EXERCISES IN SEMICLASSICAL ANALYSIS AT SNAP 2019, §7

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Exercise 7.1. This exercise finishes the details of the elliptic parametrix construction from the lecture. We assume that m_1, m_2 are order functions, $a(x,\xi;h) \in S(m_1)$, $p(x,\xi;h) \in S(m_2)$, and we have the following ellipticity condition: there exists $c > 0$ such that for all h

$$|p(x,\xi;h)| \geq c m_2(x,\xi) \quad \text{for all} \quad (x,\xi) \in \operatorname{supp} a(\bullet;h).$$

(a) Show that $a/p \in S(m_1/m_2)$. (Hint: prove first that every derivative $\partial^\alpha(p^{-1})$ is a linear combination of expressions of the form $p^{-\ell-1}\partial^{\alpha_1}p \cdots \partial^{\alpha_\ell}p$ where the multiindices $\alpha_1, \dots, \alpha_\ell$ add up to α.)

(b) Recall that $q_0 := a/p \in S(m_1/m_2)$ and $\operatorname{supp} q_0 \subset \operatorname{supp} a$. Recall from the Composition Theorem that

$$q_0 \# p = q_0 p - h r_1 + \mathcal{O}(h^2)_{S(m_1)}, \quad r_1 := i \sum_{k=1}^n (\partial_{\xi_k} q_0)(\partial_{x_k} p) \in S(m_1).$$

Put $q_1 := r_1/p$. Show that $q_1 \in S(m_1/m_2)$, $\operatorname{supp} q_1 \subset \operatorname{supp} a$, and

$$(q_0 + hq_1)\#p = a + \mathcal{O}(h^2)_{S(m_1)}.$$

(c) Iterating the argument in part (b), construct symbols $q_2, q_3, \dots \in S(m_1/m_2)$, $\operatorname{supp} q_j \subset \operatorname{supp} a$, such that for each k

$$(q_0 + hq_1 + \dots + h^{k-1}q_{k-1})\#p = a + \mathcal{O}(h^k)_{S(m_1)}.$$

(d) Using Borel's Theorem, choose

$$q \in S(m_1/m_2), \quad q \sim \sum_{j=0}^\infty h^j q_j.$$

Show that $q\#p = a + \mathcal{O}(h^\infty)_{S(m_1)}$.

Exercise 7.2. Assume that we are in the setting of Exercise 7.1 and $m_1 = 1$, $a = 1$. The elliptic parametrix construction gives two symbols $q, q' \in S(1/m_2)$ such that

$$1 = q\#p + \mathcal{O}(h^\infty)_{S(1)}, \quad 1 = p\#q' + \mathcal{O}(h^\infty)_{S(1)}.$$

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Show that $q = q' + \mathcal{O}(h^\infty)_{S(1/m_2)}$ and thus $q = p\#q + \mathcal{O}(h^\infty)_{S(1)}$. (Hint: compute the product $q\#p\#q'$.)

Exercise 7.3. Consider the Schrödinger operator on $\mathbb{R}^n$

$$P = -h^2\Delta + V(x)$$

where $V \in C^\infty(\mathbb{R}^n)$ satisfies the following assumptions for some $\ell > 0$:

- Bounded derivatives: $\partial^\alpha V(x) = \mathcal{O}(\langle x\rangle^\ell)$ for all α;
- Ellipticity at infinity: $V(x) \geq C^{-1}\langle x\rangle^\ell - C$ for some $C > 0$.

Assume that we are given a family of eigenfunctions:

$$(P - E_h)u_h = 0, \quad \|u_h\|_{L^2(\mathbb{R}^n)} = 1, \quad E_h \xrightarrow{h\to 0} E \in \mathbb{R}.$$

Define the classically allowed region

$$\Omega_E := \{x \in \mathbb{R}^n \mid V(x) \leq E\}$$

and fix an open set $U \supset \Omega_E$. Using the elliptic estimate, show that

$$\|u_h\|_{L^2(\mathbb{R}^n\setminus U)} = \mathcal{O}(h^\infty) \quad \text{as} \quad h \to 0.$$

(Hint: take $a(x,\xi) = \chi(x)$ where $\chi \in C^\infty(\mathbb{R}^n)$, $\operatorname{supp}\chi \cap \Omega_E = \emptyset$, and $\chi = 1$ on $\mathbb{R}^n \setminus U$.)

Exercise 7.4.* This advanced exercise provides estimates which may be used to establish functional calculus for pseudodifferential operators in the course on eigenfunctions. Assume that

$$P = \operatorname{Op}_h(p), \quad p \in S(m)$$

where m is an order function such that $m(x,\xi) \to \infty$ as $(x,\xi) \to \infty$, and p is real-valued and satisfies the following ellipticity at infinity assumption: there exists a constant C such that for all (x,ξ)

$$p(x,\xi;h) \geq \frac{m(x,\xi)}{C} - C.$$

Assume that $z \in \mathbb{C}$ varies in a compact set.

(a) Fix $z \in \mathbb{C} \setminus \mathbb{R}$. Show that the symbol $p - z$ is elliptic everywhere. Define the symbols $q_0, q_1, \dots \in S(1/m)$ using Exercise 7.1(c) such that for all k

$$(q_0 + hq_1 + \dots + h^{k-1}q_{k-1})\#(p - z) = 1 + \mathcal{O}(h^k)_{S(1)}. \tag{7.1}$$

(b) We now allow z to approach the real line. Show the derivative bounds for each α, j, and (x,ξ)

$$|\partial^\alpha q_j(x,\xi,z;h)| \leq \frac{C_{\alpha,j}}{|\operatorname{Im} z|^{2j+1+|\alpha|}\cdot m(x,\xi)}.$$

(Hint: first of all, for any symbol q write the composition formula in the form

$$q\#(p-z) \sim q(p-z) - \sum_{j=1}^{\infty} h^j L_j q$$

where each L_j is a differential operator of order j with z-independent coefficients which are in $S(m)$. Now, to get (7.1) we put

$$q_0 := \frac{1}{p-z}; \quad q_k := \frac{1}{p-z} \sum_{j=1}^{k} L_j q_{k-j}, \quad k \geq 1.$$

From here obtain the formula

$$q_k = \sum_{r=1}^{2k+1} \frac{\tilde{q}_{kr}}{(p-z)^r}$$

where $\tilde{q}_{kr} \in S(m^{r-1})$ are z-independent, and deduce the needed estimate.)

(c) Using L^2 boundedness (see formula (4.5.10) in Zworski's book) and analyzing the remainder in (7.1) similarly to part (b) of this exercise, show that there exists some M_k depending only on n, k such that

$$Q(z)(P-z) = I + \mathcal{O}\left(\frac{h^k}{|\operatorname{Im} z|^{M_k}}\right)_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}$$
$$\text{where} \quad Q(z) := \operatorname{Op}_h(q_0 + hq_1 + \cdots + h^{k-1}q_k).$$

(d) Show that the above statements are still true when p is not real-valued but $\operatorname{Im} p = \mathcal{O}(h)_{S(m)}$, by dividing by $\operatorname{Re} p - z$ instead of $p - z$ and putting the imaginary part of p into the next step of the iteration.