EXERCISES IN SEMICLASSICAL ANALYSIS AT SNAP 2019, §10

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Exercise 10.1. Assume that $u \in L^2(\mathbb{R}^n)$ is h-independent. Define the nonsemiclassical wavefront set $\mathrm{WF}(u) \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ as follows: a point $(x_0, \xi_0) \in \mathbb{R}^{2n}$, $\xi_0 \neq 0$, does *not* lie in $\mathrm{WF}(u)$ if there exists $\chi \in C_c^\infty(\mathbb{R}^n)$, $\chi(x_0) \neq 0$, and a conic neighborhood V of ξ_0 such that $\hat{u}(\xi) = \mathcal{O}(\langle\xi\rangle^{-\infty})$ for $\xi \in V$. Using the Fourier transform definition of the semiclassical wavefront set $\mathrm{WF}_h(u)$, show that

$$\mathrm{WF}_h(u) = \big(\operatorname{supp} u \times \{0\}\big) \cup \mathrm{WF}(u).$$

Exercise 10.2. This exercise explores basic properties of Lagrangian submanifolds and phase functions, in preparation for Wednesday's distinguished lecture. For simplicity we restrict ourselves to the setting of $\mathbb{R}^n$. An n-dimensional embedded submanifold $\Lambda \subset \mathbb{R}^{2n}$ is called *Lagrangian* if the pullback of the symplectic form $\omega = \sum_{j=1}^n d\xi_j \wedge dx_j$ to Λ is equal to 0.

(a) Assume that $U \subset \mathbb{R}^n$ is an open set and $\Phi \in C^\infty(U; \mathbb{R})$. Show that the graph of the gradient of Φ

$$\Lambda_\Phi = \{(x, d\Phi(x)) \mid x \in U\} \tag{10.1}$$

is a Lagrangian submanifold. Conversely, show that if Λ is a Lagrangian submanifold, $(x_0, \xi_0) \in \Lambda$, and $T_{(x_0,\xi_0)}\Lambda$ projects isomorphically onto the x coordinates, then Λ has the form (10.1) in a neighborhood of (x_0, ξ_0). (Hint: use that $\omega = d\alpha$ where $\alpha = \sum_{j=1}^n \xi_j\, dx_j$; for Λ_Φ given by (10.1) we have $\alpha|_{\Lambda_\Phi} = d\Phi$.)

(b) Now assume that Φ depends on additional variables $\theta \in \mathbb{R}^k$, namely $\Phi(x, \theta) \in C^\infty(U; \mathbb{R})$ where $U \subset \mathbb{R}^n_x \times \mathbb{R}^k_\theta$ is open. Define the *critical set*

$$\mathcal{C}_\Phi := \{(x, \theta) \in U \mid \partial_\theta \Phi(x, \theta) = 0\}$$

and assume that $d(\partial_{\theta_1}\Phi), \dots, d(\partial_{\theta_k}\Phi)$ are linearly independent at each point of $\mathcal{C}_\Phi$. Assume moreover that the map

$$j_\Phi : \mathcal{C}_\Phi \to \mathbb{R}^{2n}, \quad (x, \theta) \mapsto (x, \partial_x \Phi(x, \theta))$$

is an embedding. Show that the image

$$\Lambda_\Phi = j_\Phi(\mathcal{C}_\Phi) = \{(x, \partial_x \Phi(x, \theta)) \mid \partial_\theta \Phi(x, \theta) = 0\}$$

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is a Lagrangian submanifold. (Hint: show that $j_\Phi^*\alpha = d\Phi$.) We say that Λ_Φ is the *Lagrangian manifold generated by* Φ.

(c) Assume that Λ is a Lagrangian manifold, $(x_0, \xi_0) \in \Lambda$, and $T_{(x_0,\xi_0)}\Lambda$ projects isomorphically onto the ξ coordinates. Show that a neighborhood of (x_0, ξ_0) in Λ is generated by a phase function

$$\Phi(x, \theta) = \langle x, \theta\rangle - F(\theta), \quad \theta \in \mathbb{R}^n, \tag{10.2}$$

where F is some function on a neighborhood of ξ_0. (Hint: use that $\omega = -d\beta$ where $\beta = \sum_j x_j\, d\xi_j$; Λ is generated by $\Phi(x,\theta)$ of the form (10.2) if and only if $\beta|_\Lambda = dF$.)

Exercise 10.3. Assume that $\Phi(x, \theta)$ is a phase function satisfying the assumptions in Exercise 10.2(b) and Λ is the Lagrangian manifold generated by Φ. Assume next that Λ is also generated by some function $\Psi(x)$ in the sense of (10.1). Consider a family of functions of the form

$$u(x; h) = (2\pi h)^{-\frac{k}{2}} \int_{\mathbb{R}^k} e^{\frac{i}{h}\Phi(x,\theta)} a(x, \theta)\, d\theta \tag{10.3}$$

where a is a C_c^∞ function on the domain of Φ. Using the method of stationary phase, show that we can also write

$$u(x; h) = e^{\frac{i}{h}\Psi(x)} b(x; h) + \mathcal{O}(h^\infty)_{C_c^\infty(\mathbb{R}^n)}$$

for some b supported in an h-independent compact set inside the domain of Ψ, and with all derivatives bounded uniformly in h.

(This exercise shows in a special case that the class of functions of the form (10.3) does not depend on the phase function generating Λ. Functions in this class are called *semiclassical Lagrangian distributions* associated to Λ and are a key concept in semiclassical analysis.)

Exercise 10.4.* Assume that M is a compact manifold and $u = u_h \in \mathcal{D}'(M)$ is a family of distributions such that $\|u_h\|_{H_h^{-N}} \le Ch^{-N}$ for some C, N.

(a) Let $(x_0, \xi_0) \in T^*M$. Show that the following conditions are equivalent:

1. There exists $A \in \Psi_h^k(T^*M)$ such that $|\sigma_h(A)(x_0, \xi_0)| \ge c > 0$ for some h-independent constant c and $Au_h = \mathcal{O}(h^\infty)_{C^\infty}$;
2. There exists a neighborhood U of (x_0, ξ_0) such that for each $B \in \Psi_h^{\mathrm{comp}}(M)$ such that $\mathrm{WF}_h(B) \subset U$, we have $Bu_h = \mathcal{O}(h^\infty)_{C^\infty}$. (Here $\Psi_h^{\mathrm{comp}}(M), \mathrm{WF}_h(B)$ are defined in Exercise 9.3.)

(Hint: to show that (1) implies (2), use elliptic estimate.) If the above conditions hold, we say (x_0, ξ_0) does not lie in $\mathrm{WF}_h(u)$; this defines a closed subset $\mathrm{WF}_h(u) \subset T^*M$.

(b) Show that for any $A \in \Psi_h^{\mathrm{comp}}(M)$, $\mathrm{WF}_h(Au) \subset \mathrm{WF}_h(A) \cap \mathrm{WF}_h(u)$.

(c) Assume that g is a Riemannian metric on M and

$$(-h^2\Delta_g - E_h)u_h = 0, \quad E_h \to 1 \quad \text{as } h \to 0.$$

Show that $\mathrm{WF}_h(u_h) \subset S^*M = \{(x,\xi) \in T^*M \colon |\xi|_{g(x)} = 1\}$.