Fractal Uncertainty Principle and Quantum Chaos

Semyon Dyatlov (MIT)

July 12, 2022

Overview

- This talk presents several recent results in quantum chaos, including
  - lower bounds on mass of eigenfunctions and semiclassical measures
  - observability for Schrödinger equations
  - spectral gaps and exponential wave decay for open systems
- The proofs are based on the following ideas:
  - Use the classical/quantum correspondence to its limit
  - Apply the fractal uncertainty principle (FUP):

    No function can be localized in both position and frequency near a fractal set
- General FUP is only known in dimension 1, and most (but not all) results are in the setting of negatively curved surfaces

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Ya. Bazaikin


D. Borthwick


J. Bourgain


N. Burq


M. Cekić


K. Datchev


B. Delarue


G. Dyatlov


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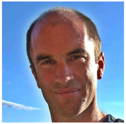
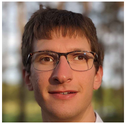

F. Faure


J. Galkowski


S. Ghosh


C. Guillarmou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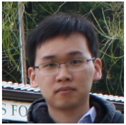
L. Jin


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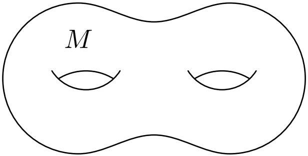
Control of eigenfunctions

- (M, g) compact negatively curved surface
- Geodesic flow on M is a standard model of classical chaos
- Eigenfunctions of the Laplacian $-\Delta_g$ studied by quantum chaos



$$(-\Delta_g - \lambda^2)u = 0, \quad \|u\|_{L^2} = 1$$

Theorem 1

Let $\Omega \subset M$ be an arbitrary nonempty open set. Then

$$\|u\|_{L^2(\Omega)} \geq c > 0$$

where c depends on M, Ω but not on λ

Constant curvature: D–Jin '18, using D–Zahl '16 and Bourgain–D '18

Variable curvature: D–Jin–Nonnenmacher '22, using Bourgain–D '18

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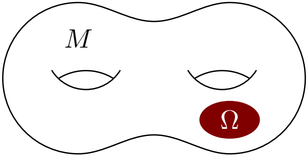

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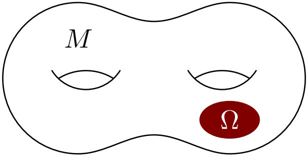

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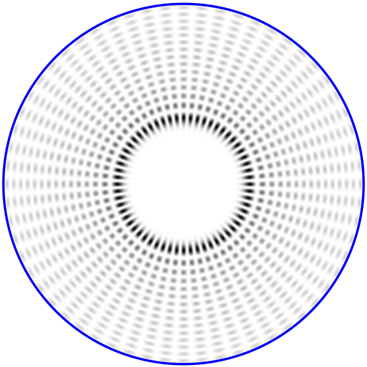
For bounded λ the estimate follows from unique continuation principle

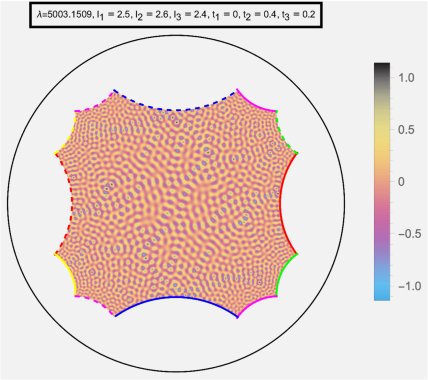
The new result is in the high frequency limit $\lambda \to \infty$

An illustration

Picture on the right courtesy of Alex Strohmaier, using Strohmaier–Uski '12



Disk (Dirichlet b.c.)
Whitespace in the middle



Hyperbolic surface
No whitespace

Applications to PDE

Theorem 2 [Jin '18, D–Jin–Nonnenmacher '22]

Let (M, g) be a compact negatively curved surface and $\Omega \subset M$ nonempty open. Then $\forall T > 0\ \exists C > 0$: any $u(t, x)$ solving the Schrödinger equation

$$(i\partial_t + \Delta_g)u(t, x) = 0, \qquad u(0, x) = u_0(x)$$

satisfies the observability estimate

$$\|u_0\|^2_{L^2(M)} \leq C \int_0^T \int_\Omega |u(t, x)|^2 \, d\,\mathrm{vol}_g(x)\, dt.$$

Previously known only for flat tori: Jaffard '90, Haraux '89, Komornik '92, Anantharaman–Macià '10, Burq–Zworski '12,'17, Bourgain–B–Z '13

Another application is to exponential energy decay for solutions to the damped wave equation: Jin '20, D–Jin–Nonnnenmacher '22

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Semiclassical measures I

- Stronger version of Theorem 1: localization in position and frequency
- Use semiclassical quantization $\mathrm{Op}_h(a) = a(x, -ih\partial_x)$ where $a(x,\xi) \in C_c^\infty(T^*M)$ and $h = \lambda^{-1}$ (here $(-\Delta_g - \lambda^2)u = 0$)
- If $(-\Delta_g - \lambda_j^2)u_j = 0$ and $\lambda_j \to \infty$, we say u_j converges semiclassically to a measure μ on the cotangent bundle T^*M if

$$\langle \mathrm{Op}_{h_j}(a)u_j, u_j\rangle_{L^2(M)} \to \int_{T^*M} a\, d\mu \qquad \text{for all} \quad a \in C_c^\infty(T^*M)$$

- The pushforward $\pi_*\mu$, $\pi : T^*M \to M$, is the weak limit of the probability measures $|u_j|^2\, d\,\mathrm{vol}_g$

Properties of semiclassical measures

- μ is a probability measure
- $\operatorname{supp}\mu \subset S^*M = \{(x,\xi) \in T^*M : |\xi|_g = 1\}$
- μ is invariant under the geodesic flow $\varphi^t : S^*M \to S^*M$

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Semiclassical measures II

Theorem 3 [D–Jin '18, D–Jin–Nonnenmacher '22]

Let (M, g) be a compact negatively curved surface and μ be a semiclassical measure associated to a sequence of eigenfunctions. Then $\operatorname{supp}\mu = S^*M$.

Previous results

- Quantum Ergodicity (QE): if φ^t is ergodic then a density 1 sequence of u_j's converges to the Liouville measure μ_L. [Shnirelman '74, Zelditch '87, Colin de Verdière '85, Zelditch–Zworski '96]
- CdV '85: conjecture that in $K < 0$ (negative sectional curvature), μ cannot be the delta measure on a closed geodesic
- Rudnick–Sarnak '94: QUE conjecture that in $K < 0$, $\mu = \mu_L$
- Lindenstrauss '06: proved QUE for arithmetic hyperbolic surfaces
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Main tool: fractal uncertainty principle (FUP)

No function can be localized in both position and frequency near a fractal set

Definition

Fix $\nu > 0$. A set $X \subset \mathbb{R}$ is ν-porous up to scale h if for each interval $I \subset R$ of length $h \leq |I| \leq 1$, there is an interval $J \subset I$, $|J| = \nu|I|$, $J \cap X = \emptyset$

Theorem 4 [Bourgain–D '18]

Assume that $X, Y \subset \mathbb{R}$ are ν-porous up to scale $h \ll 1$. Then $\exists \beta, C > 0$ depending only on ν such that for all $f \in L^2(\mathbb{R})$

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A bit about proof of Theorem 1

- Assume that $(-\Delta_g - \lambda^2)u = 0, \quad \|u\|_{L^2(M)} = 1, \quad \lambda \gg 1$, and $\|u\|_{L^2(\Omega)} \ll 1$ for some fixed nonempty open $\Omega \subset M$
- Using semiclassical quantization, can study 'localization' of u in the position-frequency space T^*M (up to a limit given by uncertainty principle)
- Using microlocal analysis, we see that this 'localization' is invariant under the geodesic flow φ^t (again, up to a certain point)
- From here we see that u is localized close to each of the two sets

$$\Gamma_\pm := \{(x,\xi) \in S^*M \mid \forall t \geq 0,\ \varphi^{\mp t}(x,\xi) \notin \Omega\}$$

of geodesics which do not pass Ω in past/future time
- The sets $\Gamma_\pm$ have porous structure in certain directions (see next slide)
- Fractal uncertainty principle (Theorem 4) implies that no function u can be localized close to both Γ_+ and Γ_-, giving a contradiction

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Illustration: Arnold cat map

Simpler model than the geodesic flow: an Arnold cat map on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$

$$\varphi : \mathbb{T}^2 \to \mathbb{T}^2, \quad \varphi(x_1, x_2) = (2x_1 + x_2, x_1 + x_2) \bmod \mathbb{Z}^2$$

Define $\Gamma_\pm(N) = \{x \in \mathbb{T}^2 \mid \forall j = 0, \dots, N,\ \varphi^{\mp j}(x) \notin \Omega\}$

$\Gamma_-(N)$, $N = 0$ Ω (in white) $\Gamma_+(N)$, $N = 0$

We see that $\Gamma_\pm(N)$ have porous structure in the stable/unstable directions

Schwartz '21: analog of Theorem 3 for quantum cat maps

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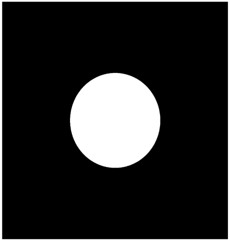

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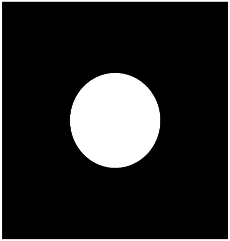
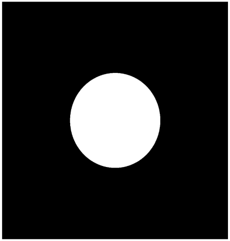


Ω (in white)



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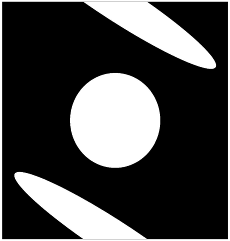
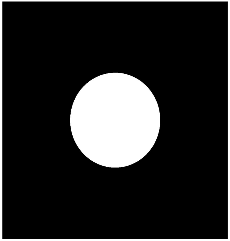


$\Gamma_-(N)$, $N = 1$



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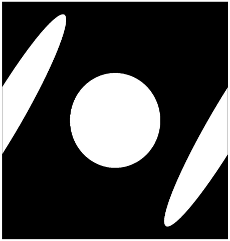

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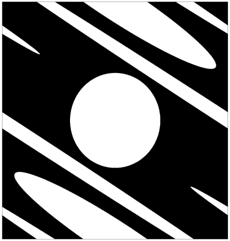
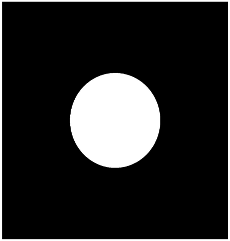
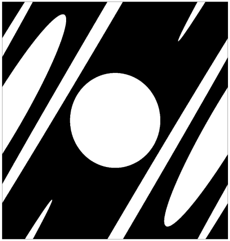


$\Gamma_-(N)$, $N = 2$



Ω (in white)



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We see that $\Gamma_\pm(N)$ have porous structure in the stable/unstable directions

Schwartz '21: analog of Theorem 3 for quantum cat maps

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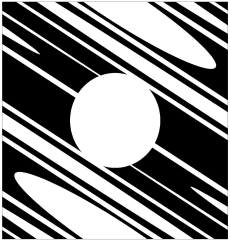

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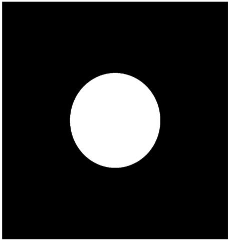

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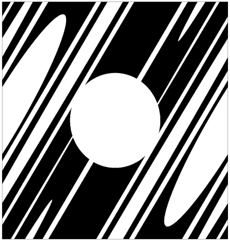

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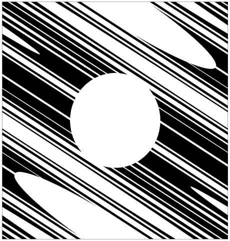

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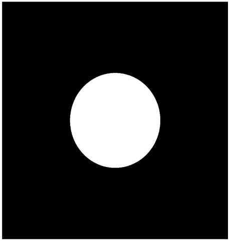

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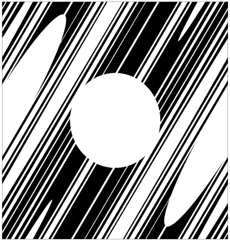

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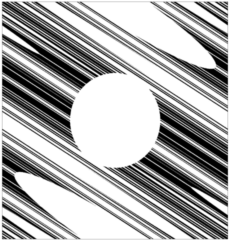

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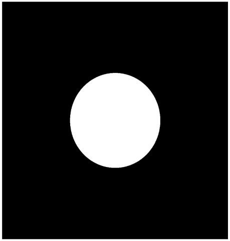

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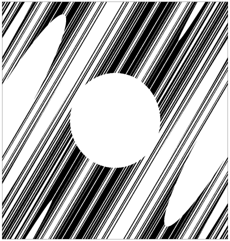

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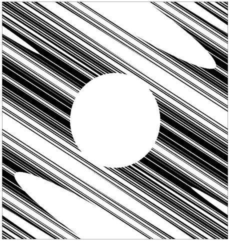

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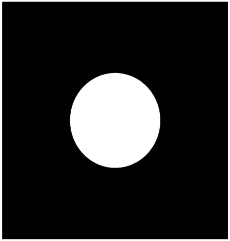

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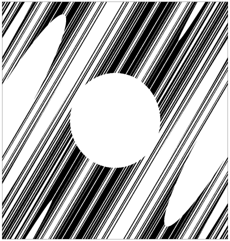

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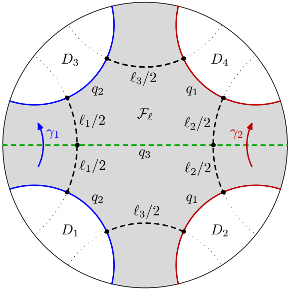
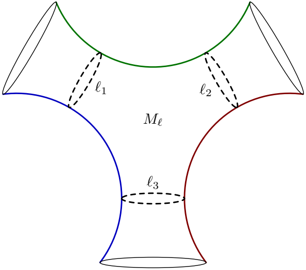
Open quantum chaos and resonances

(M,g) noncompact convex co-compact hyperbolic ($K=-1$) surface





Resonances: poles of the scattering resolvent

$$R(\lambda)=\Big(-\Delta_g-\tfrac{1}{4}-\lambda^2\Big)^{-1}:\begin{cases}L^2(M)\to L^2(M), & \operatorname{Im}\lambda>0\\ L^2_{\mathrm{comp}}(M)\to L^2_{\mathrm{loc}}(M), & \operatorname{Im}\lambda\le 0\end{cases}$$

Existence of meromorphic continuation: Patterson '75,'76, Perry '87,'89, Mazzeo–Melrose '87, Guillopé–Zworski '95, Guillarmou '05, Vasy '13

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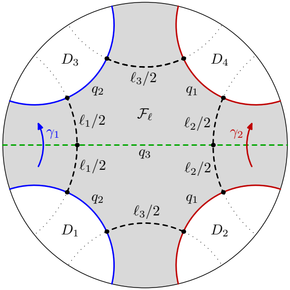

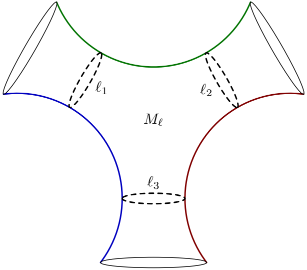

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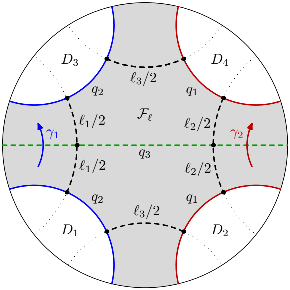

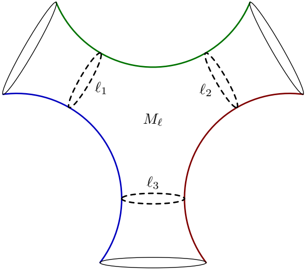

Resonances: poles of the scattering resolvent

Also correspond to zeroes of the Selberg zeta function

$$Z_M(s) = \prod_{T \in \mathscr{L}_M} \prod_{k \geq 0} (1 - e^{-(s+k)T}), \quad s = \frac{1}{2} - i\lambda$$

where $\mathscr{L}_M$ consists of lengths of primitive closed geodesics

Resonances: poles of the scattering resolvent $R(\lambda)$

Featured in resonance expansions of waves:

$$\operatorname{Re}\lambda = \text{rate of oscillation}, \quad -\operatorname{Im}\lambda = \text{rate of decay}$$

Borthwick '13, Borthwick–Weich '14: numerics for resonances

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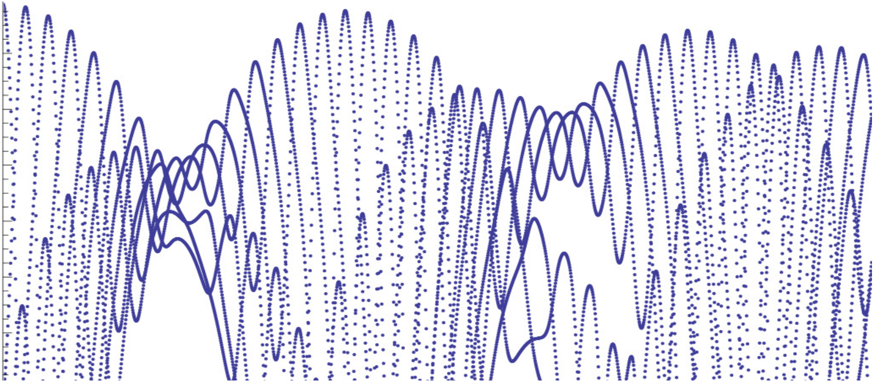

Pictures courtesy of David Borthwick

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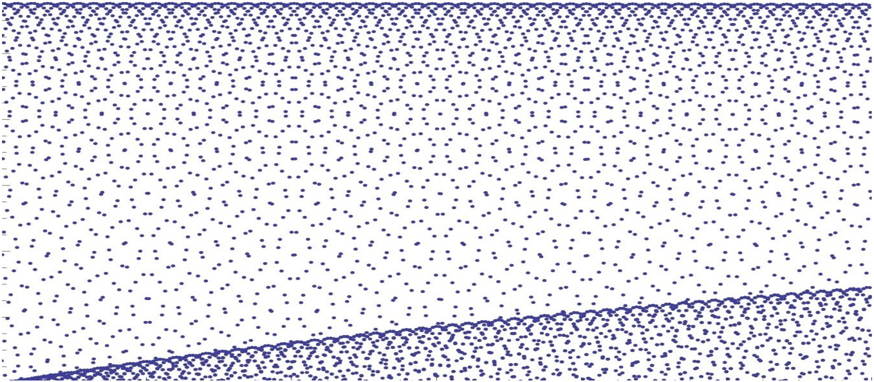

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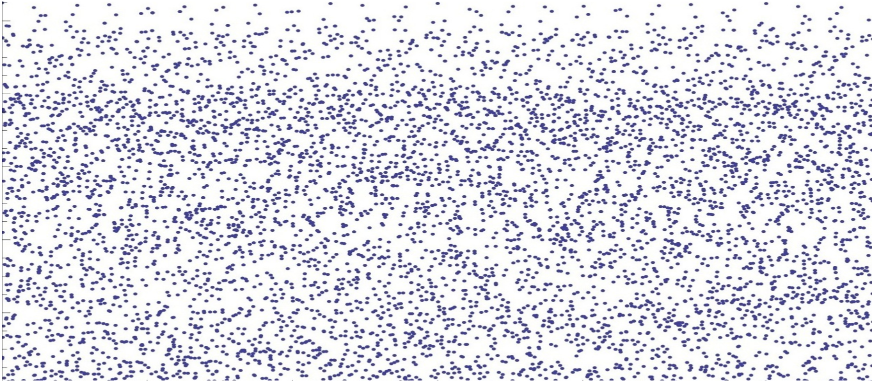

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Spectral gap

Theorem 5 [D–Zahl '16, Bourgain–D '18, D–Zworski '20]

Let (M, g) be a convex co-compact hyperbolic surface. Then it has an essential spectral gap: there exists $\beta > 0$ such that there are only finitely many resonances λ with $\operatorname{Im} \lambda > -\beta$.

- Gives $\mathcal{O}(e^{-\beta t})$ local energy decay for linear waves (at high frequency)
- Also implies Strichartz estimates: Wang '19
- Follows a long history of study of spectral gaps in this and other similar settings (e.g. obstacle scattering):
  Lax–Phillips '67, Patterson '76, Sullivan '79, Ikawa '88, Gaspard–Rice '89, Naud '05, Nonnenmacher–Zworski '09, Petkov–Stoyanov '10, Stoyanov '11 . . .

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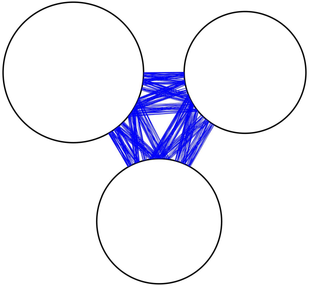
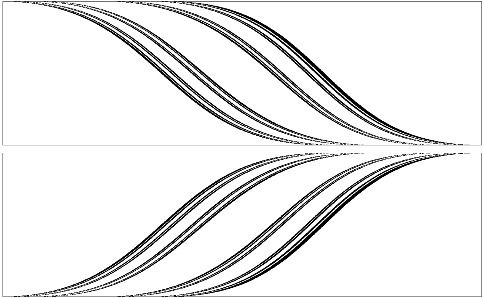
A physically relevant setting: scattering by several convex obstacles in $\mathbb{R}^n$

Resonances: poles of the meromorphic continuation of $(-\Delta_g - \lambda^2)^{-1}$





Theorem 5 [Vacossin '22, using Bourgain–D '18]

Let M be the exterior of several convex obstacles in $\mathbb{R}^2$, which satisfy the no-eclipse condition (no line intersects 3 obstacles). Then there exists $\beta > 0$ such that there are only finitely many resonances in $\{\operatorname{Im}\lambda > -\beta\}$.

Observed experimentally: Barkhofen–Weich–Potzuweit–Stöckmann–Kuhl–Zworski '13

Higher dimensional FUP?

- The results above applied to surfaces (dim $= 2$)
- To make them work for general manifolds of dim > 2, we need a fractal uncertainty principle for subsets of $\mathbb{R}^n$, $n \geq 2$
- Counterexample: $X, Y \subset \mathbb{R}^2$ are two orthogonal lines. Then $\widehat{\delta_X} = 2\pi\delta_Y$ and FUP fails

Here is what is known to date:

- Han–Schlag '20: FUP if X is a product of porous subsets of $\mathbb{R}$
- D–Jézéquel '21: Theorem 1 for certain higher dimensional quantum cat maps, still using 1D FUP
- D–Zhang '22?: FUP in 2D if X is a curve
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Thank you for your attention!