Delta_k(\psi_+)+\Delta_k(\psi_-))\right]^{\epsilon\beta}_{+}(\mathrm{Td}_{\mathbf{s}}\left(L^{-1}_{+}\otimes L^{-1}_{-}\right)-1)\\
&\qquad-\frac{1}{2}\frac{1}{\psi_++\psi_-}\left(\frac{\mathrm{ch}_{k+1}(E)}{\psi_+}+\frac{\mathrm{ch}_{k+1}(E)}{\psi_-}\right)^{\epsilon\beta}(\mathrm{Td}_{\mathbf{s}}\left(L^{-1}_{+}\otimes L^{-1}_{-}\right)-1)\\
&\qquad+\frac{(\mathrm{ch}_{k+1}(E))^{\epsilon\beta}}{2\psi_+\psi_-}(\mathrm{Td}_{\mathbf{s}}\left(L^{-1}_{+}\right)+\mathrm{Td}_{\mathbf{s}}\left(L^{-1}_{-}\right)-2)\\
&=-\frac{1}{2}\sum_{r,s}C^{\epsilon,r;\beta,s}_{k}\psi^{r}_{+}\psi^{s}_{-}-\frac{(\mathrm{ch}_{k+1}(E))^{\epsilon\beta}}{2\psi_+\psi_-}(\mathrm{Td}_{\mathbf{s}}\left(L^{-1}_{+}\otimes L^{-1}_{-}\right)-\mathrm{Td}_{\mathbf{s}}\left(L^{-1}_{+}\right)-\mathrm{Td}_{\mathbf{s}}\left(L^{-1}_{-}\right)+1)\\
&=-\frac{1}{2}\sum_{r,s}C^{\epsilon,r;\beta,s}_{k}\psi^{r}_{+}\psi^{s}_{-}-\frac{(\mathrm{ch}_{k+1}(E))^{\epsilon\beta}}{2\psi_+\psi_-}(\mathrm{Td}_{\mathbf{s}}\left(L^{-1}_{+}\otimes L^{-1}_{-}\right)-\mathrm{Td}_{\mathbf{s}}\left(L^{-1}_{+}\right)\mathrm{Td}_{\mathbf{s}}\left(L^{-1}_{-}\right))\\
&\qquad-\frac{(\mathrm{ch}_{k+1}(E))^{\epsilon\beta}}{2\psi_+\psi_-}(\mathrm{Td}_{\mathbf{s}}\left(L^{-1}_{+}\right)-1)(\mathrm{Td}_{\mathbf{s}}\left(L^{-1}_{-}\right)-1)\end{aligned}$$

Using (2.45), we can write this as

$$-\frac{1}{2}\sum_{r,s} C_k^{\epsilon,r;\beta,s}\psi_+^r\psi_-^s - \frac{1}{2}\sum_{r,s} D_k^{\epsilon,r;\beta,s}\psi_+^r\psi_-^s - \frac{1}{2}\left(\mathrm{ch}_{k+1}(E)\right)^{\epsilon\beta}\left[\frac{1}{u(-\psi_+)}\right]_+\left[\frac{1}{u(-\psi_-)}\right]_+$$

or in other words as

$$-\frac{1}{2}\sum_{r,s} C_k^{\epsilon,r;\beta,s}\psi_+^r\psi_-^s - \frac{1}{2}\sum_{r,s} D_k^{\epsilon,r;\beta,s}\psi_+^r\psi_-^s - \frac{1}{2}\sum_{r,s}(\mathrm{ch}_{k+1}(E))^{\epsilon\beta}c_r c_s\psi_+^r\psi_-^s$$

Thus the second term in (2.52) is

$$-\Big(\frac{1}{2}\sum_{r,s} C_k^{\epsilon,r;\beta,s}p_r^\epsilon p_s^\beta\Big)\mathcal{D}_{\mathbf{s}} - \Big(\frac{1}{2}\sum_{r,s} D_k^{\epsilon,r;\beta,s}p_r^\epsilon p_s^\beta\Big)\mathcal{D}_{\mathbf{s}} - \Big(\frac{1}{2}\sum_{r,s}(\mathrm{ch}_{k+1}(E))^{\epsilon\beta}c_r c_s p_r^\epsilon p_s^\beta\Big)\mathcal{D}_{\mathbf{s}}$$

Combining this with (2.52),(2.51), and (2.50), we find that

$$\begin{aligned}\frac{\partial\mathcal{D}_{\mathbf{s}}}{\partial s_k} =& -(\tfrac{1}{2}\left(\mathrm{ch}_{k+1}(E)\right)_{\alpha\beta}\bar{q}_0^\alpha\bar{q}_0^\beta)\mathcal{D}_{\mathbf{s}} - \Big(\sum_{m,n}(\Delta_{k,2m-1})^\epsilon_{\ \alpha}\bar{q}_n^\alpha p_{n+2m-1}^\epsilon\Big)\mathcal{D}_{\mathbf{s}}\\ &-\Big(\frac{1}{2}\sum_{r,s} A_k^{\alpha,r;\beta,s}p_r^\alpha p_s^\beta\Big)\mathcal{D}_{\mathbf{s}} + \Big(\frac{1}{2}\sum_{r,s} B_k^{\alpha,r;\beta,s}p_r^\alpha p_s^\beta\Big)\mathcal{D}_{\mathbf{s}}\\ &+\left(\begin{array}{c}\frac{1}{48}\int_X \mathrm{ch}_k(E)\mathbf{e}(TX) + \frac{1}{24}\int_X \mathrm{ch}_{k+1}(E)c_{D-1}(TX)\\ -\frac{1}{24}\int_X\Big(\mathrm{ch}_{k+1}(E)\sum_{l>0} s_l\,\mathrm{ch}_{l-1}(E)\Big)\mathbf{e}(TX) - \frac{s_1}{2}\int_X \mathrm{ch}_{k+1}(E)\mathbf{e}(TX)\end{array}\right)\mathcal{D}_{\mathbf{s}}\end{aligned} \tag{2.54}$$

But

$$\mathcal{G}_{\mathbf{s}} = \exp\Big(\frac{\hbar}{2}\sum_{r,s} A^{\alpha,r;\beta,s}\partial_{\alpha,r}\partial_{\beta,s}\Big)\mathcal{D}_{\mathbf{s}}$$

and

$$A_k^{\alpha,r;\beta,s} = \frac{\partial}{\partial s_k}A^{\alpha,r;\beta,s}$$

so (2.54) gives

$$\begin{aligned}\frac{\partial\mathcal{G}_{\mathbf{s}}}{\partial s_k} =& -\exp\Big(\frac{\hbar}{2}\sum_{r,s} A^{\mu,r;\nu,s}\partial_{\mu,r}\partial_{\nu,s}\Big)(\tfrac{1}{2}\left(\mathrm{ch}_{k+1}(E)\right)_{\alpha\beta}\bar{q}_0^\alpha\bar{q}_0^\beta)\mathcal{D}_{\mathbf{s}}\\ &-\exp\Big(\frac{\hbar}{2}\sum_{r,s} A^{\mu,r;\nu,s}\partial_{\mu,r}\partial_{\nu,s}\Big)\Big(\sum_{m,n}(\Delta_{k,2m-1})^\epsilon_{\ \alpha}\bar{q}_n^\alpha p_{n+2m-1}^\epsilon\Big)\mathcal{D}_{\mathbf{s}}\\ &+\exp\Big(\frac{\hbar}{2}\sum_{r,s} A^{\mu,r;\nu,s}\partial_{\mu,r}\partial_{\nu,s}\Big)\Big(\frac{1}{2}\sum_{r,s} B_k^{\alpha,r;\beta,s}p_r^\alpha p_s^\beta\Big)\mathcal{D}_{\mathbf{s}}\\ &+\left(\begin{array}{c}\frac{1}{48}\int_X \mathrm{ch}_k(E)\mathbf{e}(TX) + \frac{1}{24}\int_X \mathrm{ch}_{k+1}(E)c_{D-1}(TX)\\ -\frac{1}{24}\int_X\Big(\mathrm{ch}_{k+1}(E)\sum_{l>0} s_l\,\mathrm{ch}_{l-1}(E)\Big)\mathbf{e}(TX) - \frac{s_1}{2}\int_X \mathrm{ch}_{k+1}(E)\mathbf{e}(TX)\end{array}\right)\mathcal{G}_{\mathbf{s}}\end{aligned}$$

We know from the discussion on pages 92–94 that, roughly speaking, commuting $\bar{q}_r^\alpha$ past the exponential term turns it into q_r^α. In our situation there is also a cocycle contribution which comes from commuting the $q_0^\alpha q_0^\beta$ terms past the exponential term.

$$\begin{aligned}\frac{\partial \mathcal{G}_{\mathbf{s}}}{\partial s_k} = &-(\tfrac{1}{2}\,(\mathrm{ch}_{k+1}(E))_{\alpha\beta} q_0^\alpha q_0^\beta)\mathcal{G}_{\mathbf{s}} - \mathcal{C}\Big(\frac{1}{2}\sum_{r,s} A^{\mu,r;\nu,s} p_r^\mu p_s^\nu, \frac{1}{2}\,(\mathrm{ch}_{k+1}(E))_{\alpha\beta} q_0^\alpha q_0^\beta\Big)\mathcal{G}_{\mathbf{s}} \\ &- \Big(\sum_{m,n} (\Delta_{k,2m-1})_\alpha^\epsilon q_n^\alpha p_{n+2m-1}^\epsilon\Big)\mathcal{G}_{\mathbf{s}} + \Big(\frac{1}{2}\sum_{r,s} B_k^{\alpha,r;\beta,s} p_r^\alpha p_s^\beta\Big)\mathcal{G}_{\mathbf{s}} \\ &+ \left(\begin{array}{c} \frac{1}{48}\int_X \mathrm{ch}_k(E)\mathbf{e}(TX) + \frac{1}{24}\int_X \mathrm{ch}_{k+1}(E) c_{D-1}(TX) \\ -\frac{1}{24}\int_X \Big(\mathrm{ch}_{k+1}(E)\sum_{l>0} s_l\,\mathrm{ch}_{l-1}(E)\Big)\mathbf{e}(TX) - \frac{s_1}{2}\int_X \mathrm{ch}_{k+1}(E)\mathbf{e}(TX)\end{array}\right)\mathcal{G}_{\mathbf{s}}\end{aligned} \tag{2.55}$$

Since

$$A^{\alpha,0;\beta,0} = -s_1 g^{\alpha\beta}$$

we have

$$\begin{aligned}-\mathcal{C}\Big(\frac{1}{2}\sum_{r,s} A^{\mu,r;\nu,s} p_r^\mu p_s^\nu, \frac{1}{2}\,(\mathrm{ch}_{k+1}(E))_{\alpha\beta} q_0^\alpha q_0^\beta\Big)\mathcal{G}_{\mathbf{s}} &= \frac{s_1}{2}\,\mathrm{str}(\mathrm{ch}_{k+1}(E))\mathcal{G}_{\mathbf{s}} \\ &= \left(\frac{s_1}{2}\int_X \mathrm{ch}_{k+1}(E)\mathbf{e}(TX)\right)\mathcal{G}_{\mathbf{s}}\end{aligned}$$

This cancels with the fourth exceptional term in (2.55). Rewriting (2.55) in the notation of Example 1.3.3.1 gives

$$\begin{aligned}\frac{\partial \mathcal{G}_{\mathbf{s}}}{\partial s_k} = &\left(\frac{1}{2\hbar}\Omega_0((\Delta_k\mathbf{q})(-z), \mathbf{q}(z)) - \partial_{\Delta_k}\mathcal{G}_{\mathbf{s}} + \frac{\hbar}{2}\,(\partial \otimes_{\Delta_k} \partial)\right)\mathcal{G}_{\mathbf{s}} \\ &+ \left(\begin{array}{c} \frac{1}{48}\int_X \mathrm{ch}_k(E)\mathbf{e}(TX) + \frac{1}{24}\int_X \mathrm{ch}_{k+1}(E) c_{D-1}(TX) \\ -\frac{1}{24}\int_X \Big(\mathrm{ch}_{k+1}(E)\sum_{l>0} s_l\,\mathrm{ch}_{l-1}(E)\Big)\mathbf{e}(TX)\end{array}\right)\mathcal{G}_{\mathbf{s}}\end{aligned}$$

But Example 1.3.3.1 shows that

$$\frac{1}{2\hbar}\Omega_{\mathbf{s}}((\Delta_k\mathbf{q})(-z), \mathbf{q}(z)) - \partial_{\Delta_k}\mathcal{D}_{\mathbf{s}} + \frac{\hbar}{2}\,(\partial \otimes_{\Delta_k} \partial) = \widehat{\Delta_k}$$

and we know that

$$\begin{aligned}\Delta_k(\psi) &= \left[\mathrm{ch}(E)\frac{\mathrm{Td}^\vee(L)}{\psi}\right]_k + \frac{\mathrm{ch}_k(E)}{2} \\ &= \sum_{\substack{2m+r=k \\ r,m\geq 0}} \frac{B_{2m}}{(2m)!}\,\mathrm{ch}_r(E)\psi^{2m-1}\end{aligned}$$

Thus we have established (2.13). The proof is complete. □

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Appendix A

Many things are well-defined

Proposition A.0.1.

$$e^{F^1(\tau)}\hat{S}_\tau^{-1}\mathcal{A}_\tau$$

is well-defined as a formal function of $\mathbf{t}$ *and* τ *near* $\mathbf{t}=0$, $\tau=0$. *(Theorem 1.5.1 in fact implies that it does not depend on* τ*.)*

Proof. We work over the ground ring

$$\Lambda = \mathbb{C}[[Q]]$$

equipped with the Q-adic topology, and with the symplectic vector space

$$\mathcal{H} = \Big\{\sum_{k\in\mathbb{Z}} h_k z^k : h_k \in H^\star(X;\Lambda), h_k \to 0 \text{ in the topology of } \Lambda \text{ as } k\to\infty\Big\}$$

$S_\tau(z)$ certainly gives a well-defined linear transformation from $\mathcal{H}$ to itself. Corollary 1.4.2 shows that this is an element of the loop group, so the quantization $\hat{S}_\tau$ makes sense.

Define the $(\hbar,\bar{\mathbf{t}},\tau,Q)$-degree of a monomial

$$Q^d\hbar^{g-1}(\bar{t}_{i_1}^{\alpha_1})^{j_1}\dots(\bar{t}_{i_m}^{\alpha_n})^{j_m}(\tau^{\beta_1})^{k_1}\dots(\tau^{\beta_n})^{k_n}$$

to be $(g-1, j_1+\ldots+j_m, k_1+\ldots+k_n, d)$. For the reasons discussed on page 38, this quantity has invariant meaning. The moduli spaces $X_{0,0,0}$ and $X_{1,0,0}$ are empty, so if $\log\mathcal{A}_\tau$ contains a monomial of $(\hbar, \bar{\mathbf{t}}, \tau, Q)$-degree $(a, b, c, 0)$ then at least one of a, b, and c is strictly positive. From Proposition 1.3.2 we have that

$$(\hat{S}_\tau^{-1}\mathcal{A}_\tau)(\mathbf{q}) = \exp\left(\frac{W_{S_\tau}(\mathbf{q})}{2\hbar}\right)\mathcal{A}_\tau([S_\tau \mathbf{q}]_+)$$

The substitution

$$\bar{\mathbf{q}} = [S_\tau \mathbf{q}]_+$$

sets

$$\bar{\mathbf{t}} = [S_\tau \mathbf{t}]_+ - [S_\tau z]_+ + z$$

For any v

$$\begin{aligned}
([S_\tau z]_+, v) &= [z(S_\tau 1, v)]_+ \\
&= \left[z(1, v) + z\left\langle\!\!\left\langle \frac{1}{z - \psi}, v \right\rangle\!\!\right\rangle_{0,2}(\tau)\right]_+ \\
&= (z, v) + \sum_{n,d} \frac{Q^d}{n!}\langle 1, v, \tau, \ldots, \tau\rangle_{0,n+2,d} \\
&= (z, v) + \langle 1, v, \tau\rangle_{0,3,0} \qquad \text{(by the string equation)} \\
&= (z + \tau, v)
\end{aligned}$$

and so

$$[S_\tau z]_+ = z + \tau$$

This gives

$$\bar{\mathbf{t}} = [S_\tau \mathbf{t}]_+ - \tau$$

and therefore $\mathcal{A}_\tau([S_\tau \mathbf{q}]_+)$ is well-defined as a formal function of $\mathbf{t}$ and τ near $\mathbf{t} = 0$, $\tau = 0$. From above, we see that if $\log \mathcal{A}_\tau([S_\tau \mathbf{q}]_+)$ — regarded as a formal function of $\mathbf{t}$ and τ — contains a monomial of $(\hbar, \mathbf{t}, \tau, Q)$-degree $(a, b, c, 0)$ then at least one of a, b, and c is strictly positive.

It is clear from Proposition 1.4.1 that

$$W_{S_\tau}(\mathbf{q}) \equiv 0 \qquad \mod Q, \tau$$

Thus

$$(\hat{S}_\tau^{-1}\mathcal{A}_\tau)(\mathbf{q}) = \exp\left(\frac{W_{S_\tau}(\mathbf{q})}{2\hbar} + \log \mathcal{A}_\tau([S_\tau \mathbf{q}]_+)\right)$$

is the exponential of a power series containing only monomials of $(\hbar, \mathbf{t}, \tau, Q)$-degree (a, b, c, d) such that either at least one of a, b and c is strictly positive or $d \neq 0$. This implies that

$$e^{F^1(\tau)} \hat{S}_\tau^{-1} \mathcal{A}_\tau$$

is well-defined as a formal function of $\mathbf{t}$ and τ near $\mathbf{t} = 0$, $\tau = 0$. □

Proposition A.0.2.

$$\exp\Big(\sum_{m>0}\sum_{l\geq 0} s_{2m-1+l} \frac{B_{2m}}{(2m)!} (\mathrm{ch}_l(E) z^{2m-1})^{\wedge}\Big) \exp\Big(\sum_{l>0} s_{l-1} (\mathrm{ch}_l(E)/z)^{\wedge}\Big) \mathcal{D}_X$$

is well-defined as a formal function of $\mathbf{t}$ *which takes values in* $\Lambda = \mathbb{C}[\![Q]\!][\![s_0, s_1, \ldots]\!][\![\hbar, \hbar^{-1}]\!]$.

Proof. We work over the ground ring

$$\Lambda = \mathbb{C}[\![Q]\!][\![s_0, s_1, \ldots]\!]$$

equipped with the topology coming from the norm

$$\|Q^d s_{i_1}^{j_1} \ldots s_{i_n}^{j_n}\| = 2^{-\int_d \omega - i_1 j_1 - \ldots - i_n j_n}$$

where ω is the symplectic form on X. The symplectic vector space $\mathcal{H}$ in this context is

$$\mathcal{H} = \Big\{\sum_{k\in\mathbb{Z}} h_k z^k : h_k \in H^\star(X;\Lambda), h_k \to 0 \text{ in the topology of } \Lambda \text{ as } k \to \infty\Big\}$$

It is clear that multiplication by

$$S = \exp\Big(\sum_{l>0} s_{l-1} \frac{\mathrm{ch}_l(E)}{z}\Big)$$

defines a linear transformation from $\mathcal{H}$ to itself, and that the same is true for multiplication by

$$R = \exp\Big(\sum_{m>0}\sum_{l\geq 0} s_{2m-1+l} \frac{B_{2m}}{(2m)!} \mathrm{ch}_l(E) z^{2m-1}\Big)$$

Multiplication by a cohomology class gives a linear transformation of $H^\star(X)$ which is self-adjoint with respect to the Poincaré pairing, so multiplication by $\mathrm{ch}_l(E) z^{2m-1}$ gives an infinitesimal symplectomorphism of $\mathcal{H}$. S and R are therefore elements of the loop group, and so the quantizations $\hat{S}$ and $\widehat{R}$ make sense.

We write

$$S = TU$$

where

$$T = \exp\left(\sum_{l>1} s_{l-1} \frac{\mathrm{ch}_l(E)}{z}\right)$$

$$U = \exp\left(s_0 \frac{\mathrm{ch}_1(E)}{z}\right)$$

(We will need to treat s_0 differently from $s_1, s_2, \ldots$ since $\|s_0\| = 1$, whereas high powers of $s_1, s_2, \ldots$ have small norm.) Example 1.3.3.3 shows that the effect of the divisor flow $\widehat{U}$ on $\mathcal{D}_X$ is to replace Q^d by $Q^d \exp(s_0 \langle \mathrm{ch}_1(E), \rho \rangle)$ and then to multiply $\mathcal{D}_X$ by a function of s_0 (*cf* Corollary 1.8.2). Thus $\widehat{U}\mathcal{D}_X$ is well-defined as a formal function of $\mathbf{t}$ taking values in $\Lambda[\![\hbar, \hbar^{-1}]\!]$.

Recall that

$$(\widehat{T}\mathcal{F})(\mathbf{q}) = \exp\left(\frac{W_T(\mathbf{q})}{2\hbar}\right) \mathcal{F}([T^{-1}\mathbf{q}]_+) \tag{A.1}$$

(see Proposition 1.3.2). Making the change-of-variables

$$\mathbf{q} \rightsquigarrow [T^{-1}\mathbf{q}]_+$$

takes

$$\mathbf{t} \rightsquigarrow [T^{-1}\mathbf{t}]_+ + \sum_{l \geq 1} s_l \, \mathrm{ch}_{l+1}(E)$$

Since $\widehat{U}\mathcal{D}_X$ is well-defined as a formal function of $\mathbf{t}$ taking values in $\Lambda[\![\hbar, \hbar^{-1}]\!]$, and since the shift $\sum_{l \geq 1} s_l \, \mathrm{ch}_{l+1}(E)$ is "small",

$$(\widehat{U}\mathcal{D}_X)([T^{-1}\mathbf{q}]_+)$$

is well-defined as a formal function of $\mathbf{t}$ taking values in $\Lambda[\![\hbar, \hbar^{-1}]\!]$. It remains to deal with the $\exp(W_T(\mathbf{q})/2\hbar)$ term in (A.1). Since

$$\frac{T(-w)T^\star(-z) - I}{z + w} \equiv 0 \qquad \mod s_1, s_2 \ldots$$

we have

$$W_T(\mathbf{q}) \equiv 0 \qquad \mod s_1, s_2 \ldots$$

Thus

$$\begin{aligned}(\hat{S}\mathcal{D}_X)(\mathbf{q}) &= (\widehat{T}\widehat{U}\mathcal{D}_X)(\mathbf{q}) \\ &= \exp\left(\frac{W_T(\mathbf{q})}{2\hbar}\right)(\widehat{U}\mathcal{D}_X)([T^{-1}\mathbf{q}]_+)\end{aligned}$$

is well-defined as a formal function of $\mathbf{t}$ taking values in $\Lambda[\![\hbar,\hbar^{-1}]\!]$.

Recall further that

$$(\widehat{R}\mathcal{G})(\mathbf{q}) = \left[\exp\left(\frac{\hbar V_R(\partial_{\mathbf{q}})}{2}\right)\mathcal{G}\right](R^{-1}\mathbf{q})$$

(see Proposition 1.3.3). Since

$$\sum_{k,l}(-)^{k+l}V_{kl}w^k z^l = \frac{R^\star(w)R(z) - I}{z+w}$$

we see that

$$\|V_{kl}\| \leq 2^{-k-l-1}$$

Thus if $\mathcal{G}$ is a formal function of $\mathbf{t}$ taking values in $\Lambda[\![\hbar,\hbar^{-1}]\!]$ then

$$\exp\left(\frac{\hbar V_R(\partial_{\mathbf{q}})}{2}\right)\mathcal{G}$$

is well-defined as a formal function of $\mathbf{t}$ taking values in $\Lambda[\![\hbar,\hbar^{-1}]\!]$. The change-of-variables

$$\mathbf{q} \rightsquigarrow R^{-1}\mathbf{q}$$

takes

$$\mathbf{t} \rightsquigarrow R^{-1}\mathbf{t} - R^{-1}z + z$$

But

$$-R^{-1}z + z \equiv 0 \qquad \mod s_1, s_2, \ldots$$

(so in particular it is small) and therefore

$$(\widehat{R}\mathcal{G})(\mathbf{q}) = \left[\exp\left(\frac{\hbar V_R(\partial_{\mathbf{q}})}{2}\right)\mathcal{G}\right](R^{-1}\mathbf{q})$$

is well-defined as a formal function of $\mathbf{t}$ taking values in $\Lambda[\![\hbar,\hbar^{-1}]\!]$. Taking $\mathcal{G} = \hat{S}\mathcal{D}_X$, we are done. □

Proposition A.0.3. *$\mathcal{G}_{\mathbf{s}}$ is well-defined as a formal function of $\mathbf{t}$ which takes values in $\widetilde{\Omega}^{\star}_{MU}[[\hbar, \hbar^{-1}]]$.*

Proof. Recall that we work over the ground ring

$$\widetilde{\Omega}^{\star}_{MU} = \mathbb{C}[[Q]] \otimes \mathbb{C}[[s_1, s_2, \ldots]]$$

equipped with the topology coming from the norm

$$\|Q^d s_{i_1}^{j_1} \ldots s_{i_n}^{j_n}\| = 2^{-\int_d \omega - i_1 j_1 - \ldots - i_n j_n}$$

where ω is the symplectic form on X. Since

$$\begin{aligned} \sum_{r,s} A^{\alpha,r;\beta,s} x^r y^s &= -\left[\frac{g^{\alpha\beta}}{u(-x-y)}\right]_+ \\ &= g^{\alpha\beta} \left[\frac{\exp\left(\sum_{k>0} s_k(-x-y)^k\right)}{x+y}\right]_+ \end{aligned}$$

we see that

$$\|A^{\alpha,r;\beta,s}\| \leq 2^{-r-s-1}$$

The discussion on page 91 shows that $\mathcal{D}_{\mathbf{s}}$ is well-defined as a formal function of $\mathbf{t}$ which takes values in $\widetilde{\Omega}^{\star}_{MU}[[\hbar, \hbar^{-1}]]$. Since $\|A^{\alpha,r;\beta,s}\| < 1$, this implies that

$$\mathcal{G}_{\mathbf{s}} = \exp\left(\frac{\hbar}{2} \sum_{r,s} A^{\alpha,r;\beta,s} \partial_{\alpha,r} \partial_{\beta,s}\right) \mathcal{D}_{\mathbf{s}}$$

is also well-defined. □

Appendix B

Almost-Kähler manifolds

Suppose now that X is a compact symplectic manifold equipped with an almost-complex structure J which is tamed by the symplectic form. The foundations of Gromov–Witten theory in this setting have been laid down by several groups of authors [18, 45, 59, 56]. In this Appendix we extend many of the results proved in earlier chapters to this almost-Kähler situation; to do this we follow the approach of Siebert [59, 62]. The equivalence of the various symplectic approaches is sketched in [62]. The fact that the algebro-geometric and symplectic constructions agree in their common domain of applicability is proved in [46, 61].

B.1 An outline of Siebert's construction

As in the algebro-geometric situation, the moduli space $X_{g,n,d}$ of J-holomorphic degree-d stable maps from n-pointed genus-g complex curves to X is in general singular and of the "wrong" dimension. Siebert embeds $X_{g,n,d}$ in a finite-dimensional orbifold $Z_{g,n,d}$ and constructs a finite-rank orbibundle $F_{g,n,d}$ over $Z_{g,n,d}$ with a section $s_{g,n,d}$ such that the zero locus of $s_{g,n,d}$ is $X_{g,n,d}$. This allows him to define the virtual fundamental class of $X_{g,n,d}$ as a localized Euler class of $F_{g,n,d}$. In this section we summarize his construction [59, 62]. The first step is to realize $X_{g,n,d}$ inside a Banach orbifold of L^p stable maps.

Suppose that $p > 2$. There is a Banach orbifold $\mathcal{C}(X; p)$ consisting of equivalence classes of stable maps of Sobolev class L^p_1 from complex curves to X. A point in $\mathcal{C}(X;p)$ can be represented by a triple $(C,\mathbf{x},\varphi)$ where C is a prestable curve, $\mathbf{x}$ is an n-tuple of distinct smooth points on C and $\varphi : C \to X$ is an L^p_1 stable map. A chart on $\mathcal{C}(X;p)$ centered at $(C,\mathbf{x},\varphi)$ takes the form

$$S \times \bar{V}$$

where S is the base of an analytically semi-universal deformation $\mathcal{C} \to S$ of $(C,\mathbf{x})$ as a marked prestable curve, and $\bar{V}$ is a finite-codimension subspace of a space $V = \check{L}^p_1(C,\varphi^\star TX)$ of certain[1] L^p_1 sections of $\varphi^\star TX$. If $(C,\mathbf{x})$ is stable as a marked curve then $\bar{V} = V$. Otherwise, $S\times\bar{V}$ is a slice to the germ of the action of the identity component of $\mathrm{Aut}(C,\mathbf{x})$ on $S \times V$. Transition functions between these charts are smooth if we fix the complex structure on the domain curve, but are not smooth in general — in other words, they are differentiable relative to S. $\mathcal{C}(X;p)$ is therefore only a *topological* Banach orbifold, but it fibers in *smooth* Banach orbifolds over the (analytic) Artin stack $\mathfrak{M}$ of marked prestable curves. There is a Banach orbibundle E over $\mathcal{C}(X;p)$ with fiber at $(C,\mathbf{x},\varphi)$ equal to a space $\check{L}^p_1(C,\varphi^\star TX\otimes\Omega^{0,1}_C)$ of certain L^p_1 sections of $\varphi^\star TX \otimes \Omega^{0,1}_C$. E is smooth relative to $\mathfrak{M}$. There is an orbibundle section $s_{\bar\partial,J}$ of E, smooth relative to $\mathfrak{M}$, which sends $(C,\mathbf{x},\varphi)$ to $\bar\partial_J\varphi$. The zero locus of $s_{\bar\partial,J}$ is the space $\mathcal{C}^{\mathrm{hol}}(X,J)$ of J-holomorphic stable maps to X.

Consider a chart $S\times\bar{V}$ centered at $(C,\mathbf{x},\varphi)\in\mathcal{C}^{\mathrm{hol}}(X,J)$. The differential of $s_{\bar\partial,J}$ relative to S is Fredholm and uniformly continuous at $(0,0)\in S\times\bar{V}$. If $s_{\bar\partial,J}$ is transverse at $(C,\mathbf{x},\varphi)$, so that the relative differential σ_0 of $s_{\bar\partial,J}$ at $(0,0)$ is surjective, then the implicit function theorem (applied relative to S, see [59, section 1.3]) shows that near $(C,\mathbf{x},\varphi)$, the zero locus $\mathcal{C}^{\mathrm{hol}}(X,J)$ of $s_{\bar\partial,J}$ is a finite-dimensional topological orbifold which is smooth relative to S. A key notion from [59], which allows us to globalize this construction and simultaneously deals with problems of transversality, is that of a *Kuranishi structure*. This is a finite-rank orbibundle F defined over a neighbourhood N of $\mathcal{C}^{\mathrm{hol}}(X,J)$ in $\mathcal{C}(X;p)$, together with a map of orbibundles

$$\tau : F \to E$$

such that τ is continuously differentiable relative to S and, for any chart $S\times\bar{V}$ centered in $\mathcal{C}^{\mathrm{hol}}(X,J)$ as above, $\mathrm{im}\,\tau_{(0,0)}$ spans the cokernel of σ_0. The existence of a Kuranishi structure is established in section 6 of [59]. Let $q : F \to N$ be the bundle projection. The section

[1]See [59, section 5] for details.

$q^\star s + \tau$ of $q^\star E$ over F is transverse along $\mathcal{C}^{\mathrm{hol}}(X, J)$, which we regard as lying in the zero section of F. A neighbourhood of $\mathcal{C}^{\mathrm{hol}}(X, J)$ in F is therefore a topological orbifold Z which is smooth relative to $\mathfrak{M}$. $\mathcal{C}^{\mathrm{hol}}(X, J)$ is cut out of Z by the canonical section of the orbibundle $q^\star F$ over F (restricted to Z). Concentrating our attention on degree-d stable maps from n-pointed genus-g curves, this gives a finite-dimensional orbifold $Z_{g,n,d}$ and a finite-rank orbibundle $F_{g,n,d}$ over $Z_{g,n,d}$ together with a section $s_{g,n,d}$ such that $s_{g,n,d}^{-1}(0) = X_{g,n,d}$. The topological orbifold $Z_{g,n,d}$ is smooth relative to the Artin stack $\mathfrak{M}_{g,n}$ of prestable n-pointed, genus-g curves. A chart on $Z_{g,n,d}$ centered at $(C, \mathbf{x}, \varphi) \in X_{g,n,d}$ takes the form

$$S \times W \tag{B.1}$$

where S, as before, is the base of a semi-universal deformation $\mathcal{C} \to S$ of $(C, \mathbf{x})$, and W is a finite-dimensional vector space. Without loss of generality, we can insist that $Z_{g,n,d}$ be covered by the unit balls in finitely many such charts. We may also take $Z_{g,n,d}$ to consist of C^∞ stable maps.

By making appropriate choices in the construction of the Kuranishi structure, we may take the neighbourhood $Z_{g,n+1,d}$ of $X_{g,n+1,d}$ to be such that

$$\begin{array}{ccc} Z_{g,n+1,d} & \xrightarrow{\mathrm{ev}_{n+1}} & X \\ \pi \big\downarrow & & \\ Z_{g,n,d} & & \end{array} \tag{B.2}$$

is a family of C^∞ stable maps which restricts to give the universal family

$$\begin{array}{ccc} X_{g,n+1,d} & \xrightarrow{\mathrm{ev}_{n+1}} & X \\ \pi \big\downarrow & & \\ X_{g,n,d} & & \end{array}$$

of J-holomorphic stable maps over $X_{g,n,d}$. We may also take the "obstruction bundle" $F_{g,n+1,d}$ to be $\pi^\star F_{g,n,d}$.

B.2 K-theory and push-forwards

In order to extend our results to the almost-Kähler situation, we will need to make sense of the K-theoretic push-forward $\pi_\star : K^\star(Z_{g,n+1,d}) \to K^\star(Z_{g,n,d})$. Recall that if $f : X \to Y$ is a proper complex-oriented map between smooth manifolds then the push-forward $f_\star : K^\star(X) \to K^\star(Y)$ is defined as follows (see *e.g.* [55]). Take $N \gg 0$. Consider an embedding

$$g : X \to Y \times \mathbb{R}^N$$

which projects to f. There is a neighbourhood of $g(X)$ in $Y \times \mathbb{R}^N$ which is homeomorphic to the normal bundle ν_g, and the pushforward $f_\star$ is defined to be the composition

$$K^\star(X) \xrightarrow{\text{Thom}} K^\star(\text{Thom}(\nu_g)) \xrightarrow{\text{collapse}^\star} K^\star(\text{Thom}(Y \times \mathbb{R}^N)) \xrightarrow{\text{Thom}^{-1}} K^\star(Y)$$

For sufficiently large N, any two choices of the embedding g are isotopic through embeddings, so the push-forward is well-defined.

We are not, however, in this happy situation: the spaces $Z_{g,n,d}$ are orbifolds, and they are only smooth relative to $\mathfrak{M}_{g,n}$. We deal with the orbifold problem first.

Claim. *$Z_{g,n,d}$ is the orbifold quotient of a topological manifold $\widetilde{Z}_{g,n,d}$ by a proper action of a Lie group $G = GL_N$. The topological manifold $\widetilde{Z}_{g,n,d}$ is smooth relative to $\mathfrak{M}_{g,n}$.*

Proof. By [59, section 6.4] there is a line bundle L over X such that if

$$\begin{array}{ccc} \Gamma & \xrightarrow{\text{ev}} & X \\ \Big\downarrow{\scriptstyle \pi} & & \\ Z_{g,n,d} & & \end{array}$$

is the universal family over $Z_{g,n,d}$ then $\text{ev}^\star L$ carries the structure of a continuous family of holomorphic line bundles (see [59, section 2.4]) on the fibers of π and such that for all $(C, \mathbf{x}, \varphi) \in Z_{g,n,d}$,

$$L'_{C,\mathbf{x},\varphi} = \varphi^\star(L) \otimes \omega_C(x_1 + \ldots + x_n)$$

is ample on each component of C. Fix M sufficiently large such that for all $(C, \mathbf{x}, \varphi) \in Z_{g,n,d}$ we have:

(1) $(L'_{C,\mathbf{x},\varphi})^{\otimes M}$ is very ample

(2) $H^1(C, (L'_{C,\mathbf{x},\varphi})^{\otimes M}) = 0$

Let $N = \dim H^0(C, (L'_{C,\mathbf{x},\varphi})^{\otimes M})$. This is independent of $(C, \mathbf{x}, \varphi)$ by (2).

Consider the moduli problem for quadruples $(C, \mathbf{x}, \varphi, \{f_1, \ldots, f_N\})$ where $(C, \mathbf{x}, \varphi)$ is an L^p stable map and $\{f_1, \ldots, f_N\}$ is a basis for $H^0(C, (L'_{C,\mathbf{x},\varphi})^{\otimes M})$. The action of the group $\mathrm{Aut}(C, \mathbf{x})$ on the set of such bases is free. Repeating the construction of $Z_{g,n,d}$ for this new moduli problem therefore gives a topological *manifold* $\widetilde{Z}_{g,n,d}$ which is smooth relative to $\mathfrak{M}_{g,n}$. The quotient of $\widetilde{Z}_{g,n,d}$ by the natural action of GL_N is $Z_{g,n,d}$. □

We define the K-groups of $Z_{g,n,d}$ using finite-dimensional approximations to the classifying space BG, much as we did on page 59. Let

$$\{EG^{(r)} \to BG^{(r)} : r = 1, 2, \ldots\}$$

be approximations to the universal principal G-bundle $EG \to BG$ by finite-dimensional manifolds such that

$$\begin{array}{ccc} EG^{(r-1)} & \hookrightarrow & EG^{(r)} \\ \downarrow & & \downarrow \\ BG^{(r-1)} & \hookrightarrow & BG^{(r)} \end{array}$$

and such that $EG^{(r)} \to BG^{(r)}$ is universal for principal G-bundles on cell spaces of dimension up to r. Set

$$Z^{(r)}_{g,n,d} = (\widetilde{Z}_{g,n,d} \times EG^{(r)})/G$$

where we divide by the (free) diagonal action of G, and define

$$K^\star(Z_{g,n,d}) = \varprojlim K^\star(Z^{(r)}_{g,n,d})$$

This is in fact independent of choices — it computes the K-theory of the classifying space of the orbispace $Z_{g,n,d}$ [50]. In particular, if we have constructed $Z^{(r)}_{g,n,d}$ by considering quadruples $(C, \mathbf{x}, \varphi, \{f_1, \ldots, f_N\})$ where $\{f_1, \ldots, f_N\}$ is a basis for $H^0(C, (L'_{C,\mathbf{x},\varphi})^{\otimes M})$ as above, then we may construct $Z^{(r)}_{g,n+1,d}$ by considering quadruples $(C, \mathbf{x}', \varphi, \{f_1, \ldots, f_N\})$

where $\mathbf{x}' = \mathbf{x} \cup \{x_{n+1}\}$ and $\{f_1, \ldots, f_N\}$ is a basis for *the same space* $H^0(C, (L'_{C,\mathbf{x},\varphi})^{\otimes M})$. We will exploit this below.

Before we discuss the push-forward $\pi_\star : K^\star(Z_{g,n+1,d}) \to K^\star(Z_{g,n,d})$ note that, since the charts (B.1) on $Z_{g,n,d}$ are based on charts S on $\mathfrak{M}_{g,n}$, the argument on pages 60–61 shows that (1.13) and (1.14) give exact sequences of complex orbibundles on $Z_{g,n+1,d}$. Thus the relative cotangent orbibundle Ω_π to the map π is

$$\Omega_\pi = L_{n+1} - \sum_{i=1}^{n} \sigma_{i\star} \mathcal{O}_{Z_{g,n,d}} - i_\star \mathcal{O}_{\mathcal{Z}} \tag{B.3}$$

where $\sigma_j : Z_{g,n,d} \to Z_{g,n+1,d}$ is the section of (B.2) given by the jth marked point and $i : \mathcal{Z} \to Z_{g,n+1,d}$ is inclusion of the singular locus in the family (B.2). The equality (B.3) is as elements of the Grothendieck group of complex orbibundles on $Z_{g,n+1,d}$. It gives compatible complex orientations of the maps $\pi^{(r)}$ which are induced from π:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & Z^{(r-1)}_{g,n+1,d} & \longrightarrow & Z^{(r)}_{g,n+1,d} & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle \pi^{(r-1)}} & & \Big\downarrow{\scriptstyle \pi^{(r)}} & & \\
\cdots & \longrightarrow & Z^{(r-1)}_{g,n,d} & \longrightarrow & Z^{(r)}_{g,n,d} & \longrightarrow & \cdots
\end{array}$$

Here we use the construction of $Z^{(r)}_{g,n+1,d}$ outlined above. Now

$$\pi^{(r)} : Z^{(r)}_{g,n+1,d} \to Z^{(r)}_{g,n,d}$$

is a complex-oriented map between topological manifolds, each of which are smooth relative to $\mathfrak{M}_{g,n}$, which covers the identity map on $\mathfrak{M}_{g,n}$:

$$\begin{array}{ccc}
Z^{(r)}_{g,n+1,d} & \xrightarrow{\pi^{(r)}} & Z^{(r)}_{g,n,d} \\
\Big\downarrow & & \Big\downarrow \\
\mathfrak{M}_{g,n} & = & \mathfrak{M}_{g,n}
\end{array}$$

We define the push-forward $\pi^{(r)}_\star : K^\star(Z^{(r)}_{g,n+1,d}) \to K^\star(Z^{(r)}_{g,n,d})$ as on page 129, but using an embedding

$$g : Z^{(r)}_{g,n+1,d} \to Z^{(r)}_{g,n,d} \times \mathbb{R}^{N'}$$

which projects to $\pi^{(r)}$ and which is smooth[2] relative to $\mathfrak{M}_{g,n}$. For $N' \gg 0$, any two choices for the embedding g are isotopic through such embeddings, so $\pi_\star^{(r)}$ is well-defined.

Finally, we define the push-forward

$$\pi_\star : K^\star(Z_{g,n+1,d}) \to K^\star(Z_{g,n,d})$$

to be the map

$$\pi_\star : \varprojlim K^\star(Z^{(r)}_{g,n+1,d}) \to \varprojlim K^\star(Z^{(r)}_{g,n,d})$$

induced by $\{\pi_\star^{(r)} : r = 1, 2, \ldots\}$. The relative cotangent orbibundle Ω_π gives rise to an element of $K^\star(Z_{g,n,d})$, which we also denote by Ω_π, and the usual Riemann–Roch theorem applied to the finite-dimensional approximations $\pi^{(r)}$ gives

$$\mathrm{ch}(\pi_\star\alpha) = \pi_\star(\mathrm{ch}(\alpha)\cdot \mathrm{Td}^\vee\, \Omega_\pi) \tag{RR}$$

for all $\alpha \in K^\star(Z_{g,n,d})$. Here we used the fact that

$$\begin{aligned}\varprojlim H^\star(Z^{(r)}_{g,n,d};\mathbb{Q}) &= H^\star_G(\widetilde{Z}_{g,n,d};\mathbb{Q})\\ &= H^\star(Z_{g,n,d};\mathbb{Q})\end{aligned}$$

B.3 Quantum Riemann–Roch

Examining the proof of Theorem 1.6.5 we see that it extends to the almost-Kähler situation provided that we can establish:

- an analog of (GRR) on page 59. This is (RR) above.
- an analog of Proposition 1.6.3. This follows from (B.3).
- expressions for the normal bundles to $\sigma_i(Z_{g,n,d})$ and $\mathcal{Z}$ in terms of universal cotangent lines. These are obvious: the charts (B.1) on $Z_{g,n,d}$ are based on charts S on $\mathfrak{M}_{g,n}$ and the corresponding expressions hold on $\mathfrak{M}_{g,n}$.
- various properties of the virtual fundamental class with regard to pull-back and restriction to the singular locus. These are verified in [62].

[2]To see that such g exist one can, for example, construct appropriately smooth embeddings of $Z^{(r)}_{g,n+1,d}$ into $\mathbb{R}^{N'}$ using the standard proof of the Whitney Embedding Theorem (see *e.g.* Theorem 3.4 in [33]) and the bump functions constructed in section 6.5 of [59].

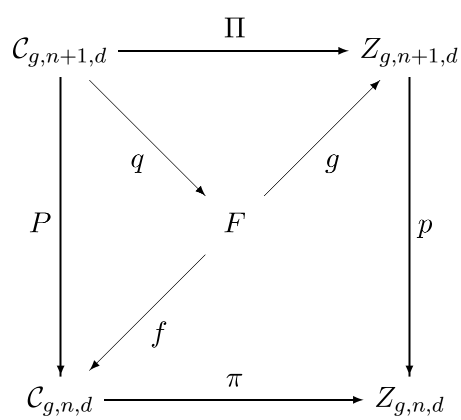
- analogs of Lemma 1.6.1 and Lemma 1.6.2, describing the behaviour of $E_{g,n,d}$ under pull-back and restriction to the singular locus. These are Lemmas B.3.1 and B.3.2 below.

Thus Theorem 1.6.5 holds for almost-Kähler manifolds. As a consequence, all of the results on pages 1–17 of Chapter 0 and all of the results of Chapter 1, except the mirror theorems in section 1.7.1, hold in the almost-Kähler setting. The mirror theorems on pages 18–20 and in section 1.7.1 rely on a comparison result [36] for algebraic virtual fundamental classes, the almost-Kähler analog of which does not seem to be known.

Lemma B.3.1. *Let $p : Z_{g,n+1,d} \to Z_{g,n,d}$ be the map that forgets the last marked point and then stabilizes. We have*

$$p^\star \operatorname{ch}(E_{g,n,d}) = \operatorname{ch}(E_{g,n+1,d})$$

Proof. Consider the diagram



where $\Pi : \mathcal{C}_{g,n+1,d} \to Z_{g,n+1,d}$ and $\pi : \mathcal{C}_{g,n,d} \to Z_{g,n,d}$ are the universal families, F is the fiber product and the map $P : \mathcal{C}_{g,n+1,d} \to \mathcal{C}_{g,n,d}$ forgets the $(n+1)$st marked point and then stabilizes. A point of the fiber of F over $(C, \mathbf{x}, \varphi) \in Z_{g,n,d}$ is a choice of two points in C — call them $\bullet$ and $\circ$, where $\bullet$ is the point corresponding to $\mathcal{C}_{g,n,d}$. We need to show that

$$p^\star \pi_\star(\operatorname{ev}_\bullet^\star(\operatorname{ch}(E)) \operatorname{Td}^\vee \Omega_\pi) = \Pi_\star(\operatorname{ev}_\bullet^\star(\operatorname{ch}(E)) \operatorname{Td}^\vee \Omega_\Pi)$$

But

$$\begin{aligned} p^\star \pi_\star(\operatorname{ev}_\bullet^\star(\operatorname{ch}(E)) \operatorname{Td}^\vee \Omega_\pi) &= g_\star f^\star(\operatorname{ev}_\bullet^\star(\operatorname{ch}(E)) \operatorname{Td}^\vee \Omega_\pi) \\ &= g_\star(\operatorname{ev}_\bullet^\star(\operatorname{ch}(E)) \operatorname{Td}^\vee \Omega_g) \end{aligned}$$

and

$$\begin{aligned}\Pi_\star(\mathrm{ev}_\bullet^\star(\mathrm{ch}(E))\,\mathrm{Td}^\vee\,\Omega_\Pi) &= g_\star q_\star(\mathrm{ev}_\bullet^\star(\mathrm{ch}(E))\,\mathrm{Td}^\vee\,\Omega_\Pi) \\ &= g_\star(\mathrm{ev}_\bullet^\star(\mathrm{ch}(E))\,\mathrm{Td}^\vee\,\Omega_g\, q_\star(\mathrm{Td}^\vee\,\Omega_q))\end{aligned}$$

so it suffices to show that

$$q_\star(\mathrm{Td}^\vee\,\Omega_q) = 1$$

Now $q : \mathcal{C}_{g,n+1,d} \to F$ is an isomorphism outside

- the codimension-2 locus Y in F where $\bullet$ and $\circ$ coincide with the same marked point, and
- the codimension-3 locus Y' in F where $\bullet$ and $\circ$ coincide with the same node.

These loci are disjoint. Since q is birational, the fundamental class of $\mathcal{C}_{g,n+1,d}$ pushes forward to the fundamental class of F, so we need to show that

$$q_\star(\mathrm{Td}\,T_q - 1) = 0 \tag{B.4}$$

The relative tangent bundle T_q vanishes away from Y and Y', so $\mathrm{Td}\,T_q - 1$ is supported near Y and Y'. In proving (B.4) we may therefore replace F by neighbourhoods U and U' of Y and Y' respectively, and replace $\mathcal{C}_{g,n+1,d}$ by $\widetilde{U} = q^{-1}(U)$ and $\widetilde{U}' = q^{-1}(U')$.

The component Y_i of Y on which $\bullet$ and $\circ$ coincide with the ith marked point is a copy of $Z_{g,n,d}$. There is a neighbourhood U_i of Y_i which is homeomorphic to a neighbourhood of the zero section in $L_i^\star \oplus L_i^\star$ and is such that $q : q^{-1}(U_i) \to U_i$ is the blow-up of U_i along Y_i.

Claim. *On U_i,*

$$q_\star(\mathrm{Td}\,T_q - 1) = 0$$

Since $\mathrm{Td}\,T_q - 1$ is supported on $q^{-1}(Y_i)$ we may take U_i to be the total space of $N = L_i^\star \oplus L_i^\star$ and $\widetilde{U}_i = q^{-1}(U_i)$ to be the total space of the tautological bundle $\mathcal{O}(-1)$ over $\mathbb{P}(N)$.

$$\begin{array}{ccc}\mathbb{P}(N) & \hookrightarrow & \mathcal{O}(-1) \\ \downarrow & & \downarrow q \\ Y & \xrightarrow{\;j\;} & N\end{array}$$

Denote the first and second copies of $L_i^\star$ in N by E_1 and E_2 respectively. Equip E_1 and E_2 with S^1-actions of distinct weight. This gives an action of the 2-torus T on $N = E_1 \oplus E_2$. Let the T-equivariant Euler classes of E_1 and E_2 be λ_1 and λ_2 respectively. We need to show that

$$\int_{\mathcal{O}(-1)} (\mathrm{Td}\, T_q - 1) q^\star \alpha = 0 \qquad \text{for all } \alpha \in H^\star(N) \text{ of compact support}$$

This will follow from the corresponding T-equivariant statement, which we prove using fixed-point localization. There are two T-fixed loci in $\mathcal{O}(-1)$, each of which is also a copy of Y:

- A_1, coming from the zero locus of N together with the line E_1 through the zero locus, with normal bundle $N_{A_1} = E_1 \oplus (E_1^\star \otimes E_2)$.
- A_2, coming from the zero locus of N together with the line E_1 through the zero locus, with normal bundle $N_{A_2} = E_2 \oplus (E_2^\star \otimes E_1)$.

We have

$$\begin{aligned} T_q|_{A_1} &= E_1^\star \otimes E_2 - E_2 \\ T_q|_{A_2} &= E_2^\star \otimes E_1 - E_1 \end{aligned}$$

Thus

$$\begin{aligned} \int_{\mathcal{O}(-1)} (\mathrm{Td}\, T_q - 1) q^\star \alpha &= \text{contribution from } A_1 + \text{contribution from } A_2 \\ &= \int_Y \frac{j^\star(\alpha)}{\lambda_1(\lambda_2 - \lambda_1)} \left(\frac{\lambda_2 - \lambda_1}{1 - e^{\lambda_1 - \lambda_2}} \frac{1 - e^{-\lambda_2}}{\lambda_2} - 1 \right) \\ &\quad + \int_Y \frac{j^\star(\alpha)}{\lambda_2(\lambda_1 - \lambda_2)} \left(\frac{\lambda_1 - \lambda_2}{1 - e^{\lambda_2 - \lambda_1}} \frac{1 - e^{-\lambda_1}}{\lambda_1} - 1 \right) \\ &= \int_Y \frac{j^\star(\alpha)}{\lambda_1 \lambda_2} \left(\frac{1 - e^{-\lambda_2}}{1 - e^{\lambda_1 - \lambda_2}} + \frac{1 - e^{-\lambda_1}}{1 - e^{\lambda_2 - \lambda_1}} \right) - \frac{j^\star(\alpha)}{\lambda_1 - \lambda_2} \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1} \right) \\ &= \int_Y \frac{j^\star(\alpha)}{\lambda_1 \lambda_2} \left(\frac{1 - e^{-\lambda_2}}{1 - e^{\lambda_1 - \lambda_2}} + \frac{e^{\lambda_1 - \lambda_2} - e^{-\lambda_2}}{e^{\lambda_1 - \lambda_2} - 1} - 1 \right) \\ &= \int_Y \frac{j^\star(\alpha)}{\lambda_1 \lambda_2} \left(\frac{1 - e^{\lambda_1 - \lambda_2}}{1 - e^{\lambda_1 - \lambda_2}} - 1 \right) \\ &= 0 \end{aligned}$$

This proves the Claim. Consequently, on the neighbourhood U of Y in F we have

$$q_\star(\mathrm{Td}\, T_q - 1) = 0$$

It remains to deal with the codimension-3 locus Y' where $\bullet$ and $\circ$ coincide with the same node. A component V' of Y' projects to a stratum V in $Z_{g,n,d}$ which consists of nodal curves, and a neighbourhood W of V in $Z_{g,n,d}$ is homeomorphic to a neighbourhood of the zero section in the bundle $L_+^\star \otimes L_-^\star$ over V. Here L_+ and L_- are the cotangent lines at the relevant node. A neighbourhood of $g(V')$ in $Z_{g,n+1,d}$ is homeomorphic to a neighbourhood of the zero section in the bundle $L_+^\star \oplus L_-^\star$ over V, and we may assume that $p : Z_{g,n+1,d} \to Z_{g,n,d}$ maps this neighbourhood to W via

$$\begin{aligned} L_+^\star \oplus L_-^\star &\to L_+^\star \otimes L_-^\star \\ (x, y) &\mapsto xy \end{aligned}$$

A similar statement is true for the map $\pi : \mathcal{C}_{g,n,d} \to Z_{g,n,d}$, and so a neighbourhood of V' in Y' consists of the intersection of the family of quadratic cones

$$Q = \{(x, y, u, v) \in L_+^\star \oplus L_-^\star \oplus L_+^\star \oplus L_-^\star : xy = uv\}$$

with a neighbourhood of the zero section of $L_+^\star \oplus L_-^\star \oplus L_+^\star \oplus L_-^\star$

The preimage of V' in $\mathcal{C}_{g,n+1,d}$ is

$$V \times \overline{\mathcal{M}}_{0,4} \cong V \times \mathbb{P}^1$$

where we choose a co-ordinate z on $\mathbb{P}^1$ such that

$$(0, 1, \infty) = (\text{node carrying } L_-, \circ, \text{node carrying } L_+)$$

A neighbourhood of $V \times \mathbb{P}^1$ in $\mathcal{C}_{g,n+1,d}$ is homeomorphic to a neighbourhood of the zero section in the bundle $\mathcal{O}(-1) \otimes (L_+^\star \oplus L_-^\star)$ over $V \times \mathbb{P}^1$, and a local model for the map $q : \mathcal{C}_{g,n+1,d} \to F$ is

$$\begin{aligned} (\mathcal{O}(-1) \otimes L_+^\star) \oplus (\mathcal{O}(-1) \otimes L_-^\star) &\to L_+^\star \oplus L_-^\star \oplus L_+^\star \oplus L_-^\star \\ (v \otimes a, w \otimes b) &\mapsto (\xi(v)a, z\xi(w)b, z\xi(v)a, \xi(w)b) \end{aligned} \tag{B.5}$$

Here ξ is the section of $\mathcal{O}(1)$ which vanishes at $z = \infty$, so $z\xi$ is the section of $\mathcal{O}(1)$ which vanishes at $z = 0$. The image of the map (B.5) is the family of cones Q; the map is an isomorphism away from the zero locus of $\mathcal{O}(-1) \otimes (L_+^\star \oplus L_-^\star)$.

We want to show that

$$q_\star \operatorname{Td} T_q = 1$$

in a neighbourhood of Y'. Since T_q vanishes outside $q^{-1}(Y')$ it suffices to prove this for the local model

$$\mathcal{O}(-1) \otimes (L_+^\star \oplus L_-^\star) \xrightarrow{q} Q \subset L_+^\star \oplus L_-^\star \oplus L_+^\star \oplus L_-^\star$$

given by (B.5). In other words, we need to show that

$$\int_{\mathcal{O}(-1)\otimes(L_+^\star \oplus L_-^\star)} (\operatorname{Td} T_q) q^\star \alpha = \int_Q \alpha \qquad \text{for all } \alpha \in H^\star(Q) \text{ of compact support} \tag{B.6}$$

Assume that L_+ and L_- carry T-actions of distinct non-zero weight, and denote the T-equivariant Euler classes of $L_+^\star$ and $L_-^\star$ by λ_+ and λ_- respectively. We will deduce (B.6) from the corresponding T-equivariant statement, which we prove using fixed-point localization. The T-fixed locus in $\mathcal{O}(-1) \otimes (L_+^\star \oplus L_-^\star)$ is a copy of $V \times \mathbb{P}^1$, with normal bundle

$$\mathcal{O}(-1) \otimes (L_+^\star \oplus L_-^\star)$$

Since Q is cut out of $L_+^\star \oplus L_-^\star \oplus L_+^\star \oplus L_-^\star$ by a section of $L_+^\star \otimes L_-^\star$, the relative tangent bundle T_q is

$$T_q = T\mathbb{P}^1 + \mathcal{O}(-1) \otimes L_+^\star + \mathcal{O}(-1) \otimes L_-^\star - 2L_+^\star - 2L_-^\star + L_+^\star \otimes L_-^\star$$

Thus, for $\alpha \in H_T^\star(Q)$,

$$\int_{\mathcal{O}(-1)\otimes(L_+^\star \oplus L_-^\star)} (\operatorname{Td} T_q) q^\star \alpha$$

equals

$$\int_{V\times\mathbb{P}^1} \frac{2P}{1-e^{-2P}} \frac{1}{1-e^{P-\lambda_+}} \frac{1}{1-e^{P-\lambda_-}} \left(\frac{1-e^{-\lambda_+}}{\lambda_+}\right)^2 \left(\frac{1-e^{-\lambda_-}}{\lambda_-}\right)^2 \frac{\lambda_+ + \lambda_-}{1-e^{-\lambda_+-\lambda_-}} j^\star \alpha$$

where $j : V \to Q$ is the inclusion of the zero section and P is the hyperplane generator for $H^\star(\mathbb{P}^1)$. But this is

$$\int_V \oint \frac{dP}{P^2} \frac{2P}{1-e^{-2P}} \frac{1}{1-e^{P-\lambda_+}} \frac{1}{1-e^{P-\lambda_-}} \left(\frac{1-e^{-\lambda_+}}{\lambda_+}\right)^2 \left(\frac{1-e^{-\lambda_-}}{\lambda_-}\right)^2 \frac{\lambda_+ + \lambda_-}{1-e^{-\lambda_+-\lambda_-}} j^\star \alpha$$

and since

$$\begin{aligned}
\oint \frac{dP}{P^2} \frac{2P}{1-e^{-2P}} \frac{1}{1-e^{P-\lambda_+}} \frac{1}{1-e^{P-\lambda_-}} &= \frac{d}{dP}\left(\frac{2P}{1-e^{-2P}} \frac{1}{1-e^{P-\lambda_+}} \frac{1}{1-e^{P-\lambda_-}}\right)\bigg|_{P=0} \\
&= \frac{1-e^{-\lambda_+-\lambda_-}}{(1-e^{-\lambda_+})^2(1-e^{-\lambda_-})^2}
\end{aligned}$$

we see that

$$\int_{\mathcal{O}(-1)\otimes(L_+^\star\oplus L_-^\star)} (\mathrm{Td}\, T_q) q^\star\alpha = \int_V \frac{\lambda_+ + \lambda_-}{\lambda_+^2\lambda_-^2} j^\star\alpha$$

$$= \int_Q \alpha$$

Thus $q_\star(\mathrm{Td}\, T_q - 1) = 0$ in a neighbourhood of Y'. The Lemma is proved. $\square$

An argument of a similar character proves:

Lemma B.3.2. *Let*

$$\widetilde{\mathcal{Z}}_{red} \coprod \widetilde{\mathcal{Z}}_{irr} \xrightarrow{\gamma_{red}\coprod\gamma_{irr}} \mathcal{Z} \xrightarrow{i} Z_{g,n+1,d}$$

where

$$\widetilde{\mathcal{Z}}_{red} = \coprod_{\substack{g=g_++g_-\\ n=n_++n_-\\ d=d_++d_-}} Z_{g_+,n_++\blacktriangle,d_+} \times_X Z_{0,1+\blacktriangle+\vartriangle,0} \times_X Z_{g_-,n_-+\vartriangle,d_-}$$

and

$$\widetilde{\mathcal{Z}}_{irr} = Z_{g-1,n+\blacktriangle+\vartriangle} \times_{X\times X} Z_{0,1+\blacktriangle+\vartriangle,0}$$

Denote by p_+ and p_- be the projections onto the first and third factors of $\widetilde{\mathcal{Z}}_{irr}$. We have

$$\gamma_{red}^\star i^\star \operatorname{ch}(E_{g,n+1,d}) = p_+^\star \operatorname{ch}(E_{g_+,n_++\blacktriangle,d_+}) + p_-^\star \operatorname{ch}(E_{g_-,n_-+\vartriangle,d_-}) - \mathrm{ev}_\vartriangle^\star \operatorname{ch}(E)$$

and

$$\gamma_{irr}^\star i^\star \operatorname{ch}(E_{g,n+1,d}) = \operatorname{ch}(E_{g-1,n+\blacktriangle+\vartriangle,d}) - \mathrm{ev}_\vartriangle^\star \operatorname{ch}(E)$$

where $\mathrm{ev}_\vartriangle$ *is the evaluation map at the point of gluing.*

B.4 Quantum cobordism

We now extend the proof of Theorem 2.4.1, and consequently of all the results in Chapter 2 and on pages 26–28 of Chapter 0, to the almost-Kähler setting. To do this, we need to establish:

- the existence of a well-defined virtual tangent bundle $\mathcal{T}_{g,n,d}^{vir} \in K^\star(Z_{g,n,d})$
- the equality

$$\mathcal{T}_{g,n+1,d}^{vir} = \pi^\star \mathcal{T}_{g,n,d}^{vir} + \Omega_\pi^\star \tag{B.7}$$

- the results of section 2.5.3

The linearization of the $\bar{\partial}$-operator $\bar{\partial}_J$ (with respect to the trivializations defined in section 6 of [59]) gives a two-term complex of Fredholm orbibundles on $Z_{g,n,d}$:

$$\sigma_J : \check{L}^p_1(C, \varphi^\star TX) \to \check{L}^p_1(C, \varphi^\star TX \otimes \Omega^{0,1}_C) \tag{B.8}$$

(this is like the linearization of the section $s_{\bar{\partial},J}$ except that we do not restrict the domain to $\bar{V} \subset \check{L}^p_1(C, \varphi^\star TX)$). We will use this complex to define the virtual tangent bundle of $Z_{g,n,d}$ relative to $\mathfrak{M}_{g,n}$. Choose a chart $S \times W$ on $Z_{g,n,d}$. Section 6.3 of [59] shows that σ_J can be written, up to a zero-order operator, as

$$\sigma_J|_{(C,\mathbf{x},\varphi)} = \bar{\partial}_{\varphi^\star TX,J} + R$$

where R is J-antilinear. Following McDuff [48, section 4], we consider a path of Fredholm operators

$$\sigma_t : \check{L}^p_1(C, \varphi^\star TX) \to \check{L}^p_1(C, \varphi^\star TX \otimes \Omega^{0,1}_C) \qquad 0 \le t \le 1$$

defined by

$$\sigma_t = \sigma_J - tR$$

Since $\sigma_1 = \bar{\partial}_{\varphi^\star TX,J}$ is J-linear, the complex

$$\sigma_1 : \check{L}^p_1(C, \varphi^\star TX) \to \check{L}^p_1(C, \varphi^\star TX \otimes \Omega^{0,1}_C)$$

defines[3] an element of $K^\star(Z_{g,n,d})$, $\mathcal{T}^{vir}_{rel}$, which we regard as the virtual tangent bundle of $Z_{g,n,d}$ relative to $\mathfrak{M}_{g,n}$. Since it is the index bundle of $\bar{\partial}_{\varphi^\star TX,J}$, the family Index Theorem gives

$$\mathrm{ch}(\mathcal{T}^{vir}_{rel}) = \mathrm{ch}((TX)_{g,n,d})$$

Recall that there is a map $\rho : Z_{g,n,d} \to \mathfrak{M}_{g,n}$. We set

$$\mathcal{T}^{vir}_{g,n,d} = \mathcal{T}^{vir}_{rel} + \rho^\star T\mathfrak{M}_{g,n}$$

[3]See [57]. The key step is the construction of a finite-rank (orbi)bundle F' over $Z_{g,n,d}$ and a map

$$\tau' : F' \to \check{L}^p_1(C, \varphi^\star TX \otimes \Omega^{0,1}_C)$$

which spans the cokernel of σ_1. This can be achieved as in section 6 of [59].

In the notation of page 21,

$$\mathrm{ch}(\mathcal{T}^{vir}_{g,n,d}) = \mathrm{ch}((TX)_{g,n,d}) + \mathrm{ch}(\mathrm{Def}(C) \ominus \mathrm{Aut}(C))$$

and so the conclusions of section 2.5.3 hold here too. Working in charts of the form (B.2, B.1), the equality (B.7) is clear. Thus Theorem 2.4.1 holds for almost-Kähler manifolds.