OVERVIEW OF CALCULUS ON MANIFOLDS

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Here is a brief overview of semiclassical pseudodifferential calculus on a manifold M. See §E.1 in the Dyatlov–Zworski book for details. (Note: the calculus here corresponds to symbols in $S^k_{1,0}$ in the notation of that book.)

- Distributions and general operators:
  - $\mathcal{D}'(M)$ distributions on M, $\mathcal{E}'(M)$ compactly supported distributions;
  - an operator $A : C_c^\infty(M) \to \mathcal{D}'(M)$ is called compactly supported, if its Schwartz kernel is compactly supported, i.e. $A = \chi A \chi$ for some $\chi \in C_c^\infty(M)$; in this case A maps $C^\infty(M) \to \mathcal{E}'(M)$;
  - $A : C_c^\infty(M) \to \mathcal{D}'(M)$ is called properly supported, if for each $\chi \in C_c^\infty(M)$, the operators χA and $A\chi$ are compactly supported; in this case A maps $C_c^\infty(M) \to \mathcal{E}'(M)$ and $C^\infty(M) \to \mathcal{D}'(M)$;
- Pseudodifferential operators:
  - $\Psi_h^k(M)$, $k \in \mathbb{R}$, the class of semiclassical pseudodifferential operators of order k on M;
  - all elements of $\Psi_h^k(M)$ map $C_c^\infty(M) \to C^\infty(M)$ and $\mathcal{E}'(M) \to \mathcal{D}'(M)$;
  - properly supported operators in $\Psi_h^k(M)$ map $C_c^\infty(M) \to C_c^\infty(M)$, $C^\infty(M) \to C^\infty(M)$, $\mathcal{E}'(M) \to \mathcal{E}'(M)$, $\mathcal{D}'(M) \to \mathcal{D}'(M)$, and thus can be multiplied with other operators;
  - $h^\infty\Psi^{-\infty} = \bigcap_k \Psi_h^k(M)$, the class of rapidly decaying smoothing operators on M: integral operators of the form $u \mapsto \int_M K(x,y;h)u(y)\,dy$ where $K \in C^\infty(M \times M)$ and each C^∞ seminorm of K is $\mathcal{O}(h^\infty)$; such operators map $\mathcal{E}'(M) \to C^\infty(M)$;
- Symbols and quantization:
  - $S^k(T^*M)$ the space of h-dependent Kohn–Nirenberg symbols of order k on the cotangent bundle T^*M (with no uniformity in x imposed when M is noncompact);
  - $\sigma_h^k : \Psi_h^k(M) \to S^k(T^*M)/hS^{k-1}(T^*M)$ the principal symbol map (we usually suppress k in notation, simply writing σ_h);
  - the kernel of σ_h^k is equal to $h\Psi_h^{k-1}(M)$;

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  - $\mathrm{Op}_h : S^k(T^*M) \to \Psi_h^k(M)$ a noncanonical quantization map;
  - $\sigma_h^k(\mathrm{Op}_h(a)) = a \mod hS^{k-1}(T^*M)$ for all $a \in S^k(T^*M)$;
  - for any $a \in S^k(T^*M)$, $\mathrm{Op}_h(a)$ is properly supported, and if a is compactly supported in x, then $\mathrm{Op}_h(a)$ is compactly supported;
  - we can choose Op_h so that $\mathrm{Op}_h(1) = I$;
  - for each $A \in \Psi_h^k(M)$ there exists $a \in S^k(T^*M)$ such that $A = \mathrm{Op}_h(a) + \mathcal{O}(h^\infty)_{\Psi^{-\infty}}$;
- Algebraic properties:
  - Product Rule: if $A \in \Psi_h^k(M)$, $B \in \Psi_h^\ell(M)$, and at least one of these operators is properly supported, then $AB \in \Psi_h^{k+\ell}(M)$, and $\sigma_h^{k+\ell}(AB) = \sigma_h^k(A)\sigma_h^\ell(B)$; equivalently, if $a \in S^k(T^*M)$, $b \in S^\ell(T^*M)$, then
$$\mathrm{Op}_h(a)\,\mathrm{Op}_h(b) = \mathrm{Op}_h(ab) + \mathcal{O}(h)_{\Psi^{k+\ell-1}(M)};$$
  - Commutator Rule: under the assumptions of the Product Rule we have $\sigma_h^{k+\ell-1}(h^{-1}[A,B]) = -i\{\sigma_h^k(A), \sigma_h^\ell(B)\}$; equivalently,
$$[\mathrm{Op}_h(a), \mathrm{Op}_h(b)] = -ih\,\mathrm{Op}_h(\{a,b\}) + \mathcal{O}(h^2)_{\Psi^{k+\ell-2}(M)};$$
  - Adjoint Rule: if we fix any smooth density on M (to fix an inner product on $L^2(M)$ and thus be able to take adjoints of operators), and $A \in \Psi_h^k(M)$, then $A^* \in \Psi_h^k(M)$ and $\sigma_h^k(A^*) = \overline{\sigma_h^k(A)}$; equivalently, if $a \in S^k(T^*M)$, then
$$\mathrm{Op}_h(a)^* = \mathrm{Op}_h(\overline{a}) + \mathcal{O}(h)_{\Psi_h^{k-1}(M)};$$
- Wavefront sets:
  - For $A \in \Psi_h^k(M)$, its wavefront set is $\mathrm{WF}_h(A) \subset \overline{T}^*M$, with $\overline{T}^*M$ the fiber-radial compactification of T^*M;
  - $\mathrm{WF}_h(A) = \emptyset \iff A = \mathcal{O}(h^\infty)_{\Psi^{-\infty}}$;
  - if $a(x,\xi;h)$ is supported in an h-independent set K, then $\mathrm{WF}_h(\mathrm{Op}_h(a)) \subset K$;
  - $\mathrm{WF}_h(A+B) \subset \mathrm{WF}_h(A) \cup \mathrm{WF}_h(B)$;
  - $\mathrm{WF}_h(AB) \subset \mathrm{WF}_h(A) \cap \mathrm{WF}_h(B)$;
  - $\mathrm{WF}_h(A^*) = \mathrm{WF}_h(A)$;
- L^2 theory, assuming for simplicity M is compact:
  - One can define semiclassical Sobolev spaces $H_h^s(M)$, with a noncanonical h-dependent norm, and $H_h^0(M) = L^2(M)$;
  - if $A \in \Psi_h^k(M)$, then $A : H_h^s(M) \to H_h^{s-k}(M)$, with the norm bounded uniformly in h;
  - $H_h^s(M)$ embeds compactly into $H_h^t(M)$ for $s > t$;

– Sharp Gårding inequality: if $a \in S^k(T^*M)$ and $\operatorname{Re} a \geq 0$ everywhere, then for each $u \in H_h^{\frac{k}{2}}(M)$ and h

$$\operatorname{Re}\langle \operatorname{Op}_h(a)u, u\rangle_{L^2} \geq -Ch\|u\|_{H_h^{\frac{k-1}{2}}(M)}$$

where C is some constant depending on a.