CLASSICAL/QUANTUM CORRESPONDENCE CHART, SNAP 2019

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Disclaimer: this is a quick reference guide omitting many necessary assumptions. See the book of Zworski for precise statements.

Classical	**Quantum**
Symbol/classical observable $a(x,\xi)$, $(x,\xi) \in \mathbb{R}^{2n}$	Pseudodifferential operator/quantum observable $\mathrm{Op}_h(a) = a(x, \frac{h}{i}\partial_x)$
$a = 1$ $a = x_j$ $a = \xi_j$	$\mathrm{Op}_h(a) = I$ $\mathrm{Op}_h(a) = x_j$ $\mathrm{Op}_h(a) = \frac{h}{i}\partial_{x_j}$
Polynomial symbol $a(x,\xi) = \sum_\alpha a_\alpha(x)\xi^\alpha$	Differential operator $\mathrm{Op}_h(a) = \sum_\alpha a_\alpha(x)(hD_x)^\alpha$
Symbol $a \in C^\infty(T^*M)$ on the cotangent bundle of a manifold M	Operator $\mathrm{Op}_h(a)$ on functions on M
Length-square of covectors w.r.t. Riemannian metric, $p(x,\xi) = \lvert\xi\rvert^2_{g(x)^{-1}}$	Laplace–Beltrami operator $-h^2\Delta_g = \mathrm{Op}_h(p) + \mathcal{O}(h)$
Product of symbols	Composition of operators
Product Rule: $\mathrm{Op}_h(a)\,\mathrm{Op}_h(b) = \mathrm{Op}_h(ab) + \mathcal{O}(h)$	
Poisson bracket of symbols: $\{a,b\} = \sum_j(\partial_{\xi_j}a \cdot \partial_{x_j}b - \partial_{x_j}a \cdot \partial_{\xi_j}b)$	Commutator of operators: $[A,B] = AB - BA$
Commutator Rule: $[\mathrm{Op}_h(a), \mathrm{Op}_h(b)] = -ih\,\mathrm{Op}_h(\{a,b\}) + \mathcal{O}(h^2)$	
Complex conjugate of a symbol	Adjoint of an operator
Adjoint Rule: $\mathrm{Op}_h(a)^* = \mathrm{Op}_h(\overline{a}) + \mathcal{O}(h)$	
Bounded symbol: $\sup\lvert a\rvert \le \gamma$	Bounded operator on $L^2(\mathbb{R}^n)$: $\lVert \mathrm{Op}_h(a)\rVert \le \gamma + \mathcal{O}(h)$
Symbol converging to 0: $\lim_{(x,\xi)\to\infty} a(x,\xi) = 0$	Compact operator $\mathrm{Op}_h(a)$ on $L^2(\mathbb{R}^n)$
Nonnegative bounded symbol: $\sup\lvert a\rvert \le C$, $a \ge 0$	Sharp Gårding inequality: $\mathrm{Re}\langle \mathrm{Op}_h(a)u, u\rangle_{L^2} \ge -C'h\lVert u\rVert^2_{L^2}$
Classical evolution: $e^{tH_p} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, $H_p a = \{p, a\}$, p real-valued	Quantum evolution: $U(t) = \exp(-itP/h)$, $P = \mathrm{Op}_h(p) + \mathcal{O}(h)$, $P^* = P$
Egorov's Theorem: $U(-t)\,\mathrm{Op}_h(a)U(t) = \mathrm{Op}_h(a \circ e^{tH_p}) + \mathcal{O}(h)$	
Essential support: $(x_0,\xi_0) \notin \text{ess-supp}(a) \;\Leftrightarrow\; \exists U(x_0,\xi_0) : a\rvert_U = \mathcal{O}(h^\infty)$	Wavefront set of a pseudodifferential operator, $\mathrm{WF}_h(\mathrm{Op}_h(a))$
Value of an observable at a point, $a(x_0,\xi_0)$	Average value of an observable on a wave function, $\langle \mathrm{Op}_h(a)u, u\rangle_{L^2}$

<table>
<tr><td>Integral of a classical observable</td><td>Trace of a quantum observable</td></tr>
<tr><td colspan="2">Trace formula: $\operatorname{tr} \mathrm{Op}_h(a) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} a(x,\xi)\, dx d\xi + \mathcal{O}(h^{1-n})$</td></tr>
<tr><td>Composition with a smooth function: $f(p)$, p real-valued</td><td>Functional calculus: $f(P)$, $P = \mathrm{Op}_h(p) + \mathcal{O}(h)$, $P^* = P$</td></tr>
<tr><td colspan="2">Semiclassical functional calculus: $f(P) = \mathrm{Op}_h(f(p)) + \mathcal{O}(h)$</td></tr>
<tr><td>Volume of preimages $p^{-1}([\alpha,\beta])$, p real-valued</td><td>Asymptotics of spectrum $\mathrm{Spec}(P)$, $P = \mathrm{Op}_h(p) + \mathcal{O}(h)$, $P^* = P$</td></tr>
<tr><td colspan="2">Weyl Law: $\#(\mathrm{Spec}(P) \cap [\alpha,\beta]) = (2\pi h)^{-n} \operatorname{vol}(p^{-1}([\alpha,\beta])) + o(h^{-n})$</td></tr>
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