SEMICLASSICAL ANALYSIS

LECTURES 1–7: SEMYON DYATLOV

WEBSITE: http://math.mit.edu/~dyatlov/semisnap/

- Lecture notes
- Exercises

Books: [ZW] Zworski, "Semiclassical Analysis", AMS, 2012

[DZ] Dyatlov-Zworski, "Mathematical Theory of Scattering Resonances", AMS, 2019
http://math.mit.edu/~dyatlov/res/,
Appendix E

§1. A motivating example

Semiclassical analysis is a theory underlying

- classical/quantum correspondence
- particle/wave duality
- geometric optics approximation ("light travels in straight lines")
- semiclassical approximation in quantum mechanics

Closely related to Microlocal Analysis: a different flavor of the same theory (no h)

Here are some aspects of classical/quantum correspondence (more on the website):
using 1D for simplicity

CLASSICAL MECHANICS	QUANTUM MECHANICS
Hamiltonian $p(x,\xi)$, $x,\xi \in \mathbb{R}$; $p: \mathbb{R}^2 \to \mathbb{R}$ "Total energy of the classical system" x = position of the particle ξ = momentum	Semiclassical differential operator/<u>quantization</u> of p $P = p(x, hD_x)$ <u>NOTATION</u>: $D_x = \frac{1}{i}\partial_x = -i\partial_x$

Role of $h>0$: effective wavelength

h^{-1}: effective frequency

Note: if f oscillates at frequency $\sim h^{-1}$
then $hD_x f \sim$ same size as f

e.g. $f(x) = e^{\frac{ix}{h}}$, $hD_x f = e^{\frac{ix}{h}}$.

We take $\underline{\underline{h \to 0}}$.

HAMILTONIAN FLOW	SCHRÖDINGER PROPAGATOR
$e^{tH_p}: \mathbb{R}^2 \to \mathbb{R}^2$ $e^{tH_p}(x_0,\xi_0) = (x(t), \xi(t))$ $x(0) = x_0$, $\xi(0) = \xi_0$ Hamilton's $\begin{cases} \dot{x}(t) = \partial_\xi p(x(t),\xi(t)) \\ \dot{\xi}(t) = -\partial_x p(x(t),\xi(t)) \end{cases}$ ← <u>ordinary</u> differential equation "Evolution of classical particle"	$e^{-itP/h}: L^2(\mathbb{R}) \circlearrowleft$ $e^{-itP/h} u_0 = u(t)$ $\begin{cases} hD_t u = -Pu \\ u(0) = u_0 \end{cases}$ ← Schrödinger's <u>partial</u> differential equation "Evolution of a quantum particle" $u(t)$ = wave function at time t

Now we present a <u>basic example</u>:

free particle on $\mathbb{R}$

CLASSICAL	QUANTUM
$p(x,\xi) = \xi^2$ (kinetic energy)	$P = (hD_x)^2 = -h^2\partial_x^2$
$\begin{cases} \dot{x}(t) = 2\xi(t) \\ \dot{\xi}(t) = 0 \end{cases}$ $e^{tH_p}(x_0,\xi_0) = (x_0 + 2t\xi_0, \xi_0)$ "Newton's I-st Law"	$-ih\partial_t u(t,x) = h^2\partial_x^2 u(t,x)$ Schrödinger's Equation (S.E.)

Let's take initial data

$$u_0(x) = \chi\left(\frac{x}{h}\right) \quad \text{where} \quad \chi \in \mathcal{S}(\mathbb{R})$$

<u>NOTATION</u>: $\mathcal{S}$ = Schwartz functions

(we should normalize so that $\|u_0\|_{L^2} = 1$ but it won't matter below]

Originally the particle is localized near $x=0$.

What happens with time?

MOVIE TIME

(see website)

The movie looked curious...
It appeared that

$$\boxed{u(t,x) = 0 \quad \text{unless} \quad x \in [-2t, 4t]} \quad (SPT_u)$$

(actually we'll set $u(t,x) = O(h^\infty)$ there)

Turns out $\underbrace{\text{I took}}_{\text{Why so?}}$ a special χ, with

$$\boxed{\operatorname{supp} \hat{\chi} \subset [-1, 2]} \quad (SPT\hat{\chi})$$

Let's write $u(t,x)$ in terms of $\hat{\chi}$:

$$u_0(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{\frac{ix\xi}{h}} \hat{\chi}(\xi)\, d\xi \quad \text{(Fourier Inversion Formula)}$$

$$\Downarrow$$

$$u(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{\frac{i}{h}(x\xi - t\xi^2)} \hat{\chi}(\xi)\, d\xi \qquad (*)$$

(exercise: check that $(*)$ satisfies (S.E.))

To show (SPT_u) will use

<u>Theorem [Method of nonstationary phase]</u> Assume

$$I = I(h) = \int_U e^{\frac{i\varphi(y)}{h}} a(y)\, dy \quad \text{where:}$$

- $U \subset \mathbb{R}^n$ open set
- $\varphi \in C^\infty(U; \mathbb{R})$ (<u>phase function</u>)
- $a \in C^\infty(U)$ $(= C^\infty(U; \mathbb{C}))$ (<u>amplitude</u>)

Assume φ has no critical points (a.k.a. stationary points) on supp a:

$$d\varphi \neq 0 \quad \text{on} \quad \text{supp}\, a.$$

Then $I(h) = O(h^\infty)$.

__NOTATION:__ $O(h^\infty)$ means that

$$\forall N \; \exists C_N \; \forall h \in (0,1] : \; |I(h)| \leq C_N h^N$$

[A proof will be given later, in §2]

Now we can get a weaker (but correct!) form of (SPT_u):

__Proposition__ Assume ($SPT_{\hat{\chi}}$). Fix $t \geq 0$ and $x \notin [-2t, 4t]$. Then $u(t,x) = O(h^\infty)$.

__Proof__ By (*) write

$$u(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{\frac{i}{h}\varphi(\xi)} a(\xi)\, d\xi \quad \text{where}$$

$$\varphi(\xi) = x\xi - t\xi^2, \quad a(\xi) = \hat{\chi}(\xi).$$

Critical points of φ, i.e. $\cancel{\partial_\xi \varphi = 0}$, : $\partial_\xi \varphi = 0$:

$$0 = \partial_\xi \varphi = x - 2t\xi \;\Rightarrow\; x = 2t\xi.$$

Does not happen ~~for~~ $\xi \in \text{supp}\, a \overset{(SPT_{\hat{\chi}})}{\subset} [-1, 2]$.

By Method of Nonstationary Phase, $u(t,x) = O(h^\infty)$.

$\square$

The initial data u_0 was localized
in position (to $x \approx 0$)
& in frequency (to $h\xi \in [1,2]$).

But ~~the~~ the last proposition only gives localization of $u(t,x)$ in position.

We now get localization for $u(t,x)$ in position & frequency (also known as microlocalization) using the concept of wavefront set!

Wavefront sets

Semiclassical Fourier transform:

Notation: $C_c^\infty = C^\infty$ compactly supported

$\mathcal{F}_h : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ unitary,

$$\mathcal{F}_h f(\xi) = (2\pi h)^{-\frac{n}{2}} \hat{f}(\xi/h) = (2\pi h)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{ix\xi}{h}} f(x)\,dx$$

Definition Assume $u = u_h \in L^2(\mathbb{R}^n)$ family of functions indexed by $h \to 0$.

Define $WF_h(u) \subset \mathbb{R}^{2n}$ as follows:
for $(x_0, \xi_0) \in \mathbb{R}^{2n}$, $(x_0,\xi_0) \notin WF_h(u)$ iff
$\exists \chi \in C_c^\infty(\mathbb{R}^n)$, $\chi(x_0) \neq 0$,
$\exists$ neighborhood $V \subset \mathbb{R}^n$ of ξ_0:

$$\mathcal{F}_h(\chi u_h)(\xi) = O(h^\infty) \text{ uniformly in } \xi \in V$$

localization in position to near $x = x_0$

localization in frequency to $\xi \approx \xi_0$

(More on wavefront sets later in the course...)

Proposition Assume (SPT$_{\hat{x}}$). Fix $t \geq 0$.
Then $WF_h(u(t)) \subset \{(2t\xi, \xi) \mid \xi \in [-1,2]\}$.

Remark Recall the Hamiltonian flow

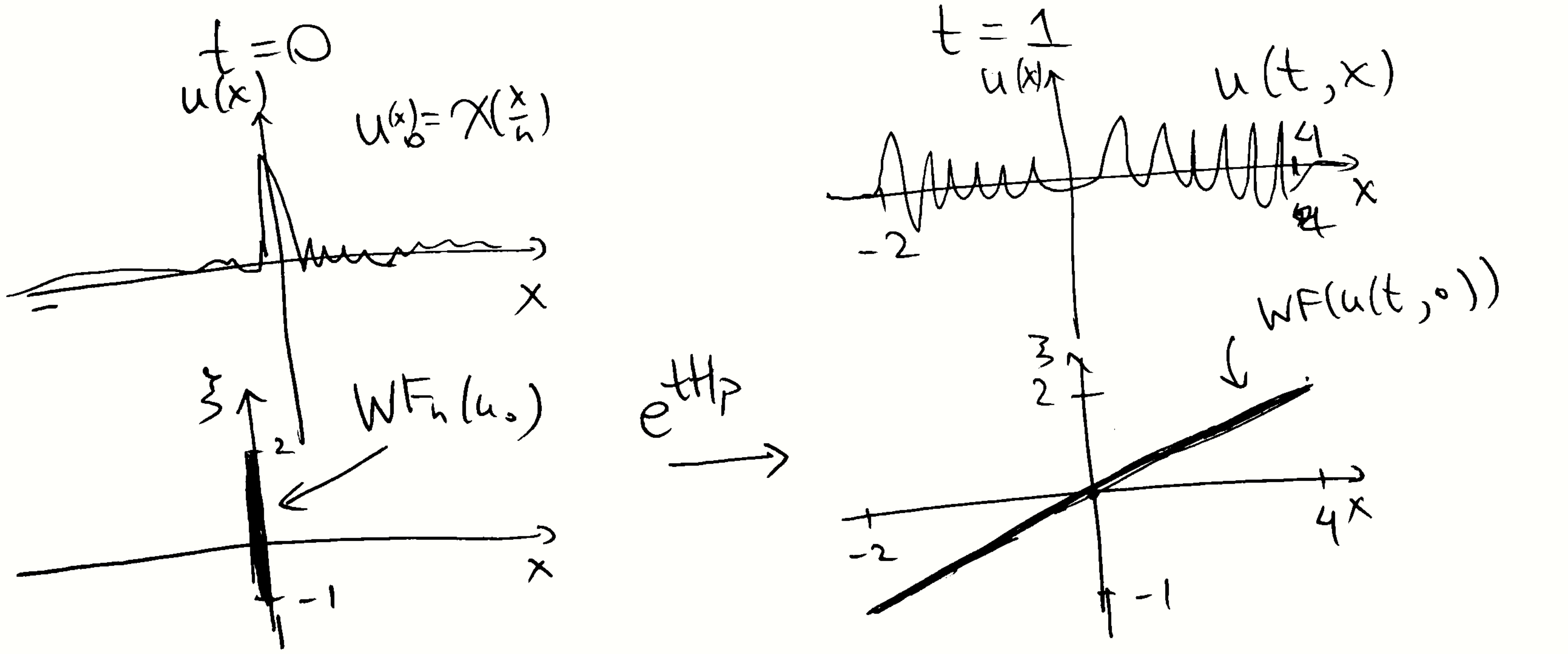
$e^{tH_p}(x,\xi) = (x + 2t\xi, \xi)$. Then

$$WF_h(u(t)) \subset e^{tH_p}(\{(0,\xi) \mid \xi \in [-1,2]\}).$$

In fact we ~~can~~ ~~use~~ have a more general statement called "propagation of singularities":

$$WF_h(u(t)) = e^{tH_p}(WF_h(u_0)) \text{ for any } u_0.$$

This is a form of classical/quantum correspondence: "WF set of the quantum particle ~~follows~~ evolves according to trajectories of classical particles"



Proof 1. Take first any

$\psi \in C_c^\infty(\mathbb{R})$, $\xi \in \mathbb{R}$.

By $(*)$ we compute

$$\psi(x)u(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{\frac{i}{h}(x\eta - t\eta^2)} \psi(x)\hat{\chi}(\eta)\,d\eta$$

$$\mathcal{F}_h(\psi u(t))(\xi) = \frac{1}{(2\pi)^{3/2}\sqrt{h}} \int_{\mathbb{R}^2} e^{\frac{i}{h}(x\eta - t\eta^2 - x\xi)} \psi(x)\hat{\chi}(\eta)\,d\eta\,dx$$

$$= \frac{1}{(2\pi)^{3/2}\sqrt{h}} \int_{\mathbb{R}^2} e^{\frac{i}{h}\varphi(x,\eta)} a(x,\eta)\,d\eta\,dx \quad \text{where}$$

$$\varphi(x,\eta) = x\eta - t\eta^2 - x\xi$$

$$a(x,\eta) = \psi(x)\hat{\chi}(\eta) \in C_c^\infty(\mathbb{R}^2).$$

Critical points of φ: $\partial_x \varphi = 0 \iff \eta = \xi$

$\partial_\eta \varphi = 0 \iff x = 2t\eta$

So by Method of Nonstationary Phase,

$\mathcal{F}_h(\psi u(t))(\xi) = O(h^\infty)$ as long as

$(\star)$ there is no $x \in \operatorname{supp}\psi$, $\eta \in [-1,2]$ such that $x = 2t\eta$, $\xi = \eta$.

<u>2.</u> Now fix $(x_0,\xi_0) \notin \{(2t\xi,\xi) \mid \xi \in [-1,2]\}$

Then we can find

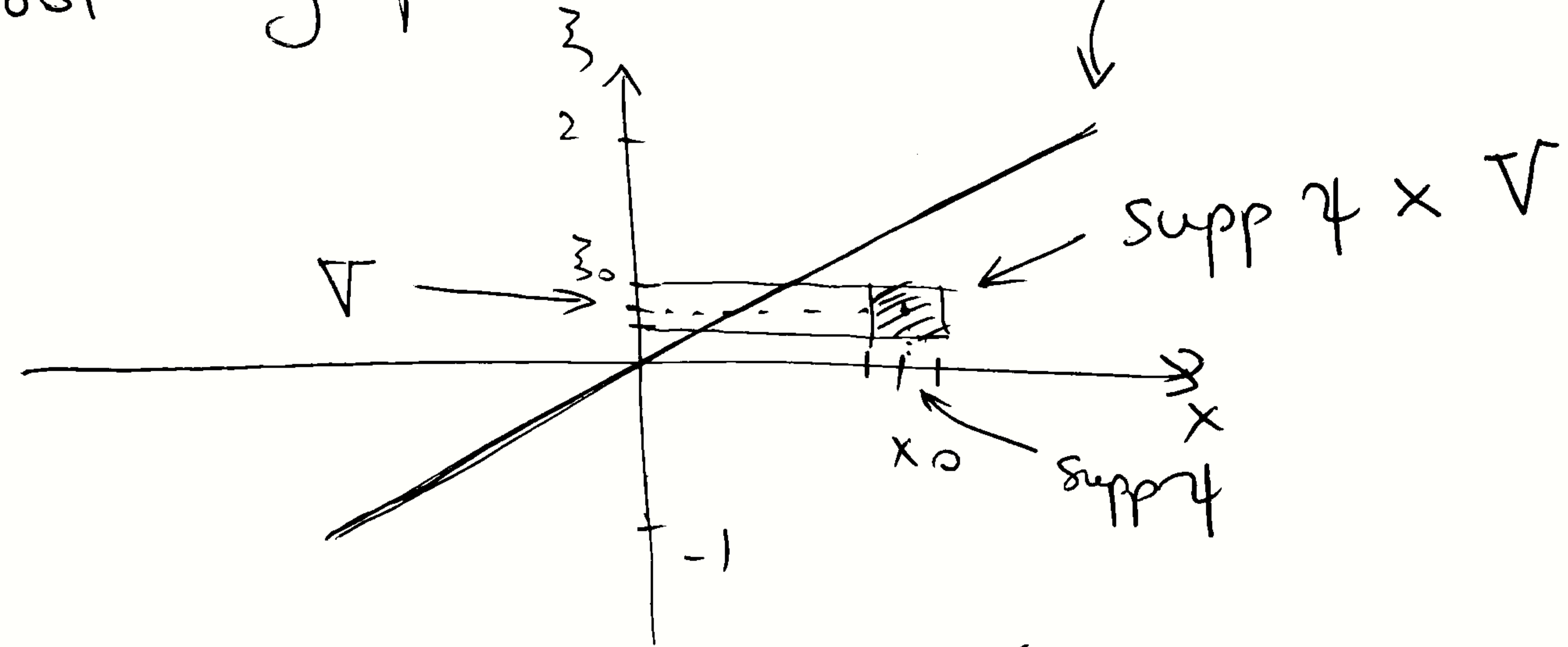
$\chi \in C_c^\infty(\mathbb{R})$, $\chi(x_0) \neq 0$,

nbhd $V(\xi_0)$, such that

(★) holds for all $\xi \in V$.

Proof by picture:

$\{(2t\xi,\xi) \mid \xi \in [-1,2]\}$



So $\mathcal{F}_h(\chi u(t))(\xi) = O(h^\infty)$

uniformly in $\xi \in V$ $\Rightarrow$ $(x_0,\xi_0) \notin WF_h(u(t))$.

("Method of Nonstationary Phase is uniform in parameters")

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