§6. L^2 THEORY

§6.1. Boundedness

Recall from §5 the symbol class

$$S(m) = \{ a(x,\xi;h) : |\partial^\alpha_{(x,\xi)} a(x,\xi;h)| \leq C_\alpha m(x,\xi) \}$$

where m is an order function

Theorem (Calderón–Vaillancourt)

Assume that $a \in S(1)$. Then $\forall h$, $Op_h(a)$ defines a bounded operator on $L^2(\mathbb{R}^n)$ for all $h \in (0,1]$ and $\exists C = C(a)$ such that

$$\| Op_h(a) \|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq C$$

Proof We only show the uniform norm bound for $a \in \mathcal{S}(\mathbb{R}^{2n})$. For the harder case of $a \in S(1)$, see Zworski's book, Theorem 4.23

1. We use the following general fact, known as Schur's inequality:

if $Au(x) = \int_{\mathbb{R}^n} K_A(x,y) u(y)\, dy$, $K_A \in \mathcal{S}(\mathbb{R}^{2n})$,

and $C_1 := \sup_x \int_{\mathbb{R}^n} |K_A(x,y)|\, dy$,

$C_2 := \sup_y \int_{\mathbb{R}^n} |K_A(x,y)|\, dx$,

then $\|A\|_{L^2 \to L^2} \leq \sqrt{C_1 \cdot C_2}$

To prove Schur's inequality, take $u \in L^2(\mathbb{R}^n)$ and write

$$\|Au\|_{L^2}^2 = \int_{\mathbb{R}^{3n}} K_A(x,y)\,\overline{K_A(x,z)}\,u(y)\,\overline{u(z)}\,dx\,dy\,dz$$

$$\le \int_{\mathbb{R}^{3n}} |K_A(x,y)|\cdot|K_A(x,z)|\cdot\frac{1}{2}\left(|u(y)|^2+|u(z)|^2\right)dx\,dy\,dz$$

Now we bound

only instance of z

$$\int_{\mathbb{R}^{3n}} |K_A(x,y)|\cdot|K_A(x,z)|\cdot|u(y)|^2\,dx\,dy\,dz$$

$$\le C_1 \int_{\mathbb{R}^{2n}} |K_A(x,y)|\cdot|u(y)|^2\,dx\,dy$$

only instance of x

$$\le C_1 C_2 \int_{\mathbb{R}^n} |u(y)|^2\,dy = C_1 C_2 \|u\|_{L^2}^2$$

We handle $\int \cdots |u(z)|^2 \cdots$ similarly, giving Schur's inequality.

2. Now take $A = Op_h(a)$, $a \in \mathcal{S}(\mathbb{R}^{2n})$.

Then $K_A(x,y) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y,\xi\rangle} a(x,\xi)\,d\xi$

$$= (2\pi h)^{-n}\,\tilde{F}a\left(x, \frac{x-y}{h}\right) \quad \text{where}$$

$\tilde{F}a(x,z) = \int_{\mathbb{R}^n} e^{i\langle z,\xi\rangle} a(x,\xi)\,d\xi$ ← partial Fourier transform

Since $a \in \mathcal{S}(\mathbb{R}^n)$, we have $\tilde{F}a \in \mathcal{S}(\mathbb{R}^n)$ too.

We have $\int_{\mathbb{R}^n} |K_A(x,y)|\,dy = (2\pi)^{-n} \int_{\mathbb{R}^n} |\tilde{F}a(x,z)|\,dz \le C_1$

And $\int_{\mathbb{R}^n} |K_A(x,y)|\,dx = (2\pi)^{-n} \int_{\mathbb{R}^n} |\tilde{F}a(y+hz,z)|\,dz \le C_1$. □

§6.2. Compactness

Definition A bounded operator $A : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is called compact, if $\forall$ bounded sequence $u_j \in L^2(\mathbb{R}^n)$, the sequence Au_j has a convergent subsequence.

Basic property: if A_k are compact and
$$\|A - A_k\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \xrightarrow[k \to \infty]{} 0$$
then A is also compact.

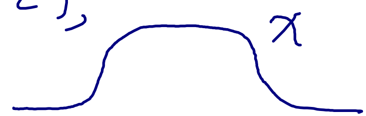
Theorem
Assume $a \in S(m)$ where m is an order function and $m(x, \xi) \to 0$ as $(x, \xi) \to \infty$.
Then $Op_h(a) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is compact $\forall h$.

Proof We only give a sketch; see the book of Zworski, §4.6 for details.

1 Assume first that $a \in \mathcal{S}(\mathbb{R}^n)$. Then $Op_h(a) : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. So if u_j is bounded in L^2, then $Op_h(a)u_j$ is bounded in $\mathcal{S}(\mathbb{R}^n)$.
In particular, $|\partial_x^\alpha Op_h(a)u_j(x)| \le C_N \langle x\rangle^{-N}$ $\forall \alpha, N$.
Then $Op_h(a)u_j$ has a subsequence converging in $L^2(\mathbb{R}^n)$ by the Arzela-Ascoli theorem...

2. We now consider the general case.

Take $\chi \in C_c^\infty(\mathbb{R}^{2n})$: $\operatorname{supp} \chi \subset B(0,2)$,
$\chi = 1$ on $B(0,1)$



Take large R and put
$$a_R(w) := \chi\left(\frac{w}{R}\right) a(w), \quad w = (x,\xi)$$

Then $a_R \in \mathscr{S}(\mathbb{R}^{2n}) \Rightarrow Op_h(a_R)$ is compact by Step 1.

Now, each $S(1)$ seminorm of $a - a_R$ goes to 0 as $R \to \infty$ (see exercises; here we use that $m(x,\xi) \to 0$ as $(x,\xi) \to \infty$).

Thus by the L^2 boundedness statement,
$$\|Op_h(a) - Op_h(a_R)\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \xrightarrow[R\to\infty]{} 0.$$

By the basic property above, $Op_h(a)$ is compact.

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§6.3. Sharp Gårding inequality

Theorem Assume that $a \in S(1)$ and
$$a(x,\xi) \geq 0 \text{ for all } (x,\xi).$$
Then $\exists\, C = C(a)$: $\forall h \in (0,1]$, $\forall u \in L^2(\mathbb{R}^n)$
$$\langle Op_h(a) u, u \rangle_{L^2(\mathbb{R}^n)} \geq -Ch \|u\|^2_{L^2(\mathbb{R}^n)}.$$

Proof We only do the simple special case when $\boxed{a = b^2}$ for some real-valued $b \in S(1)$.

For the general case, see the book of Zworski, Theorem 4.32

By the Adjoint Rule + L^2 boundedness

$$Op_h(b)^* = Op_h(b) + O(h)_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)}.$$

Then by the Product Rule + L^2 boundedness

$$Op_h(b)^* Op_h(b) = Op_h(\underbrace{b^2}_{\shortparallel\, a}) + O(h)_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)}$$

Thus

$$\langle Op_h(a)u, u\rangle = \langle Op_h(b)^* Op_h(b)u, u\rangle + O(h)\|u\|^2_{L^2(\mathbb{R}^n)}$$

$$= \underbrace{\|Op_h(b)u\|^2_{L^2(\mathbb{R}^n)}}_{\geq 0} + O(h)\|u\|^2_{L^2(\mathbb{R}^n)}.$$

$\square$