Exercises supplementing those in Ian Stewart and David Tall's *Complex Analysis*

URL: http://math.berkeley.edu/~gbergman/ug.hndts/185_S+T_exs.ps

These are exercises that I put on homework sheets in my class in Complex Analysis, Berkeley's Math 185, this past Spring, supplementing those in the text (*plus a few I thought of subsequently*). *Some of them I included in the assignments to be handed in, but most were shown as for students who wanted to think about ''further interesting and/or more challenging problems''.*

They are of various sorts. Some are standard results which are not given in the text but can be obtained without much work from the results that are given; some would be most appropriate to assign in an honors course (*I gave ''Chapter 10, Exercise 25'' as an extra credit problem*)*; a few* (*e.g., ''Chapter 2, Exercise 18''*) *are tangential rather than directly relevant to the material. But I think that many of them constitute good additional exercises for a regular course.*

Often when an exercise has successive parts (a), (b), ... , *these are steps building up to some result. In other cases, the first one or two parts yield the main result, while later parts give optional applications or related results* (*e.g., they may ask for an example showing that the main result is sharp, something it is usually more difficult for students to come up with than the proof of the result*)*; or they may point to a generalization of the main result. In many such cases I only assigned the first parts to my class. An instructor using these sheets should consider, case by case, which parts of a given exercise to assign.*

A few exercises depend on earlier exercises. When this is so I have tried to make it explicit, writing ''Prove using the result of —'', but I may have missed some cases.

I have given the exercises numbers that indicate the relevant chapters of the text, beginning where the numbering of exercises in each chapter leaves off. These numbers are shown in quotation marks, to distinguish them from the exercises in the text.

There are probably mistakes among these exercises. I have already found and corrected many; I would welcome having any further cases pointed out to me!

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Chapter 2, ''Exercise 12'': Show that if S is an open subset of $\mathbb{C}$, and T is a relatively open subset of S, then T is an open subset of $\mathbb{C}$.

Chapter 2, ''Exercise 13'': Show that if z_1 and z_2 are points of a neighborhood $N_r(z_0)$, then the line segment $[z_1, z_2]$ is also contained in $N_r(z_0)$.

Chapter 2, ''Exercise 14'': Our text's proof of the Paving Lemma shows that among all ways that one might try to ''pave'' a path in an open set S, there are some that reach the end without getting stuck; but it does not actually show how to get such a paving. This exercise indicates how to do so, following the spirit of the text's proof fairly closely. The next exercise also constructs a paving, but in a somewhat different way.

If our set S is all of $\mathbb{C}$, we can pave any path $\gamma\colon [a,b] \to S$ by just using the subdivision $a = t_0 < t_1 = b$, and taking an open disc D_1 centered at 0 and large enough to contain the whole path γ. So for the rest of this exercise, let S be a *proper* open subset of $\mathbb{C}$.

(a) Show that for every $z \in S$, there exists a largest $r > 0$ such that $N_r(z) \subseteq S$ (i.e., a positive real number r which has this property, but such that no larger number has the same property).

Now let $\gamma\colon [a,b] \to S$ be a path. We start with an easy fact, to be called on at the end of our construction:

(b) Show that there exists $d \in [a,b)$ such that γ carries $(d,b]$ into an open disc contained in S.

Another general fact we will need, but less trivial, is given in:

(c) Show that for every $s \in [a, b)$ and positive real number r, there exists a largest $t \in (s, b]$ such that γ carries $[s, t)$ into $N_r(\gamma(s))$.

We can now describe our paving. Starting with $t_0 = a$, we define an increasing sequence of values in $[a, b]$ as follows. Suppose recursively (which is like ''inductively'' except that the word is used in giving constructions rather than proofs) that for some $m \geq 0$ we have found $a = t_0 < \ldots < t_m \in [a, b]$ such that the image of each interval $[t_{r-1}, t_r]$ under γ lies in some open disc in S. Let d be as in (b) above. Then,
(i) if $t_m > d$, we let $n = m+1$ and $t_{m+1} = b$, and consider the construction finished, while
(ii) otherwise, we choose r to be (as in (a)) the largest real number such that $N_r(\gamma(t_m)) \subseteq S$, let t be (as in (c)) the largest real number such that γ carries $[t_m, t)$ into $N_r(\gamma(s))$, and let $t_{m+1} = (t_m + t)/2$.

(d) Show that in both cases (i) and (ii) above, the sequence $t_0 < \ldots < t_{m+1}$ is indeed a paving of $[a, t_{m+1}]$. (Why couldn't we have taken $t_{m+1} = t$ in (ii), instead of $(t_m + t)/2$, and gotten the same conclusion?)

(e) Show that the above construction eventually terminates; i.e., that at some stage, we must get $t_m > d$, and so land in case (i), finishing the construction.

(Suggestion: If the contrary is true, let t_∞ be the least upper bound of the values t_m, look at the neighborhood $N_r(\gamma(t_\infty))$ constructed as in (a), note that γ will map some open interval $(t_\infty - \varepsilon, t_\infty + \varepsilon)$ into this neighborhood, and show that there exists an m such that the construction of t_{m+1} from t_m as in (ii) above gives a value $t_{m+1} > t_\infty$, contradicting the definition of t_∞.)

Chapter 2, ''Exercise 15'': Here is another ''constructive'' version of the proof of the Paving Lemma (cf. introductory paragraph of preceding exercise).

Let S be an open subset of $\mathbb{C}$, and $\gamma\colon [a, b] \to S$ a path. Clearly, if the image of γ is contained in an open disc in S, then by taking $a = t_0 < t_1 = b$ we get a one-step paving; so we will assume below that it is not.

(a) For each $t \in [a, b]$, show that there is a largest $\delta > 0$ such that the image under γ of $[a, b] \cap (t - \delta, t + \delta)$ is contained in a disc centered at $\gamma(t)$ and lying in S.

(b) Show that the function associating to every $t \in [a, b]$ the δ of part (a) is continuous. Deduce that it assumes a smallest value, c, and that $c > 0$.

(c) Show that every closed subinterval of $[a, b]$ of length $< 2c$ is contained in a disc.

(d) Show that if we take c as above, let n any integer $> (b - a)/2c$, and divide $[a, b]$ into n equal closed intervals, the result will be a paving.

Remark: Though this construction looks satisfyingly concrete, whether one can actually calculate the ''smallest value'' referred to in step (b), or even a lower bound for that value, depends on how S and γ are given.

Chapter 2, ''Exercise 16'': Show by example that if in the statement of the Paving Lemma (p.39) one replaces the path $\gamma\colon [a, b] \to S$ by a continuous function on the corresponding open interval, $\gamma\colon (a, b) \to S$, and replaces the initial and final intervals $[t_0, t_1]$ and $[t_{n-1}, t_n]$ in the conclusion by $(t_0, t_1]$ and $[t_{n-1}, t_n)$, the resulting statement is *not* true.

Chapter 2, ''Exercise 17'': Suppose S_1 and S_2 are path connected subsets of $\mathbb{C}$.

(a) Show that if $S_1 \cap S_2 = \varnothing$, then $S_1 \cup S_2$ is not path connected.

(b) Show that if $S_1 \cap S_2 \neq \varnothing$, then $S_1 \cup S_2$ is path connected.

(c) Give an example where $S_1 \cap S_2 \neq \varnothing$, but $S_1 \cap S_2$ is not path connected.

Chapter 2, ‘‘Exercise 18’’: Let S be a nonempty subset of $\mathbb{C}$. We shall call S *topologically disconnected* if it can be written as the union $S_1 \cup S_2$ of two nonempty sets, each of which is disjoint from the closure of the other; we shall call it *topologically connected* if it not topologically disconnected. We will use the term *path connected* as in the text.

(a) Show that every interval $[a, b]$ is topologically connected. (If you’ve seen this proved, e.g., in Math 104, then go on to the next part. Hint on how to start if you haven’t: Suppose $[a, b]$ were topologically disconnected, say equal to $S_1 \cup S_2$ as in the definition. Assume without loss of generality that $a \in S_1$. Deduce from the assumption on S_1 and S_2 that there is some $c \in (a, b]$ such that $[a, c) \subseteq S_1$. Let d be the least upper bound of all c with that property. Get a contradiction if $d \neq b$ and another if $d = b$.)

(b) Deduce from (a) that if $S \subseteq \mathbb{C}$ is path connected, then it is topologically connected. (Hint: A path is a function defined on an interval.)

(c) Show that if S is the union of two topologically connected nonempty subsets A and B, then S is topologically connected if and only if one of A, B contains a point in the closure of the other.

(d) Let P be the line segment $[-i, i]$ and Q the set $\{t + i\sin(t^{-1}) \mid t \in (0, (2\pi)^{-1}]\}$. Show that $P \cup Q$ is topologically connected but not path connected.

(e) Show that if S is open, then it is path connected if and only if it is topologically connected.

The contrasting results (d) and (e) are the main point of this exercise; but here are a few more results that illuminate these concepts:

(f) Show by example that there exist sets A and B, each of which is topologically disconnected, but whose union is topologically connected, and in fact, path connected.

(g) Show that S is topologically connected if and only if the only subsets of S that are both relatively open and relatively closed in S are $\varnothing$ and S.

(h) Show that S is topologically connected if and only if every continuous function from S to the set $\{0, 1\}$ is constant.

Terminological note: The condition called ‘‘topological connectedness’’ above is what mathematicians usually call connectedness, while the condition that our authors call ‘‘path connectedness’’, but often shorten to ‘‘connectedness’’, is properly known by the full term path connectedness (or ‘‘pathwise’’ connectedness). However, our authors’ frequent shortening of ‘‘path connected’’ to ‘‘connected’’ is partially justified by (e) above, since the application of the concept in this course is to open sets. The difficulty of proving (a) above, which is used in proving (b) and (e), is probably their reason for not introducing the standard definition.

Chapter 4, ‘‘Exercise 17’’: Recall from multivariable calculus that if a is a real-valued function on an open subset S of the plane $\mathbb{R}^2$, and is differentiable in the sense of multivariable calculus, then the *gradient* ∇a of a means the vector-valued function on S defined by $\nabla a(x, y) = (\partial a(x, y)/\partial x,\ \partial a(x, y)/\partial y)$.

Now suppose f is a function on an open subset S of $\mathbb{R}^2 = \mathbb{C}$ which is differentiable in the sense of complex analysis. Let u, v be its real and imaginary parts, $u(z) = \mathrm{re}(f(z))$, $v(z) = \mathrm{im}(f(z))$.

Express the gradients of u and v at a point $z \in S$ in terms of the complex derivative $f'(z)$ (regarding the ordered pairs in the definition of gradient as complex numbers).

Chapter 4, ‘‘Exercise 18’’: Suppose f is a nowhere zero differentiable function on a domain D, such that $\arg(f(z))$ is constant. Show that f is constant.

Chapter 5, ''Exercise 19'': The text proves the identity $\sin^2 z + \cos^2 z = 1$ ((20) p.85) as an instance of the formula for $\cos(z_1 - z_2)$. Give two more proofs: one by substituting the formulas (14) and (15) of that page into the desired identity, the other by differentiating the left-hand-side of that identity. (The second method also requires that we know the value of that left-hand side for some particular z. The value at $z = 0$ is easy to find.)

Chapter 5, ''Exercise 20'': The most noticeable feature of the graph of the cotangent function of a real variable is that it ''goes to infinity'' at all integer multiples of π, with negative values to the left and positive values to the right of each of these points.

Now if we look at $\cot(t) + \cot(t - \frac{\pi}{2})$, we see that this goes to infinity twice as often, namely at every integer multiple of $\pi/2$, again with negative values to the left and positive values to the right. This suggests that this function might equal a scalar multiple of $\cot(2t)$.

(a) Prove that this is true; not just for this function of a real variable, but for the function $\cot(z) + \cot(z - \frac{\pi}{2})$ of a complex variable z; and find the scalar factor.

(b) More generally, show that for every positive integer n, we have an identity

$$\cot(z) + \cot(z - \tfrac{\pi}{n}) + \cot(z - 2\tfrac{\pi}{n}) + \ldots + \cot(z - (n-1)\tfrac{\pi}{n}) \;=\; a_n \cot(nz)$$

for an appropriate constant a_n, which you should find.

(c) The cosecant function similarly goes to infinity at the integer multiples of π, changing sign at each of these points. But for this function, positive values occur to the right of the *even* multiples of π, and to the *left* of the odd multiples. (Unlike the cotangent, this function does not change sign by going through 0 between one multiple of π and the next.)

Clearly, we can get the same sort of alternating pattern of infinities with the function $\cot(t) - \cot(t - \frac{\pi}{2})$, again with a doubling of the frequency. Show that this function is in fact equal to a scalar multiple of $\csc(2t)$. Again prove your conclusion for functions of a complex variable z, not just a real variable t.

(iv) Find a simple formula for the function

$$\cos(z) + \cos(z - \tfrac{\pi}{n}) + \cos(z - 2\tfrac{\pi}{n}) + \ldots + \cos(z - (n-1)\tfrac{\pi}{n}),$$

for $n > 1$. (Since the cosine does not go to infinity anywhere, you can't use the idea of the preceding parts to guess the answer. However, the answer turns out to be very simple.)

Chapter 7, ''Exercise 23'': In this exercise, we will anticipate some results of Chapter 10 by proving that they hold for any function f that is given by a power series. The proofs in that chapter will only need f to be a differentiable function, and will in fact lead to the amazing result that every differentiable function of a complex variable has a power series expansion. The present exercise is not that powerful, but it will give us some sense of what to expect of these functions.

For any positive real number r, let C_r denote the circle of radius r around the origin, parametrized by letting $C_r(t) = re^{it}$ $(0 \leq t \leq 2\pi)$. We begin with the case of polynomials.

(a) Suppose $f(z) = \Sigma_{n=0}^{k} a_n z^n$ is a polynomial with coefficients in $\mathbb{C}$. Show that

$$\int_{C_r} z^{-1} f(z)\, dz \;=\; 2\pi i\, a_0.$$

(Hint: Substitute the definition of f into this integral, note that all but one of the terms you get have antiderivatives, and that the integral of that one is easy to evaluate.)

(b) Now let $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ be a power series, and r a positive real number less than the radius of convergence of this series. Obtain the same result as in (a) for this function. (Suggestion: Write f as a polynomial plus a remainder term, apply part (a) to the polynomial, and apply ε-δ arguments to the remainder.)

(c) Substitute $z = re^{it}$ in the formula you got in (b), and translate the result into a relation between $f(0)$ and the average value of $f(z)$ on the circle C_r.

(d) Deduce from the result of (c) the corresponding result for a function with a power series expansion about an arbitrary point z_0, $f(z) = \Sigma_{n=0}^{\infty} a_n (z-z_0)^n$. I.e., relate its value at z_0 to its values on a circle about z_0.

(e) Generalize the ideas of (a) and (b) to $\int_{C_r} z^{-m-1} f(z)\,dz$ for arbitrary $m \geq 0$, obtaining a formula for the mth derivative of f at 0; and, as in (d) above, extend this to functions with power series expansions about an arbitrary point z_0.

Chapter 7, ''Exercise 24'': Here is a result not in the text that may be obtained using the idea of the preceding exercise.

Suppose $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ and $g(z) = \Sigma_{n=0}^{\infty} b_n z^n$ are power series, and r a real number less than the radius of convergence of each of them. Show that

$$\int_{C_r} (f(z)\,\overline{g(z)}/z)\,dz \;=\; 2\pi i\ \Sigma_{n=0}^{\infty} a_n \overline{b_n}\, r^{2n}.$$

Suggestion: Note that on the circle C_r we have $z\bar{z} = r^2$. Solve this equation for $\bar{z}$, substitute the result into the expression for $\overline{g(z)}$, and regard the expression you get for $f(z)\,\overline{g(z)}$ as the restriction to the circle C_r of a certain differentiable function on a domain containing C_r. Then treat the integral as in parts (a) and (b) of the preceding exercise.

Chapter 8, ''Exercise 8'': The equation $L(\partial T_1) = ½\,L(\partial T)$ in the last display on p.144 is clear geometrically. However, prove it without calling on geometric knowledge for the case $T_1 = T^{(1)}$ and discuss briefly how the cases $T_1 = T^{(2)}, T^{(3)}, T^{(4)}$ are related to that case. (Hint: The text proves the formula for the length of a line-segment, $L([z_1, z_2])$.)

Chapter 8, ''Exercise 9'': (a) Let D be a star domain, and S the set of all star centers of D. Show that if $z_0, z_1 \in S$, then $[z_0, z_1] \subseteq S$. (I.e., S is a *convex* set.)

(b) Show by example that S need not be open. In fact, find an example where S consists of a single point.

Chapter 8, ''Exercise 10'': Suppose D_1 and D_2 are star domains in $\mathbb{C}$.

(a) Show that if there is a point z_* which is both a star center for D_1 and a star center for D_2, then both $D_1 \cap D_2$ and $D_1 \cup D_2$ are star domains.

(In the remaining parts, D_1 and D_2 are still star domains, but the condition of (a) is not assumed.)

(b) Given an example where $D_1 \cap D_2$ is nonempty, but is not path connected.

(c) Given an example where $D_1 \cap D_2$ is nonempty and path connected, but is not a star domain.

(d) Given an example where $D_1 \cap D_2$ is nonempty, so that by ''Chapter 2, Exercise 17(b)'' above, $D_1 \cup D_2$ is path connected, but where $D_1 \cup D_2$ is not a star domain.

Chapter 8, ''Exercise 11'': In §8.3 the authors redefine $\mathrm{Log}\, z_1$ for $z_1 \in \mathbb{C}_\pi$ as $\int_{[1,z_1]} 1/z\ dz$, and then show, by doing the integration along a more convenient path, that this has the property

$$\mathrm{Log}(re^{i\theta}) \;=\; \log(r) + i\theta \quad \text{for} \quad n > 0 \quad \text{and} \quad \theta \in (-\pi, \pi),$$

and is therefore equal to the function they had named ''Log'' in the preceding chapter. Their development is a little messy, however, because they consider separately the cases $r > 1$ and $r < 1$, and likewise the cases $\theta > 0$ and $\theta < 0$ (neglecting, incidentally, the cases $r = 1$ and $\theta = 0$).

In part (a) below, we will see how to avoid dividing into cases, and in part (b), we give a still simpler way to get this result.

(a) Given any $r > 0$ and $\theta \in (-\pi, \pi)$, define $\gamma_1, \gamma_2\colon [0,1] \to \mathbb{C}$ by $\gamma_1(t) = e^{t \log r}$ and $\gamma_2(t) =$

$re^{ti\theta}$. Show that $\gamma_1 + \gamma_2$ is a contour from 1 to $re^{i\theta}$ in $\mathbb{C}_\pi$, and by integrating $1/z$ over this contour, conclude that $\mathrm{Log}(re^{i\theta})$, as defined in §8.3, is indeed $\log(r) + i\theta$.

(b) Given r, θ as in part (a), define $\gamma\colon [0,1] \to \mathbb{C}$ by $\gamma(t) = e^{t\,(\log(r) + i\theta)}$. Show that γ is a contour from 1 to $re^{i\theta}$ in $\mathbb{C}_\pi$, and integrate $1/z$ over this contour to obtain the same conclusion as in part (a).

Chapter 8, ''Exercise 12'': (a) Suppose σ is a closed step path, and let us subdivide the plane into (finite and infinite) rectangles based on this path as on pp.150-152. Let L be some line segment in that subdivision, and let R_r and R_s be the two rectangles of our subdivision that have L as a side. Suppose that, after cancelling opposite line segments, L occurs exactly q times in σ. Show that v_r and v_s differ by exactly q.

(b) Describe how to determine whether $v_r - v_s$ is $+q$ or $-q$ in terms of the direction of L (upward, downward, right or left) and which side of L the rectangle R_r lies on (left or right if L points upward or downward; top or bottom if L points right or left).

(The analog of the above results apply to general contours. In brief: ''When you cross the contour, the winding number changes by 1, unless you cross at a point through which it goes multiple times, in which case the integer by which it changes is obtained by a summation of the appropriate terms ± 1''. Cf. Fig. 7.8, p.132.)

Chapter 8, ''Exercise 13'': Show that if D_1 and D_2 are simply connected domains, then every connected component of $D_1 \cap D_2$ is a simply connected domain.

Chapter 8, ''Exercise 14'': Show that if D is a bounded domain and $\mathbb{C} \setminus D$ is path connected, then D is simply connected. (Bounded means contained in $N_R(0)$ for some real number R.)

Chapter 8, ''Exercise 15'': Suppose D_1 and D_2 are simply connected domains such that $D_1 \cap D_2$ is path connected. Drawing a few pictures, one can convince oneself that $D_1 \cup D_2$ must also be simply connected, but I don't know of any simpler proof (using the methods of this course) than the lengthy one below. (If you can find one, show me!)

Let $\gamma\colon [a,b] \to D_1 \cup D_2$ be a closed path, which we wish to show has winding number 0 about every point not in $D_1 \cup D_2$. We start by taking care of an easy case:

(a) Show that if γ lies wholly in D_1 or wholly in D_2, then it has winding number 0 about every point not in $D_1 \cup D_2$.

(b) Deduce that if γ contains no point of $D_1 \cap D_2$, then it has winding number 0 about every point not in $D_1 \cup D_2$.

Throughout what follows, we may thus assume that γ contains a point of $D_1 \cap D_2$.

(c) Show that, by a reparametrization, we can assume without loss of generality that the point $\gamma(a) = \gamma(b)$ lies in $D_1 \cap D_2$. I.e., show that γ will have a reparametrization with this property, and that if we can prove that that reparametrization has the desired winding-number property, then so does γ.

Hence it suffices to prove our claims for contours γ with $\gamma(a) = \gamma(b) \in D_1 \cap D_2$, and we shall assume in the remainder of this proof that γ has that property.

(d) Show that one can find finitely many values $a = t_0 < t_1 < \ldots < t_n = b$ such that each subpath $\gamma([t_{r-1}, t_r])$ is contained either in D_1 or in D_2.

As in the proof of the Paving Lemma (p.39), for $r = 1, \ldots, n$ let us write γ_r for the subpath gotten by restricting γ to $[t_{r-1}, t_r]$; thus $\gamma = \gamma_1 + \ldots + \gamma_n$. If $n = 1$, we see from (a) that γ has winding number zero about all points not in $D_1 \cup D_2$. Hence we suppose $n > 1$, and will prove our conclusion, assuming inductively that it is known to hold for all paths γ' that can be decomposed as above into $< n$

subpaths each lying in one of our domains.

Clearly, $t_{n-1}, t_n \in D_1 \cap D_2$, so by path connectedness of that set, there exists a path $\phi: [t_{n-1}, t_n] \to D_1 \cap D_2$ from $\gamma(t_{n-1})$ to $\gamma(t_n)$.

(e) Show that $\phi - \gamma_n$ is a closed path in $D_1 \cup D_2$ which has winding number zero about all points not in that set.

(f) Comparing the closed paths $\gamma = \gamma_1 + \ldots + \gamma_{n-1} + \gamma_n$ and $\gamma' = \gamma_1 + \ldots + \gamma_{n-1} + \phi$, deduce that $w(\gamma, z_0) = w(\gamma', z_0)$ for all $z_0 \in \mathbb{C} \setminus D_1 \cup D_2$.

(g) Show that the path γ' of part (f) can be decomposed into $< n$ subpaths each of which lies in one of our domains. Combining the result of (f) with our inductive hypothesis, conclude that γ has winding number 0 about all points not in $D_1 \cup D_2$, as desired.

Chapter 8, ''Exercise 16'': The converse of the result of the preceding exercise is also true: If D_1 and D_2 are simply connected domains such that $D_1 \cup D_2$ is simply connected, then $D_1 \cap D_2$ is path connected. But I don't see how to prove this using the methods of this course.

I think a proof should go essentially as follows: If $D_1 \cap D_2$ is not path connected, take p and q in different connected components of that set, let γ_1 be a path from p to q in D_1, and let γ_2 be a path from q to p in D_2. Then $\gamma_1 + \gamma_2$ will be a closed path in $D_1 \cup D_2$, which intuitively should have nonzero winding number about some point of $\mathbb{C} \setminus D_1 \cup D_2$. But I don't see a way to show this. Do you?

Chapter 9, ''Exercise 12'': Let D be a domain in $\mathbb{C}$. If γ_1 and γ_2 are paths in D, then where the book says ''γ_1 and γ_2 *are homotopic* via closed paths'' (definition at the start of §9.5), let us temporarily use the wording ''γ_1 *is homotopic to* γ_2 via closed paths,'' to avoid assuming the symmetry condition you will prove in (b) below. Symbols γ, γ_1 etc. will denote closed paths in D.

(a) Show that every γ is homotopic to itself via closed paths in D.

(b) Show that if γ_1 is homotopic to γ_2 via closed paths in D, then γ_2 is homotopic to γ_1 via closed paths in D.

(c) Show that if γ_1 is homotopic to γ_2 via closed paths in D and γ_2 is homotopic to γ_3 via closed paths in D, then γ_1 is homotopic to γ_3 via closed paths in D.

(If you have seen the concept of *equivalence relation* you will recognize that what you have proved is that homotopy via closed paths is an equivalence relation on the set of closed paths in D. By essentially the same arguments, fixed end-point homotopy is likewise an equivalence relation on the set of paths from z_1 to z_2, for any $z_1, z_2 \in D$.)

Chapter 9, ''Exercise 13'': Let's fill in some details from the Example on p.169.

(a) Show that for $z_0, K, \rho, \varepsilon$ as in that example, the curves γ_0 and γ_1 lie in $D = N_K(0) \setminus \{z_0\}$.

(b) Show that function ϕ defined in the display at the end of that example likewise takes values in D. (Hint: For $s \in [0,1]$, look at the curve $\{\phi(t,s) \mid t \in [0, 2\pi]\}$.)

Chapter 9, ''Exercise 14'': Let D be a domain. Show that every closed path γ in D is closed path homotopic in D to a closed step path σ. (Suggestions: Let σ be obtained from γ as in the proof of Lemma 8.4 and let $\phi(s,t)$ be constructed from γ and σ as in the Example on p.169. In showing that $\phi(s,t)$ takes on values in D, ''Chapter 2, Exercise 13'' above may be useful.)

Chapter 9, ''Exercise 15'': The book defines two kinds of homotopy among paths in a domain D: closed path homotopy and fixed-end homotopy. Let us show why it would not be of interest to define homotopy between paths without either sort of restriction. Given a domain D, call paths γ_0, γ_1: $[a,b] \to D$ *free-end homotopic* if there merely exists a continuous map $\phi: [a,b] \times [0,1]$ such that for all

$t \in [a,b]$, $\gamma_0(t) = \phi(t,0)$ and $\gamma_1(t) = \phi(t,1)$.

Show that *any* two paths $[a,b] \to D$ are free-end homotopic.

Chapter 9, ‘‘Exercise 16’’: The book’s definition of simple connectedness is nonstandard. Under the standard definition, a set $S \subseteq \mathbb{C}$ is called simply connected if it is connected, and every closed path in S is homotopic to zero. (This definition is applicable, more generally, to subsets S of $\mathbb{R}^n$ for any n, still more generally to metric spaces S, and yet more generally to topological spaces S.) For open subsets S of the plane, the standard definition can be proved equivalent to the definition in our text, but the full proof is hard. One direction, however, is easy:

Prove that if a domain D in $\mathbb{C}$ is simply connected in the standard sense (i.e., if every closed path in D is homotopic to zero), then it is simply connected in the sense of our text (p.157, first paragraph of §8.7).

Of course, outside of this exercise, we will continue to use the book’s definition rather than the standard one.

Chapter 10, ‘‘Exercise 21’’: Suppose D is a domain, L is a line in $\mathbb{C}$, and f is a continuous function on D which is differentiable on $D \setminus L$. Show that f is differentiable on all of D.

(Suggestion: Note why it suffices to prove that given any point $z_0 \in D \cap L$ and any neighborhood $N = N_\varepsilon(z_0)$ contained in D, the restriction of f to N is differentiable at z_0. Now such an N is a star domain with star center z_0; given a triangle T in N with one vertex at z_0, show how to approximate the integral of f around T using integrals around triangles which lie entirely on one or the other side of L. Deduce that the integral of f around T is zero, and apply the method of proof of Theorem 8.2 and its Corollary.)

Chapter 10, ‘‘Exercise 22’’: Prove Chapter 14, Exercise 21 (p.286), replacing ‘‘Morera’s Theorem’’ in the authors’ hint by ‘‘Chapter 10, Exercise 21’’ above.

(The result of that exercise is quite nice, but so far as I can see, the version of Morera’s Theorem proved in the text is too weak to use in proving it. I don’t know why the authors left this result to Chapter 14. In the statement of that exercise, the phrase ‘‘continuous, analytic on U^+’’ means ‘‘continuous on the whole set where it is defined, and analytic on the subset U^+’’.)

Chapter 10, ‘‘Exercise 23’’: Let $f(z)$ be a differentiable function on a domain D, let z_0 be a point of D, and let R be the radius of convergence of the Taylor series of $f(z)$ about z_0.

(a) Show that the radius of convergence of the Taylor series of $e^{f(z)}$ about z_0 is $\geq R$.

(b) Find an example where the radius of convergence of $e^{f(z)}$ about z_0 is strictly greater than R.

Chapter 10, ‘‘Exercise 24’’: Suppose D is a domain, and $S_1 \subseteq S_2$ are subsets of D such that S_1 has at least one limit point in D. Call a $\mathbb{C}$-valued function h on S_1 or S_2 ‘‘extensible’’ if there exists a differentiable function f on D which extends h.

(a) Show that if h_1 is an extensible function on S_1, then there exists one and only one extensible function h_2 on S_2 which is an extension of h_1.

(b) Let $D = N_1(0)$ and $S = \{1/n \mid n \in \mathbb{Z},\ n > 1\}$. Deduce from (a) that if h is a function on S such that $h(1/n) = 1/n$ for infinitely many integers $n > 1$, but not for all such integers, then h is not extensible.

Chapter 10, ‘‘Exercise 25’’: The assertion in the next-to-last paragraph of p.189, beginning ‘‘it is easy to see’’ (second line of that paragraph) is not clearly stated, and when one states it clearly, it is not as trivial to prove as the authors claim. Give a precise proof of the result below, which is the hard part of that assertion:

Suppose $\Sigma_{n=0}^{\infty} a_n(z-z_0)^n$ is a Taylor series with radius of convergence R, where $0 < R < \infty$, and f is the differentiable function on $N_R(z_0)$ which it defines. If w is a point with $|w-z_0| = R$, let us say that f ''can be extended at w'' if there exists a positive real number ε and a differentiable function h on $N_R(z_0) \cup N_\varepsilon(w)$ which extends f.

Show that there exists some w with $|w-z_0| = R$ such that f *cannot* be extended at w.

Chapter 10, ''Exercise 26'': Let f be a differentiable function whose domain of definition contains a disc $N_R(z_0)$, and let $0 \leq r_1 < r_2 < R$. Let C_1 and C_2 denote the circles of radius r_1 and r_2 respectively about z_0. Show that $\sup_{z\in C_1}|f(z)| \leq \sup_{z\in C_2}|f(z)|$, with equality if and only if f is constant.

Chapter 11, ''Exercise 23'': Let $f(z)$ be a differentiable function on an annulus $\{z\in\mathbb{C} \mid r < |z-z_0| < R\}$ with Laurent expansion $f(z) = \Sigma_{-\infty}^{\infty} c_n(z-z_0)^n$ on that annulus. Show that on that same annulus, $f'(z)$ has the Laurent expansion $\Sigma_{-\infty}^{\infty} (n+1)c_{n+1}(z-z_0)^n$.

Chapter 11, ''Exercise 24'': Let $z_0 \in \mathbb{C}$, let $0 \leq r < \rho < R \leq \infty$, and let $w_1, \ldots, w_k \in \mathbb{C}$ be points all satisfying $|w_j - z_0| = \rho$. We shall write $A_{r,R}(z_0)$ for the annulus $\{z\in\mathbb{C} \mid r < |z-z_0| < R\}$, and use the corresponding notation for annuli determined by other pairs of radii.

Suppose f and g are functions differentiable on $A_{r,R}(z_0) - \{w_1, \ldots, w_k\}$. Let the Laurent series for $f(z)$ in $A_{r,\rho}(z_0)$ be $\Sigma\, a_n(z-z_0)^n$, and the series in $A_{\rho,R}(z_0)$ be $\Sigma\, A_n(z-z_0)^n$; and let the corresponding two series for $g(z)$ be $\Sigma\, b_n(z-z_0)^n$ and $\Sigma\, B_n(z-z_0)^n$ respectively. (All these summations are from $n = -\infty$ to ∞.)

Show that if the singularities of $f(z) - g(z)$ at $w_1, \ldots, w_k$ are all removable, then for all $n\in\mathbb{Z}$,

$$A_n - a_n = B_n - b_n.$$

(Hint: What will be the relation between the Laurent series for of $f - g$ on the two annuli in question?)

The above result can be used in the following way. Suppose we have a function f for which we know the Laurent expansion in $A_{r,\rho}(z_0)$ and want to find the expansion in $A_{\rho,R}(z_0)$. If we can find some g whose expansions we can calculate in both annuli, and such that the singularities of $f - g$ on the circle between the annuli are removable, then we can get the expansion of f where we did not know it.

We shall use this method in ''Chapter 12, Exercise 30'' to study the Laurent series of the function $\sec z$ in the annulus about 0 between its smallest and second-smallest poles. (We could do this after reading §11.3, but §§12.1-12.2 will give us some convenient language and techniques for describing the behavior of functions at poles.)

Chapter 11, ''Exercise 25'': (a) What kind of singularity does $e^{z^{-1}}/e^{(\sin z)^{-1}}$ have at $z = 0$?

(b) What about $e^{z^{-1}}/(1+e^{(\sin z)^{-1}})$?

Of course, you must justify your answers.

Chapter 11, ''Exercise 26'': Suppose S and D are domains, $g: S \to D$ is a differentiable function, w_0 and z_0 are points of S and D respectively such that $g(w_0) = z_0$, and f is a differentiable function defined on D except possibly for a singularity at z_0.

(a) If $g'(w_0) \neq 0$, show that $f\circ g$ ''behaves at w_0 as f does at z_0'', in the sense that it has a zero of order m at w_0 if and only if f has a zero of order m at z_0, it has a pole of order m at w_0 if and only if f has a pole of order m at z_0, it has an essential singularity at w_0 if and only if f has an essential singularity at z_0, and it has neither a zero nor a singularity at w_0 if and only if f has neither a zero nor a singularity at z_0.

(b) Suppose instead that g' has a zero of finite order d at w_0. State and prove results describing the behavior of $f\circ g$ at w_0 in terms of that of f at z_0.

Remark: If we count a point where a differentiable function is nonzero as a ''zero of order 0'', then the answer to (a) will be the ''$m = 0$'' case of the answer to (b).

Chapter 11, ''Exercise 27'': Let f be a function differentiable on a ''punctured neighborhood N of *infinity*'', i.e., on $N = \{z \in \mathbb{C} \mid |z| > r\}$ for some real number r. Regarding N as an annulus centered at 0 with infinite outer radius, let the Laurent expansion of f in that annulus be $f(z) = \Sigma_{-\infty}^{\infty} c_n z^n$.

Let m be a positive integer. Prove the following two criteria for f to have a zero of order m at ∞, and two criteria for it to have a pole of order m there.

(a) f has a zero of order m at ∞ if and only if $c_n = 0$ for all $n > -m$ and $c_{-m} \neq 0$.

(b) f has a zero of order m at ∞ if and only if $\lim_{|z| \to \infty} z^m f(z)$ exists and is finite and nonzero.

(c) f has a pole of order m at ∞ if and only if $c_n = 0$ for all $n > m$ and $c_m \neq 0$.

(d) f has a pole of order m at ∞ if and only if $\lim_{|z| \to \infty} z^{-m} f(z)$ exists and is finite and nonzero.

Chapter 11, ''Exercise 28'': Again let f be a function differentiable on $N = \{z \in \mathbb{C} \mid |z| > r\}$ for some real number r. Let a and b be complex numbers, with $b \neq 0$. Assuming ''Exercise 26'' above, prove

(a) $f(z+a)$ has a zero of order m at ∞ if and only if $f(z)$ does.

(b) $f(bz)$ has a zero of order m at ∞ if and only if $f(z)$ does.

(c) $f(z+a)$ has a pole of order m at ∞ if and only if $f(z)$ does.

(d) $f(bz)$ has a pole of order m at ∞ if and only if $f(z)$ does.

Chapter 11, ''Exercise 29'': On p.207, the authors say that a $\mathbb{C}$-valued function f defined on a neighborhood of ∞ will be called differentiable at ∞ if $f(1/z)$ is differentiable at 0. If f is a $\mathbb{C} \cup \{\infty\}$-valued function on a domain $D \subseteq \mathbb{C}$ containing a point z_0 such that $f(z_0) = \infty$, let us similarly say that f is differentiable at z_0, as a function to the extended complex plane, if the function $1/f(z)$, which takes the value 0 at z_0, is differentiable there.

Show that for any domain D, the functions $D \to \mathbb{C} \cup \{\infty\}$ which are everywhere differentiable in this sense are precisely the meromorphic functions on D, and the constant function ∞.

Remark: Of course, the most complete definition would be for $\mathbb{C} \cup \{\infty\}$-valued functions on a domain $D \subseteq \mathbb{C} \cup \{\infty\}$. I've restricted D to be contained in $\mathbb{C}$ because we haven't gotten a set of definitions for the extended complex plane as complete as what we have for $\mathbb{C}$; e.g., we haven't defined a ''path'' in $\mathbb{C} \cup \{\infty\}$, which is needed to define ''path-connected'', and hence ''domain'' (though the definition is what we would expect).

Important note: The definitions in this exercise will only be used when we say explicitly that we are considering differentiability of functions *as* $\mathbb{C} \cup \{\infty\}$*-valued functions*. Unless this is stated, a ''differentiable function'' on a domain will always mean a differentiable $\mathbb{C}$-valued function, and poles will be considered nonremovable singularities. Though the ideas of this exercise are elegant, they create a danger of serious confusion between the extended and standard meanings of terms unless used with care.

Chapter 12, ''Exercise 27'': (a) Suppose that D is a domain containing a simple loop γ and all points inside γ, that f is a function differentiable in D, and that w is a point inside γ. Prove that Cauchy's integral formula, the displayed formula of Lemma 10.1, holds with γ in place of C_r. (You will not have to repeat the proof of Lemma 10.1; you should be able to get this easily using later results.)

(b) Assume the generalization of Cauchy's Residue Theorem obtained by replacing ''simple loop'' by ''closed path'', and replacing each summand $\operatorname{res}(f, z_r)$ by the product $w(\gamma, z_r)\operatorname{res}(f, z_r)$ in the resulting formula. (This generalized statement is true; the proof is almost identical to that of the theorem. Cf. Exercise 7 of Chapter 8, p.158, though that exercise is not very clearly worded.)

With the help of the above result, state and prove a generalization of the result of part (a), for closed paths γ which are not necessarily simple loops.

Chapter 12, ‘‘Exercise 28’’: Let $f(z) = \alpha_0 + \alpha_1 z + ... + \alpha_m z^m$ and $g(z) = \beta_0 + \beta_1 z + ... + \beta_n z^n$ be polynomials over the complex numbers, of degrees m and n respectively. (Thus, the two coefficients α_m and β_n are assumed nonzero.) Let $r(z)$ denote the rational function $f(z)/g(z)$.

Let Σ denote the sum of the residues of $r(z)$ at all of its singularities in $\mathbb{C}$. We shall obtain some tricks for computing Σ.

(a) Show that if $n - m \geq 2$, then $\Sigma = 0$. (Hint: Verify that $\lim_{z\to\infty} z\, r(z) = 0$. Deduce, using the Estimation Lemma, that $\lim_{R\to\infty} \int_{C_R} r(z)\, dz = 0$, where C_R is the circle of radius R centered at 0.)

(b) Deduce that if $n - m = 1$, then $\Sigma = \alpha_m/\beta_{m+1}$. (One approach: Show that $r(z) - (\alpha_m/\beta_{m+1})/z$ satisfies the conditions of part (a).)

(c) Find Σ if $f(z) = (z+1)^2/(z^2+1)$. (Hint: Subtracting a constant will reduce this rational function to one of the sort described in (b). What residues does a constant have?)

(d) Find Σ if $f(z) = z^3/(z+1)^2$.

Chapter 12, ‘‘Exercise 29’’: Let f be a function differentiable in a ‘‘punctured neighborhood of infinity,’’ $N = \{z\in\mathbb{C} \mid |z| > r\}$ for some real number r. If the Laurent expansion of $f(z)$ in N is $\Sigma_{-\infty}^{\infty} c_n z^n$, let us define the ‘‘residue of f at ∞’’ by $\mathrm{res}(f, \infty) = -c_{-1}$.

(Because of the minus sign, this has the property that for any counterclockwise circle C about 0 in N, we have $\int_C f(z)\, dz = -2\pi i\, \mathrm{res}(f, \infty)$. One way of justifying this reversal of sign is to note that on the Riemann sphere, a counterclockwise circle about 0 goes clockwise around ∞. Residues at infinity are referred to in Chapter 12, Exercise 2, second line, but as far as I can see they are not defined in the text.)

This concept differs from the residue at a finite point in one peculiar way:

(a) Show that f may have nonzero residue at ∞ without having a pole there. In fact, show that if f has a simple zero at ∞, then it will have nonzero residue there. Show, however, that if f has a zero of degree 2 or higher at ∞, then it has zero residue there.

The next part shows that residues at ∞ cooperate nicely with residues at points of $\mathbb{C}$.

(b) Suppose f is differentiable at all of $\mathbb{C}$ except for finitely many singularities $z_1, \ldots, z_r$, and let us also write $z_{r+1} = \infty$. Show that $\Sigma_{k=1}^{r+1} \mathrm{res}(f, z_k) = 0$.

Remark: Combining parts (a) and (b) above with ‘‘Chapter 12, Exercise 28(a)’’, it follows that if f has a zero of order at least 2 at ∞, then the sum of its residues at all points of $\mathbb{C}$ is zero.

Chapter 12, ‘‘Exercise 30’’: Suppose as in ‘‘Chapter 11, Exercise 26’’ that S and D are domains in $\mathbb{C}$, that $g\colon S \to D$ is a differentiable function, that w_0 and z_0 are points of S and D respectively such that $g(w_0) = z_0$, and that f is a differentiable function defined on D except possibly for a singularity at z_0.

In that exercise we saw that if g' was nonzero at w_0, then the behavior of $f\circ g$ at w_0 was ‘‘essentially the same as’’ that of f at z_0, in terms of order of poles, zeroes, etc.. However –

(a) Show by example that the residue of $f\circ g$ at w_0 may not equal that of f at z_0. (Suggestion: take for f the easiest example you know of a function with a pole at a point, and try out various possibilities for g.)

However, if we think of residues as things that are intimately related to integration, this suggests a different relationship that might be true. Suppose we take the expression $f(z)\, dz$ and make the substitution $z = g(w)$ in the way you learned in freshman calculus. Then we get the expression $f(g(w))\, g'(w)\, dw$. This suggests

(b) Show that for f and g as in the first two paragraphs of this exercise, the residue of $(f\circ g)\cdot g'$ at

w_0 is equal to the residue of f at z_0. (Hint: To go back to the definition of residue would lead to a messy calculation with Laurent series. Use instead results relating residues and integrals.)

The next two parts each assume you have done part (b) above, but are independent of one another.

(c) Suppose f and g are as in the first paragraph of this exercise, but that g' has a zero of finite order d at w_0. State and prove a relation between the residues of f at z_0 and of $(f\circ g)\cdot g'$ at w_0 in this case.

(As in ''Chapter 11, Exercise 26 part (b)'', you should get a result such that, if we regard a point where a differentiable function is nonzero as a ''zero of order 0'', the result of the preceding part is just the ''$m = 0$'' case of your result.)

(d) Suppose now that f is a differentiable function whose domain includes a punctured neighborhood of ∞, i.e., which is defined for all z with $|z| > R$ for some real number R. Let us take $g(z) = z^{-1}$, $w_0 = 0$ and $z_0 = \infty$, and use the formula of part (b) to *define* the residue of f at ∞. Does the resulting definition agree with the one proposed in ''Chapter 12, Exercise 29'' above? Show your calculation.

Chapter 12, ''Exercise 31'': In this exercise we shall relate the expansions of $\sec z$ in two domains.

(a) Find the Taylor expansion of $\sec z$ about 0 through the z^6 term, by ''long division'' of 1 by the series for $\cos z$. (Though this computation is not difficult, there is no simple formula for the nth term of the result. Long division of Taylor series is justified because by Theorem 3.9, Taylor series multiply in the expected way, and long division is simply an algorithm for finding a series which, when multiplied formally by a given series, will yield another given series.)

(b) Show that all singularities of $\sec z$ in $\mathbb{C}$ are simple poles, find the residues at all these poles, and find a rational function $g(z)$ whose only singularities are simple poles at the two complex numbers of smallest absolute value which are poles of $\sec z$, and such that the residues of g there are equal to the residues of $\sec z$ there.

(c) Combine the result of ''Chapter 11, Exercise 24'' with that of (b) above to get an expression for the Laurent coefficients of $\sec z$ on the annulus $A_{\pi/2,\,3\pi/2}(0)$ in terms of the Laurent coefficients of $\sec z$ on the annulus $A_{0,\,\pi/2}(0)$.

(d) Combine (a) and (c) to calculate the terms of the Laurent expansion of $\sec z$ from the z^{-8} term to the z^6 term.

Here are some variants of this computation:

(a′)-(d′) Do the analogous computations for $\tan z$.

(a′′)-(d′′) Do the analogous computations for $\csc z$. The one difference is that it will have a pole at 0 as well, so in part (a′′), instead of a Taylor series you will have a Laurent series.

(a′′′)-(d′′′) Do the analogous computations for $\sec^2 z$. Here there will be a difference in part (b′′′): The singularities will still be poles, but not simple poles; so you need to find not only the residues, but all negative-exponent terms in the expansions about those poles, and then a rational function whose Laurent series has the same negative-exponent terms as these.

Chapter 12, ''Exercise 32'': Let D be a domain, z_0 a point of D, and f and g functions differentiable on D with zeroes at z_0 of orders m and n respectively, where $0 < m < n$.

We would like to compare the residues of f^{-1} and $(f+g)^{-1}$ at z_0. We begin with an observation which will simplify the computations.

(a) Show that $f(z)+g(z) = f(z)\,h(z)$, where h is a differentiable function whose Taylor series expansion about z_0 has the form $1 + c(z-z_0)^{n-m}$ + terms of higher degree in $z-z_0$, with $c \neq 0$.

(b) Deduce that if $n \geq 2m$, then the residues of f^{-1} and $(f+g)^{-1}$ at z_0 are equal.

(c) Deduce, on the other hand, that if $n = 2m-1$, then these residues are unequal.

(d) Show by examples that if $n < 2m-1$, then either conclusion may be true. In fact, give a point z_0, integers m and n such that $0 < m < n < 2m-1$, a function f for which z_0 is a zero of order m, and two functions g_1 and g_2 for both of which z_0 is a zero of order n, such that

$$\mathrm{res}(f^{-1}, z_0) \ = \ \mathrm{res}((f+g_1)^{-1}, z_0) \ \neq \ \mathrm{res}((f+g_2)^{-1}, z_0).$$

Chapter 12, ‘‘Exercise 33’’: Suppose f is a differentiable function on a punctured neighborhood $N_\varepsilon(z_0) \setminus \{z_0\}$ of a point $z_0 \in \mathbb{C}$,

(a) Show that if $f(z)$ has a zero of order n at z_0, then $f'(z)/f(z)$ has a simple pole there with residue n.

(b) Show that if $f(z)$ has a pole of order n at z_0, then $f'(z)/f(z)$ has a simple pole there with residue $-n$.

Remarks: The function f'/f is called the ‘‘logarithmic derivative of f’’; note that where f does not have a zero or pole, it is indeed the derivative of any branch of $\log f(z)$. Intuitively, if f has a zero of order n at z_0, then in the neighborhood of z_0 the function f ‘‘looks like’’ $c(z-z_0)^n$, and similarly if it has a pole of order n, it ‘‘looks like’’ $c(z-z_0)^{-n}$; so a logarithm of $f(z)$ should look like $\log c \pm n \log(z-z_0)$, so its derivative should look like $\pm n/(z-z_0)$, which has the asserted residue. Because of the impossibility of defining the logarithm continuously in a neighborhood of 0 or ∞, it takes work to turn these ideas into a proof, while a direct proof of (a) and (b) above is easy; nevertheless, thinking of f'/f as roughly the logarithm of the derivative can provide helpful intuition.

Along with (a) and (b) above there are some cases we have not considered. If $f(z)$ has a removable singularity with nonzero value at z_0, it is immediate that $f'(z)/f(z)$ has a removable singularity there. What if f has an essential singularity at z_0? If it has zeros in every neighborhood of that point, then $f'(z)/f(z)$ has a non-isolated singularity there, and we can’t speak of its Laurent expansion. In the contrary case various behaviors can occur; you might find it interesting to work out what one gets for $f(z) = e^{z^{-1}}$ and for $f(z) = e^{e^{z^{-1}}}$.

Chapter 12, ‘‘Exercise 34’’: Suppose f is a function with a simple pole at 0, and Laurent expansion $c_{-1}z^{-1} + c_0 + c_1 z + \ldots$ about that point. For each positive integer k, let $r_k = \mathrm{res}(f(z)^k, 0)$.

(a) Find formulas for r_1, r_2, r_3 in terms of the coefficients $c_{-1}, c_0, \ldots$.

(b) Solve these formulas, and so express c_{-1}, c_0 and c_1 in terms of r_1, r_2 and r_3.

Chapter 12, ‘‘Exercise 35’’: For $a \in (-1, 1)$, derive an explicit formula for an antiderivative of the real function $1/(a - \cos t)$ in terms of real functions familiar from calculus, with the help of the substitution of method (I), pp.217-218. (Suggestion: After making that substitution, use the partial fraction decomposition of the rational function you get, and the expressions we saw in earlier chapters for various complex functions in terms of familiar real functions.)

Chapter 12, ‘‘Exercise 36’’: Let a be a complex number and ϕ a differentiable function on $\mathbb{C}$ with only finitely many singularities. The third display on p.227 gives a condition that must be satisfied for method (V) (a trick for integrating $e^{ax}/\phi(e^x)$ from $-\infty$ to ∞), to work. Let us examine when that condition holds.

(a) For all $C > 0$, let $m(C) = \max_{|z|=C} |\phi(z)|$. Show that sufficient conditions for the limit-statement of the third display on p.227 to hold is that $\lim_{C\to 0} C^{\mathrm{re}(a)}/m(C)$ and $\lim_{C\to\infty} C^{\mathrm{re}(a)}/m(C)$ both be 0.

To state the next part neatly, we introduce some terminology: If ϕ is differentiable in a punctured neighborhood of $z_0 \in \mathbb{C}$ and does not have an essential singularity there, let the *order* of ϕ at z_0 be the smallest n such that the Laurent expansion $\Sigma_{-\infty}^{\infty} c_k(z-z_0)^k$ of ϕ around z_0 has $c_n \neq 0$. (Thus, if ϕ has a zero there, its order is the order of that zero, if ϕ has a pole there, its order is *minus* the order of the pole, while if ϕ has neither, the order is 0.) We also define the order of f at ∞ to be the order of $f(z^{-1})$ at 0. (You can easily translate this into a description in terms of the Laurent series of f. The characterization of the order of f at a point in terms of zeroes and poles at that point also clearly holds at ∞.)

(b) Show that the conditions given in (a) hold if and only if ϕ does not have an essential singularity at 0 or ∞, and has order less than $\mathrm{re}(a)$ at 0 and less than $-\mathrm{re}(a)$ at ∞.

(c) Let m be a positive integer. For what values of a will the conditions of (a) be satisfied if $\phi(z) = z^m+1$? If $\phi(z) = (z^m+1)^{-1}$? If $\phi(z) = z^m$?

Remark: As mentioned in class, another condition needed for the method described on that page to work is that $a \notin \mathbb{Z}$, so that $1 - e^{2\pi a i}$ will be nonzero. But that is independent of the conditions considered here.

Chapter 12, ''Exercise 37'': (a) Suppose f and g are nonconstant differentiable functions on a domain D, such that f has at least one zero in D. Show that there exists a positive real number ε such that for all $c \in N_\varepsilon(0)$, the function $f(z) + c\,g(z)$ has at least one zero in D.

(b) Given a nonconstant differentiable function f on a domain D and a point z_0 of D, apply the result of part (a), with $f(z) - f(z_0)$ in place of f and the constant function 1 in place of g, and deduce that $f(D)$ is an open subset of $\mathbb{C}$.

(c) From the result of (b) prove that if f is a nonconstant differentiable function on a domain D and U is an open subset of D, then $f(U)$ is an open subset of $\mathbb{C}$. (Hint: Express U as a union of domains.) This result is called the Open Mapping Theorem.

Chapter 12, ''Exercise 38'': (a) Show how the proof of Rouché's Theorem (Theorem 12.6, p.232) can be adapted to prove:

> THEOREM 12.6′
> Suppose that f and g are differentiable in a domain S which includes a simple loop γ and all points inside γ. Suppose also that for all $z = \gamma(t)$, the values $f(z)$ and $g(z)$ are nonzero, and $f(z)/g(z)$ is not a negative real number. Then f and g have the same number of zeroes (counting multiplicities) inside γ.

(b) Show that any functions f and g which satisfy the hypothesis of Rouché's Theorem also satisfy the hypothesis of the above result. (Hence, that result is a generalization of Rouché's Theorem.)

(c) Show that if in the hypothesis of (a) we change $f(z)/g(z)$ to $f(z)\,g(z)$, then we can conclude that neither f nor g has any zeroes inside γ.

(d) Show that if in part (a) we replace the assumption that f and g are differentiable by the assumption that they are meromorphic, and the assumption that they have no zeroes at points of γ by the assumption that they have neither zeroes nor poles at such points. Show that the same conclusion holds, with ''number of zeroes'' replaced by ''number of zeroes minus number of poles''.

What can we conclude if we make the same change in the hypothesis of part (c)?

Chapter 13, ''Exercise 20'': In this exercise we will write $\mathbb{R}/2\pi\mathbb{Z}$ where the text uses $\mathbb{R}/2\pi$. Both notations are used, but I thought it would be good to accustom you to the more precise one.

For $r, s \in \mathbb{R}/2\pi\mathbb{Z}$, let us define the set $\mathrm{dist}(r,s) = \{|x-y| \mid x \in r, y \in s\} \subseteq \mathbb{R}$.

(a) Show that for all $r, s \in \mathbb{R}/2\pi\mathbb{Z}$, the set $\mathrm{dist}(r,s)$ has a unique element belonging to the interval $[0,\pi]$. (Recall that any nonempty set of nonnegative integers has a least member. You might begin by applying this fact to the set of n such that $\mathrm{dist}(r,s) \cap [n\pi, (n+1)\pi] \neq \emptyset$.)

Below, we shall write $d(r,s)$ for the unique element of $\mathrm{dist}(r,s) \cap [0,\pi]$.

(b) Show that for $r,s \in \mathbb{R}/2\pi\mathbb{Z}$, one has $d(r,s) = 0$ if and only if $r = s$.

(c) Show that for $r,s \in \mathbb{R}/2\pi\mathbb{Z}$, one has $d(r,s) = d(s,r)$.

(d) Show that for $r,s,t \in \mathbb{R}/2\pi\mathbb{Z}$, one has $d(r,t) \leq d(r,s) + d(s,t)$.

(If you have seen the concept of *metric space*, you will realize that these three conditions show that d is a metric on the set $\mathbb{R}/2\pi\mathbb{Z}$.)

(e) Show that the function $\mathrm{arc}: \mathbb{C}\setminus\{0\} \to \mathbb{R}/2\pi\mathbb{Z}$ defined on p.240 has the property that for all $z_0 \in \mathbb{C}\setminus\{0\}$ and all $\varepsilon > 0$ in $\mathbb{R}$ there exists $\delta > 0$ in $\mathbb{R}$ such that for all z with $|z - z_0| < \delta$, we have $d(\mathrm{arc}(z), \mathrm{arc}(z_0)) < \varepsilon$. This is the precise meaning of the book's claim that the function arc is continuous.

Chapter 13, ''Exercise 21'': (a) Show that for any three distinct points z_0, z_1, $z_\infty \in \mathbb{C} \cup \{\infty\}$, there exists a Möbius transformation f such that $f(z_0) = 0$, $f(z_1) = 1$ and $f(z_\infty) = \infty$. (Suggestion: First find a Möbius transformation taking z_∞ to ∞, then see whether, by a Möbius transformation that sends ∞ to itself, you can move the images of z_0 and z_1 to 0 and 1 respectively.)

(b) Show that the only Möbius transformation f such that $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$ is the identity transformation.

(c) Deduce from (a) and (b) that for any three distinct points z_0, z_1, $z_\infty \in \mathbb{C} \cup \{\infty\}$ there exists a *unique* Möbius transformation f such that $f(z_0) = 0$, $f(z_1) = 1$ and $f(z_\infty) = \infty$.

Chapter 13, ''Exercise 22'': This exercise continues the preceding one, ''Chapter 13, Exercise 21''.

Suppose p, q, r, s are four distinct elements of $\mathbb{C} \cup \{\infty\}$. By the preceding exercise, there exists a unique Möbius transformation f such that $f(p) = 0$, $f(q) = 1$ and $f(r) = \infty$. Let $\mathrm{cr}(p, q, r, s)$ denote $f(s)$ for this Möbius transformation f; $\mathrm{cr}(p, q, r, s)$ is called the *cross-ratio* of p, q, r and s.

(a) Obtain a formula expressing $\mathrm{cr}(p, q, r, s)$ in terms of p, q, r and s in the case where none of these equals ∞. (That formula will also work when one of them is ∞, if properly interpreted; but I won't make you check the details of this.)

(b) Prove (without computations, just using the definition of the cross-ratio and the facts proved in ''Exercise 21'' above) that if f is any Möbius transformation and p, q, r, s are any four distinct complex numbers, then

$$\mathrm{cr}(f(p), f(q), f(r), f(s)) \;=\; \mathrm{cr}(p, q, r, s).$$

One expresses this by saying that the cross ratio is an *invariant* of a 4-tuple (p, q, r, s) under Möbius transformations.

(c) Show that no nonconstant function $C(p, q)$ or $C(p, q, r)$ of two or three distinct variables in $\mathbb{C} \cup \{\infty\}$ is invariant under Möbius transformations.

(d) Investigate the relationship between $\mathrm{cr}(p, q, r, s)$ and the other 23 expressions, $\mathrm{cr}(p, q, s, r)$, $\mathrm{cr}(p, r, q, s)$, etc., that one gets from it by permuting the variables.

Chapter 13, ''Exercise 23'': Let $f(z) = (az+b)/(cz+d)$ be a Möbius transformation, and suppose $c \neq 0$. Show that f maps all lines which contain the point $-d/c$ and also all circles which contain the point $-d/c$ to lines, and maps all other lines and all other circles to circles.

(You do not have to go back to first principles. Taking it as known that f maps $\{\text{lines}\} \cup \{\text{circles}\}$ to itself, you can show why the only elements of that set which can possibly be mapped to a line, respectively to a circle, are those indicated.)

Chapter 13, ''Exercise 24'': Let D be a domain, z_0 a point of D, and R a positive real number such that $\{z \in \mathbb{C} \mid |z| \le R\} \subseteq D$. Show that if u is a harmonic function on D, then

$$u(z_0) \;=\; \int_0^1 u(z_0 + R\,e^{2\pi i t})\,dt,$$

i.e., that the value of u at z_0 equals the average of its values over a circle around that point. (Cf. Chapter 10, Exercise 12.)

Chapter 14, ''Exercise 22'': The properties of the logarithm that we explored in Chapter 7 can be stated as follows in terms of the ideas of §14.3: Although the logarithm function is the antiderivative of the uniform analytic function $1/z$ on the domain $\mathbb{C} \setminus \{0\}$, it cannot itself be made a uniform analytic function on that set, but it *can* be made a multiform function.

It is natural to ask whether every complete analytic function (uniform or multiform) has an antiderivative in the form of a complete analytic function. As the example of the logarithm shows, we cannot expect the antiderivative to necessarily be definable on the same domains as the original function. But with this proviso, the answer is yes. Namely –

Show that if F is a complete analytic function, then there is a complete analytic function G which is an antiderivative of F, in the sense that for every member (f, S) of the equivalence class F, and every point $z_0 \in S$, there exists a member (g, T) of the equivalence class G such that $z_0 \in T \subseteq S$, and such that $g' = f$ on T.

Chapter 14, ''Exercise 9, continued'':

(v) $\sqrt{z+1} + \sqrt{z-1}$.

(vi) $\sqrt{z^2-1}$.

(vii) $\sqrt{\sqrt{z}-1}$.

(I would recommend doing (v) and (vi) between (i) and (ii), and (vii) between (ii) and (iii).)

Chapter 14, ''Exercise 23'': As Stewart and Tall say on p.271, we would like to define complex exponentiation so that all the standard ''laws of exponents'' hold. In trying to do so, we are led to multiform power functions; but the authors never return to the question of what becomes of the laws of exponents. Since one has a *set* of values for z^a, one has to pose those laws in terms of quantification, saying ''For all members of this set'' and/or ''For some member of that set''. This exercise will concern the question of which quantifications make the various laws hold.

For any complex numbers z and a with $z \ne 0$, let us write $\mathrm{expset}(z, a)$ for ''the set of values of z^a''; formally,

$$\mathrm{expset}(z, a) \;=\; \{e^{a(\mathrm{Log}(z) + 2\pi i n)} \mid n \in \mathbb{Z}\}.$$

In what follows, z, z_1, z_2, a and b will denote arbitrary complex numbers, with z, z_1, z_2 assumed nonzero. A complication is that whether a given law of exponents holds may depend on the exponents. We could bring this question into the game by considering laws quantified over the exponents as well over the choice of elements from the ''expsets''; but for simplicity, let us say below that a law holds only if it holds for *every* value of all the *given* quantities. Thus, to show that a law holds, you must prove it for all z, a, etc., while to show that it fails, it suffices to find a counterexample for some such values. The laws we state will also contain symbols w, w_1, etc. representing values occurring in our expsets; these you should *not* assume quantified by ''for all''; rather, their quantifications will be part of the stated laws.

The next three parts consider, successively, the three basic laws of exponents. For simplicity, I only ask you to test five quantifications of each.

(a) To obtain complex versions of the law $z^a z^b = z^{a+b}$, consider elements

$$w_1 \in \mathrm{expset}(z, a)$$
$$w_2 \in \mathrm{expset}(z, b)$$
$$w_3 \in \mathrm{expset}(z, a+b).$$

Determine whether the equation $w_1 w_2 = w_3$ gives a true statement if prefixed as follows

(a.1) $(\exists\, w_1)\ (\exists\, w_2)\ (\exists\, w_3)\ \ w_1 w_2 = w_3.$
(a.2) $(\forall\, w_1)\ (\exists\, w_2)\ (\exists\, w_3)\ \ w_1 w_2 = w_3.$
(a.3) $(\exists\, w_1)\ (\forall\, w_2)\ (\exists\, w_3)\ \ w_1 w_2 = w_3.$
(a.4) $(\exists\, w_1)\ (\exists\, w_2)\ (\forall\, w_3)\ \ w_1 w_2 = w_3.$
(a.5) $(\forall\, w_1)\ (\forall\, w_2)\ (\exists\, w_3)\ \ w_1 w_2 = w_3.$

where w_1, w_2, w_2 are, in each case, restricted to the expset specified in the first display (the values of z, a and b being assumed given).

(b) To study similarly the law $z_1^a z_2^a = (z_1 z_2)^a$, answer the same question for the *same five quantifications*, where this time

$$w_1 \in \mathrm{expset}(z_1, a),$$
$$w_2 \in \mathrm{expset}(z_2, a),$$
$$w_3 \in \mathrm{expset}(z_1 z_2, a)$$

and where the relation being quantified is again $w_1 w_2 = w_3$. (Call the resulting laws (b.1), ... , (b.5).)

(c) To study the law $(z^a)^b = z^{ab}$, answer the same questions for

$$w_1 \in \mathrm{expset}(z, a)$$
$$w_2 \in \mathrm{expset}(w_1, b)$$
$$w_3 \in \mathrm{expset}(z, ab)$$

and, this time, the relation $w_2 = w_3$. (Call the resulting laws (c.1), ... , (c.5).)

Above we considered five quantifications in each case; obviously, we could get 8 quantifications by using all combinations of choices of $\forall$ and $\exists$. Moreover, there are 6 *orders* in which we can quantify 3 elements w_1, w_2, w_3 (remember that $(\forall\, x)(\exists\, y)$ is not the same as $(\exists\, y)(\forall\, x)$), making a total of 48 arrangements of the quantifiers – except in case (c), where the range of w_2 depends on w_1, so that the latter must be quantified before the former, and there are only 24 cases. (Certain quantifications are logically equivalent to certain others, reducing the number of logically distinct statements in each set, but we will not worry about this.) You might like to try

(a′), (b′), (c′): For each of the relations given in (a), (b), (c), find and prove some simple rule characterizing precisely those of the 48 or 24 quantifications that yield valid formulas.

In a different direction, you might explore

(a′′), (b′′), (c′′): For various cases that you found *not* to hold in general, study what values, if any, of the exponents a, b make them hold.

A case of (c′′) that has an interesting answer is the following: For what pairs of complex numbers (a, b) is the version of (c) with quantification $(\forall\, w_1)\ (\forall\, w_2)\ (\exists\, w_3)$ true? I.e., for what a and b is it true that for all z, every bth power of an ath power of z is an abth power of z?

Chapter 14, ‘‘Exercise 24’’: Describe the image of the quadrant $S = \{z \in \mathbb{C} \mid \mathrm{re}(z) \geq 0,\ \mathrm{im}(z) \geq 0\}$ under the map $f(z) = z^{1+i}$, evaluated with the help of the principal value of the logarithm. In particular, describe and sketch the images under f of the positive real and imaginary axes, noting how they behave as one approaches 0 and ∞, and also the images under f of the portions of the circles $|z| = c$ lying in the quadrant S.

Chapter 14, ''Exercise 25'': On p.282 the authors mention *doubly periodic* analytic functions. This exercise will indicate how such functions can be constructed; though I won't make you go through the details of the proof that the functions we will construct are in fact analytic.

Let us consider functions that have 1 and i as periods, i.e., which satisfy

$$f(z+1) = f(z) = f(z+i)$$

for all z at which f is defined. Let $\mathbb{Z}+i\mathbb{Z} = \{m+ni \mid m,n\in\mathbb{Z}\}$. We first show that we cannot have any interesting analytic functions of this sort without singularities.

(a) Suppose f is an analytic function on $\mathbb{C}$ that has 1 and i as periods, i.e., satisfies the equation displayed above. Show that f is constant. (Suggestion: Note why it must be bounded on the square $\{z \mid 0 \le \mathrm{re}(z) \le 1,\ 0 \le \mathrm{im}(z) \le 1\}$, then deduce from this that it is bounded everywhere.)

The mildest kind of nontrivial singularities a function can have are poles. Clearly, if a function with periods 1 and i has a pole at some point z_0, it will have poles at $z_0 + m + ni$ for all integers m and n. We shall now construct doubly periodic functions by taking $z_0 = 0$ and ''putting together'' such a set of poles. The method we shall use turns out to fail if the poles have too low order; it works starting with order 3. Namely,

(b) Let k be a positive integer and z a complex number not belonging to $\mathbb{Z}+i\mathbb{Z}$, and let us write

$$f_k(z) = \Sigma_{m+ni}\,(z-(m+ni))^{-k},$$

where the summation is over all $m+ni \in \mathbb{Z}+i\mathbb{Z}$, equivalently over all $m,\ n\in\mathbb{Z}$. Show that this sum converges absolutely if $k \ge 3$, but not if $k = 1$ or 2. Assuming $k \ge 3$, show that $f_k(z)$ has 1 and i as periods.

As I said, I will not make you go through a proof that $f_k(z)$ is differentiable. It comes down to showing that in such a sum, one can ''differentiate under the summation sign''. The proof is more or less like that of Theorem 4.14 (differentiation of power series).

A slightly more challenging version of this exercise is

(c) Suppose that α and β are nonzero complex numbers such that α/β is not real, and let $\alpha\mathbb{Z}+\beta\mathbb{Z} = \{m\alpha+n\beta \mid m,n\in\mathbb{Z}\}$. Show how to construct, similarly, functions analytic on $\mathbb{C}$ except for poles at the points of $\alpha\mathbb{Z}+\beta\mathbb{Z}$, and having α and β as periods. (Proving that the summations you define converge will be a little more complicated than before, because the absolute value of $z-(m\alpha+n\beta)$ is not as simple to write down as that of $z-(m+ni)$.)