Pollicott–Ruelle resonances

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I. Correlations

① Anosov flows

- M compact connected manifold
- $X \in C^\infty(M; TM)$
  nonvanishing vector field on M
- $\varphi^t = e^{tX} : M \circlearrowleft$ the flow of X

Defn We say φ^t is an Anosov flow if there exists a φ^t-invariant continuous splitting

$$T_xM = E_0(x) \oplus E_u(x) \oplus E_s(x), \quad x \in M$$

such that $\exists\, C, \theta > 0$ $\forall (x,v) \in TM$

$$\boxed{|d\varphi^t(x)v| \leq C e^{-\theta|t|} |v|,}$$ for

$v \in E_u(x)$ and $t \leq 0$ or

$v \in E_s(x)$ and $t \geq 0$

(flow / unstable / stable decomposition)

Fundamental example:

- (Σ, g) compact Riemannian mfld
- $M = S\Sigma$ unit tangent bundle
- $\varphi^t : M \circlearrowleft$ the geodesic flow on (Σ, g)
- If (Σ, g) has negative sectional curvature then φ^t is an Anosov flow.

A simpler example: a suspension

(special case w/ constant roof function)

- N compact mfld
- $\Phi : N \circlearrowleft$ an Anosov map, i.e. a diffeomorphism with an unstable/stable decomposition (no flow direction)
- For example, $N = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and $\Phi(x) = Ax \mod \mathbb{Z}^2$ where $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

  (a "cat map")

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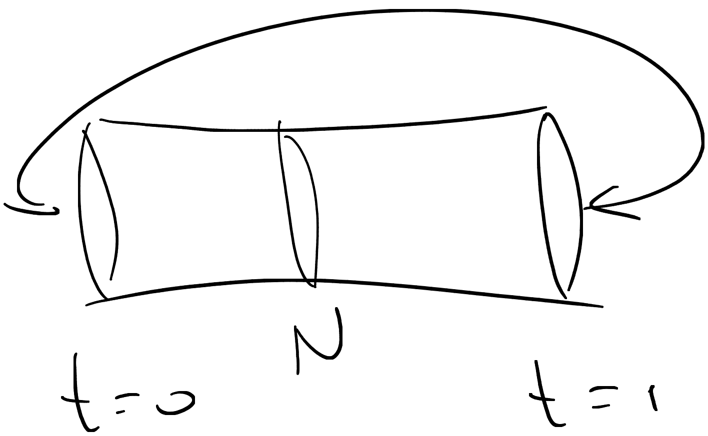
- Put $M := [0,1]_s \times N_x /\sim$

where $(1, x) \sim (0, \Phi(x))$

That's a compact manifold which fibers over $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ via

$(s, x) \mapsto s \bmod \mathbb{Z}$



glue via Φ

- $X = \partial_t$
- Then φ^t is an Anosov flow (suspension flow of Φ)

2. Correlations

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Let $\varphi^t : M \circlearrowleft$ be an Anosov flow

We assume that it is

volume preserving:

$\exists\, \mu$ a smooth probability measure on M

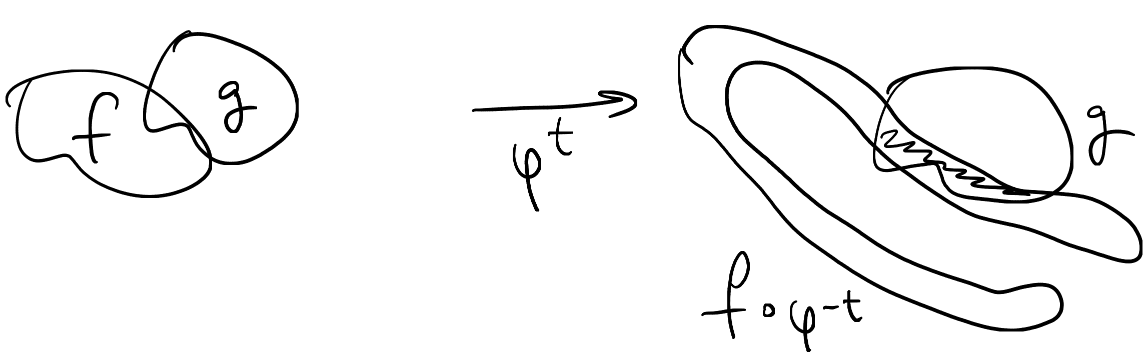
s.t. $\mu(\varphi^t(A)) = \mu(A)$ $\forall t$, $A \subset M$

(i.e. $\mathcal{L}_X \mu = 0$)

(example: geodesic flows, μ = Liouville measure)

For $f, g \in L^2(M)$, define correlation

$$\boxed{\rho_{f,g}(t) = \int_M (f \circ \varphi^{-t})\, g \, d\mu, \quad t \in \mathbb{R}}$$



Thm. [Mixing]

Assume that φ^t is an Anosov flow and is not a suspension.

Then $\forall f, g \in L^2(M)$ we have

$$\rho_{f,g}(t) \xrightarrow[t \to \infty]{} \int_M f \, d\mu \int_M g \, d\mu$$

($f \circ \varphi^{-t}$, g become independent as $t \to \infty$)

We'll give a scheme of the proof of the Mixing Theorem.

(3.) Power spectrum

For $\lambda \in \mathbb{C}$, $\operatorname{Re} \lambda > 0$, define the power spectrum

$$\hat{\rho}_{f,g}(\lambda) = \int_0^\infty e^{-\lambda t} \rho_{f,g}(t) \, dt$$

The integral converges since $|\rho_{f,g}(t)| \le C$ and $\hat{\rho}_{f,g}(\lambda)$ is holomorphic in $\{\operatorname{Re} \lambda > 0\}$

Define for $f \in L^2(M)$, $\operatorname{Re}\lambda > 0$

the function

$$R(\lambda) f := \int_0^\infty e^{-\lambda t} (f \circ \varphi^{-t})\, dt \in L^2(M)$$

Recall X, the generator of φ^t.

- We think of X as a 1^{st} order differential operator on M.
- Since μ is φ^t-invariant, $P := -iX : C^\infty(M) \circlearrowleft$ extends to an unbounded <u>self-adjoint operator</u> on $L^2(M) = L^2(M, d\mu)$

and $\boxed{f \circ \varphi^{-t} = e^{-tX} f}$,

where e^{-tX} is defined using the functional calculus of P

So $R(\lambda) f = \int_0^\infty e^{-\lambda t} e^{-tX} f\, dt = (X+\lambda)^{-1} f$

and $\hat{\rho}_{f,g}(\lambda) = (R(\lambda) f, g)$

where $(f,g) := \int_M f g \, d\mu$

We see that for $\mathrm{Re}\,\lambda > 0$,

$$R(\lambda) = (X+\lambda)^{-1}$$

is the L^2 resolvent of X,

namely $u = R(\lambda) f$ is the only solution in $L^2(M)$ to the equation $(X+\lambda)u = f$ (understood in the sense of distributions)

For the next theorem, we briefly review distributions:

$\mathcal{D}'(M) =$ dual to $C^\infty(M)$ consisting of all continuous linear functionals $C^\infty(M) \to \mathbb{C}$

We embed $L^1(M) \subset \mathcal{D}'(M)$ by the rule

$$f(\varphi) := \int_M f \varphi \, d\mu \quad \text{for} \quad \begin{matrix} f \in L^1(M) \\ \varphi \in C^\infty(M) \end{matrix}$$

Write $(u, \varphi) = u(\varphi)$ for $\begin{matrix} u \in \mathcal{D}'(M) \\ \varphi \in C^\infty(M) \end{matrix}$

Thm [Meromorphic continuation]

The operator $R(\lambda) = (X+\lambda)^{-1} : L^2(M) \circlearrowleft$, $\underline{\operatorname{Re}\lambda > 0}$

has a meromorphic continuation with poles of finite rank

to an operator $R(\lambda) : C^\infty(M) \to \mathcal{D}'(M)$, $\underline{\lambda \in \mathbb{C}}$

In particular, if $f, g \in C^\infty(M)$ then $\hat{\rho}_{f,g}(\lambda)$ continues to a meromorphic function of $\lambda \in \mathbb{C}$.

Remark If we had exponential mixing,

i.e. $\rho_{f,g}(t) = \underbrace{\int_M f\,d\mu \int_M g\,d\mu}_{=:\,c_{f,g}} + r(t)$, $\quad r(t) = O(e^{-\delta t})$ for some $\delta > 0$

then we would get meromorphic continuation to $\{\operatorname{Re}\lambda > -\delta\}$:

$$\hat{\rho}_{f,g}(\lambda) = c_{f,g} \int_0^\infty e^{-\lambda t}\,dt + \int_0^\infty e^{-\lambda t}\, r(t)\,dt$$

$$= \frac{c_{f,g}}{\lambda} + \hat{r}(\lambda), \quad \hat{r} \text{ holomorphic in } \{\operatorname{Re}\lambda > -\delta\}$$

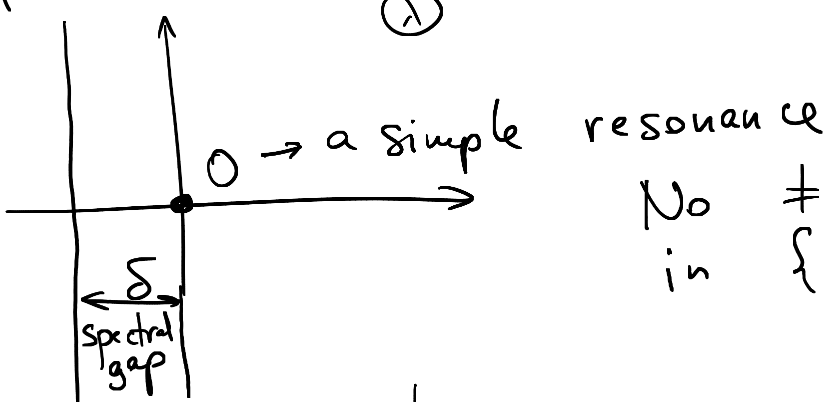
So then $R(\lambda) = \dfrac{1 \otimes 1}{\lambda} + \left(\text{holomorphic in } \{\operatorname{Re}\lambda > -\delta\}\right)$, $\quad (1\otimes 1)f = \left(\int f\,d\mu\right) 1$

But typically in proofs of exponential mixing (e.g. Liverani '04 for geodesic flows) one uses meromorphic continuation as an ingredient in the proof

<u>Defn.</u> We call the poles of $R(\lambda)$ the <u>Pollicott-Ruelle resonances</u>

They are all in $\{\operatorname{Re} \lambda \leq 0\}$ (as $R(\lambda)$ is holomorphic in $\{\operatorname{Re} \lambda > 0\}$)

If there's exponential mixing, then we have the following picture



No $\neq 0$ resonances in $\{\operatorname{Re} \lambda > -\delta\}$

4. From meromorphic continuation to mixing

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Thm ("Limiting Absorption Principle" / LAP Dyatlov–Zworski '17)

Assume that λ is a Pollicott-Ruelle resonance and $\boxed{\operatorname{Re}\lambda = 0}$.

Then $\exists\, u \in C^\infty(M)$, $u \not\equiv 0$, such that $(X+\lambda)u = 0$.

u called a resonant state

We now give a sketch of the proof that

Meromorphic Continuation + LAP $\Rightarrow$ Mixing

Step 1: Assume that φ^t is an Anosov flow & it is not a suspension flow.

We claim that

(a) $\lambda = 0$ is a simple resonance, where the only resonant state is $u = \text{const}$

(b) there are no other resonances with $\operatorname{Re}\lambda = 0$.

Proof of ⓐ:

by LAP we have $u \in C^\infty(M)$ and $Xu = 0$, so $u = u \circ \varphi^t$.

Apply the Chain Rule:

$\forall x \in M \ \forall v \in T_x M$

$$du(x) \cdot v = du(\varphi^t(x)) \cdot d\varphi^t(x) \cdot v$$

Now

$v \in E_s(x)$ (stable space) $\Rightarrow |d\varphi^t(x) \cdot v| \xrightarrow[t \to \infty]{} 0$

$\Rightarrow du(x) \cdot v = 0$.

$v \in E_u(x) \Rightarrow |d\varphi^t(x) \cdot v| \xrightarrow[t \to -\infty]{} 0$

$\Rightarrow du(x) \cdot v = 0$

$v \in E_0(x) \Rightarrow du(x) \cdot v = 0$ as $Xu = 0$, $E_0(x) = \mathbb{R} X(x)$

Since $T_x X = E_0(x) \oplus E_u(x) \oplus E_s(x)$, get

$\boxed{du = 0} \Rightarrow u = \text{const}$ (as M is connected)

From here one can show that near $\lambda = 0$

$$R(\lambda) = \frac{1 \otimes 1}{\lambda} + (\text{holomorphic})$$

Proof of (b):

Assume that $\lambda = -is$, $s \in \mathbb{R} \setminus 0$ is a resonance. By LAP, there exists $u \in C^\infty(M)$, $u \not\equiv 0$, $\boxed{Xu = isu}$

Put $v = |u|^2 \in C^\infty(M)$, then $\boxed{Xv = 0}$

By part (a) $v = \text{const}$, so WLOG $\boxed{|u| \equiv 1.}$

Thus $u = e^{i\theta}$ for some $\theta \in C^\infty(M; \mathbb{S}^1)$ and $X\theta \equiv s$, a nonzero constant.

So $d\theta \neq 0$ everywhere, i.e. $\theta : M \to \mathbb{S}^1$ is a fibration, and $\varphi^t = e^{tX}$ is a suspension of the map $\varphi^{2\pi/s}$ on the fiber $N := \theta^{-1}(0)$.

Remark: it was crucial for both (a) & (b) that resonant states are smooth. For λ with $\operatorname{Re} \lambda < 0$, resonant states will never be smooth.

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<u>Step 2</u>: use spectral theory for the self-adjoint operator

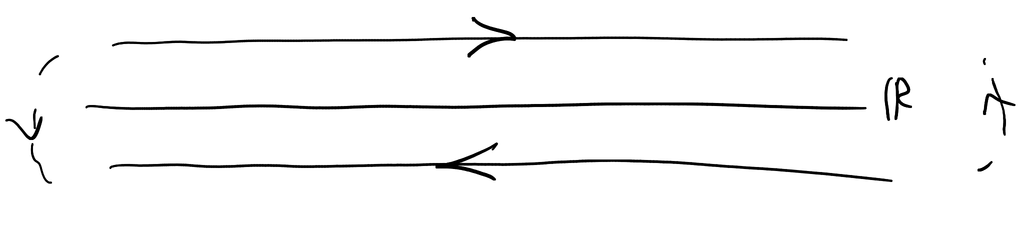
$$\underline{P} := -iX \quad \text{on } L^2(M)$$

$\forall f \in L^2(M)$, the spectral measure ν_f is a finite Borel measure on $\mathbb{R}$ such that $\forall$ bounded measurable F,

$$\langle F(\underline{P})f, f\rangle_{L^2} = \int_{\mathbb{R}} F(\omega)\, d\nu_f(\omega)$$

By analogy to Cauchy's Integral formula we expect that if F were holomorphic near $\mathbb{R}$ then

$$F(P) = \frac{1}{2\pi i}\left[\int_{\mathbb{R}+i\varepsilon} F(\omega)(P-\omega)^{-1} d\omega - \int_{\mathbb{R}-i\varepsilon} F(\omega)(P-\omega)^{-1} d\omega\right]$$



For general $F \in C^0(\mathbb{R})$ bdd we have <u>Stone's Formula</u>:

$$F(\underline{P}) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0+} \int_{\mathbb{R}} (P-\omega-i\varepsilon)^{-1} - (P-\omega+i\varepsilon)^{-1} d\omega .$$

Since $\underline{P}^* = \underline{P}$, we have $(\underline{P}-\omega+i\varepsilon)^{-1} = ((\underline{P}-\omega-i\varepsilon)^{-1})^*$

So if $f \in L^2(M)$ then

$$\langle F(\underline{P})f,f\rangle_{L^2} = \frac{1}{\pi} \lim_{\varepsilon \to 0+} \int_{\mathbb{R}} F(\omega) \operatorname{Im} \langle (\underline{P}-\omega-i\varepsilon)^{-1} f,f\rangle_{L^2} d\omega$$

Recall that $\underline{P} = -iX$, so

$$(\underline{P}-\omega-i\varepsilon)^{-1} = i(X-i\omega+\varepsilon)^{-1} = i\,R(-i\omega+\varepsilon)$$ So

$$\int_{\mathbb{R}} F(\omega)\, d\nu_f(\omega) = \langle F(P)f,f\rangle_{L^2}$$

$$= \frac{1}{\pi} \lim_{\varepsilon \to 0+} \int_{\mathbb{R}} F(\omega) \operatorname{Re} \langle R(-i\omega+\varepsilon)f,f\rangle_{L^2} d\omega$$

$$= \frac{1}{\pi} \lim_{\varepsilon \to 0+} \int_{\mathbb{R}} F(\omega) \operatorname{Re} \hat{\rho}_{f,f}(-i\omega+\varepsilon)\, d\omega$$

Therefore, we have (in the sense of weak limit)

$$d\nu_f(\omega) = \frac{1}{\pi} \lim_{\varepsilon \to 0+} \operatorname{Re} \hat{\rho}_{f,f}(-i\omega+\varepsilon)\, d\omega .$$

Step 3: putting it together

By density etc. it suffices to show that $\forall f \in C^\infty(M)$, $\int f\, d\mu = 0$

$$\rho_{f,f}(t) \xrightarrow[t\to\infty]{} 0.$$

We have $(P = -iX)$

$$\rho_{f,f}(t) = \langle e^{-tX} f, f\rangle_{L^2} = \langle e^{-itP} f, f\rangle_{L^2}$$
$$= \int_{\mathbb{R}} e^{-it\omega}\, d\nu_f(\omega).$$

By Step 1, we see that $\hat{\rho}_{f,f}(\lambda)$ is holomorphic in $\{\operatorname{Re}\lambda \geq 0\}$.

So by Step 2, we have

$$d\nu_f(\omega) = \frac{1}{\pi} \operatorname{Re} \hat{\rho}_{f,f}(-i\omega)\, d\omega = G(\omega)\, d\omega$$

where $G \in C^\infty(\mathbb{R})$ and, since ν_f is a finite measure, $G \in L^1(\mathbb{R})$. Now

$$\rho_{f,f}(t) = \int_{\mathbb{R}} e^{-it\omega} G(\omega)\, d\omega \xrightarrow[t\to\infty]{} 0$$ by the Riemann-Lebesgue Lemma. $\square$

⑤ Meromorphic continuation on an example

Our example will have M non-compact. The theorem actually still applies but we have to replace C^∞ by C_c^∞

General references:
- Liverani '04
- Butterley–Liverani '07
- Baladi–Tsujii '07
- Faure–Sjöstrand '11
- Dyatlov–Zworski '16
- Dyatlov–Guillarmou '16

Here's the example:

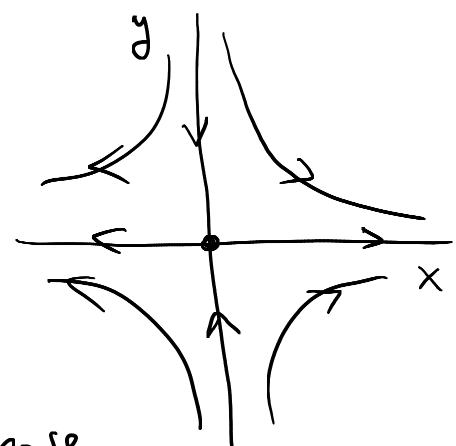
$$M = \mathbb{R}^2_{x,y} \times \mathbb{S}^1_\theta, \quad \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$$

$$X = x\partial_x - y\partial_y + \partial_\theta$$

$$\varphi^t(x,y,\theta) = (e^t x, e^{-t} y, \theta + t)$$

$E_u = \mathbb{R}\partial_x$ (really only make sense at $x=y=0$)

$E_s = \mathbb{R}\partial_y$



Let us take $f, g \in C_c^\infty(M)$ and decompose into Fourier series:

$$f(x,y,\theta) = \sum_{k\in\mathbb{Z}} f_k(x,y)e^{2\pi i k\theta}, \quad g(x,y,\theta) = \sum_{k\in\mathbb{Z}} g_k(x,y)e^{2\pi i k\theta}$$

We compute the <u>correlation</u>

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$$\rho_{f,\bar{g}}(t) = \langle f\circ\varphi^{-t}, g\rangle_{L^2}$$

$$= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^2} f_k(e^{-t}x, e^{t}y)\, e^{-2\pi i k t}\, \overline{g_k(x,y)}\, dx\,dy$$

$$= \sum_{k\in\mathbb{Z}} \rho_k(t)\, e^{-2\pi i k t} \quad \text{where}$$

$$\rho_k(t) = \int_{\mathbb{R}^2} f_k(e^{-t}x, e^{t}y)\, \overline{g_k(x,y)}\, dx\,dy.$$

We write an asymptotic expansion of $\rho_k(t)$ as $t\to\infty$ using the Taylor expansions of f, g: $\forall N$

$$\rho_k(t) = e^{-t}\int_{\mathbb{R}^2} f_k(e^{-t}x, y)\, \overline{g_k(x, e^{-t}y)}\, dx\,dy$$

$$= \sum_{\substack{j,\ell\geq 0\\ j+\ell<N}} \frac{1}{j!\,\ell!}\, e^{-(j+\ell+1)t} \int_{\mathbb{R}^2} \partial_x^j f_k(0,y)\, x^j\, \partial_y^\ell \bar{g}_k(x,0)\, y^\ell\, dx\,dy$$

$$+ O(e^{-(N+1)t})$$

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We can rewrite this as

$$\rho_k(t) = \sum_{\substack{j,\ell\geq 0\\ j+\ell\leq N}} \frac{e^{-(j+\ell+1)t}}{j!\,\ell!}\left(\int_{\mathbb{R}} y^\ell \partial_x^j f_k(0,y)\,dy\right)\left(\int_{\mathbb{R}} x^j \partial_y^\ell \bar{g}_k(x,0)\,dx\right) + O(e^{-(N+1)t})$$

$$= \sum_{\substack{j,\ell\geq 0\\ j+\ell\leq N}} \frac{e^{-(j+\ell+1)t}}{j!\,\ell!\,(-1)^{j+\ell}} \langle f_k, \partial_x^j \delta_0(x) \otimes y^\ell\rangle_{L^2} \langle x^j \otimes \partial_y^\ell \delta_0(y), g_k\rangle_{L^2} + O(e^{-(N+1)t})$$

$$= \sum_{\substack{j,\ell\geq 0\\ j+\ell\leq N}} \frac{e^{-(j+\ell+1)t}}{j!\,\ell!\,(-1)^{j+\ell}} \langle f, \partial_x^j \delta_0(x) \otimes y^\ell \otimes e^{2\pi i k\theta}\rangle_{L^2} \cdot \langle x^j \otimes \partial_y^\ell \delta_0(y) \otimes e^{-2\pi i k\theta}, g\rangle_{L^2} + O(e^{-(N+1)t})$$

Summing over k, we get

$$\rho_{f,\bar{g}}(t) = \sum_{\substack{k\in\mathbb{Z}\\ j+\ell\leq N}} \frac{e^{-(j+\ell+1)t}e^{-2\pi i k t}}{j!\,\ell!\,(-1)^{j+\ell}} \langle f, \dots\rangle\langle \dots, g\rangle + O(e^{-(N+1)t})$$

Now, $\hat{\rho}_{f,\bar{g}}(\lambda) = \int_0^\infty e^{-\lambda t} \rho_{f,\bar{g}}(t)\,dt$.

The $O(e^{-(N+1)t})$ piece gives something holomorphic in $\{\mathrm{Re}\,\lambda > -N-1\}$

And each $e^{-(j+\ell+1)t - 2\pi i k t}(\cdots)$ piece

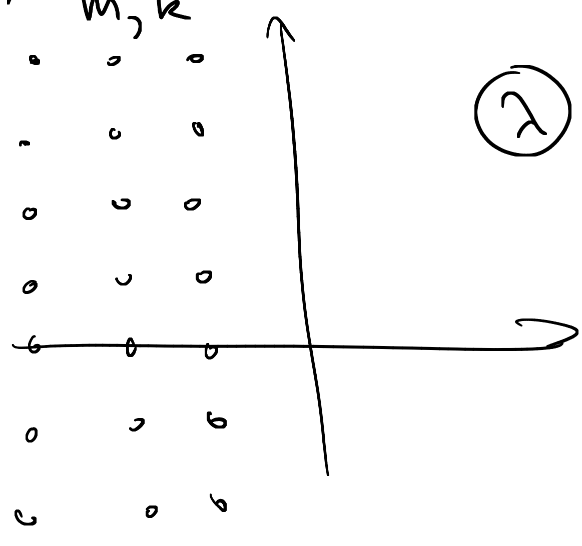
gives $\displaystyle\int_0^\infty e^{-(j+\ell+1)t - 2\pi i k t - \lambda t}\,dt\,(\cdots)$

$$= \frac{(\cdots)}{\lambda + (j+\ell+1) + 2\pi i k}.$$

So $\hat{\rho}_{f,\bar{g}}(\lambda)$ continues meromorphically

to $\lambda \in \mathbb{C}$ with poles at

$$\lambda_{m,k} = -m - 2\pi i k, \quad m, k \in \mathbb{Z},\ m \geq 1$$



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(6.) Some remarks on meromorphic continuation in general

There are several proofs,
I am unsurprisingly biased towards a microlocal proof by Dyatlov-Zworski 2016
(which followed the strategy of Faure-Sjöstrand 2011)

The general strategy is to find spaces $\mathcal{H}, \mathcal{D}$ such that

$$X+\lambda : \mathcal{D} \to \mathcal{H} \text{ is a Fredholm operator}$$

Then $(X+\lambda)^{-1} : \mathcal{H} \to \mathcal{D}$ is meromorphic and we show that $R(\lambda) = (X+\lambda)^{-1}$, $\operatorname{Re} \lambda > 0$

The spaces $\mathcal{H}, \mathcal{D}$ used are
<u>anisotropic Sobolev spaces</u>
tailored to the stable/unstable decomposition for φ^t
(in particular, these spaces work to study φ^t as $t \to \infty$, but the "opposite" spaces are needed for $t \to -\infty$)

The proof of the Fredholm property uses a bunch of microlocal analysis, including

- Elliptic regularity
- Propagation of singularities (Hörmander)
- Radial point estimates (Melrose)

Instead of talking more about the proof, we look here at a few of its corollaries:

Wavefront set

For a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$, we say a point $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ does <u>not</u> lie in the <u>wavefront set</u> $WF(u)$, if $\exists$

- a cutoff fn $\chi \in C_c^\infty(\mathbb{R}^n)$, $\chi(x_0) \neq 0$,
- an open conical set $V \subset \mathbb{R}^n \setminus 0$, $\xi_0 \in V$,

such that $\underbrace{\widehat{\chi u}}_{\text{Fourier transform}}(\xi) = O((1+|\xi|)^{-N})\ \forall N$ when $\xi \in V$

The wavefront set $WF(u) \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ tells us at which points (x) and frequencies (ξ) u is <u>not</u> in C^∞. In particular

$$WF(u) = \emptyset \iff u \in C^\infty(\mathbb{R}^n).$$

If M is a manifold and $u \in \mathcal{D}'(M)$ then one can define

$$\mathrm{WF}(u) \subset \underset{\substack{\uparrow \\ \text{cotangent bundle}}}{T^*M} \setminus \underset{\text{zero section}}{0}$$

which is a closed conic set.

Resonant states

Assume $\lambda \in \mathbb{C}$ is a pole of $R(\lambda)$, i.e. a Pollicott-Ruelle resonance.

Then $\exists$ a resonant state at λ, which is a distribution $u \in \mathcal{D}'(M)$ such that $u \neq 0$,

$(X+\lambda)u = 0$ (in the sense of distributions),

and $\mathrm{WF}(u) \subset E_u^*$.

Here $E_u^* = \{(x,\xi) \in T^*M : \xi \perp E_0(x) \oplus E_u(x)\}$

<u>NOTE</u>: u lies in the range of the residue $\mathrm{Res}_\lambda R(\lambda)$

As an *example*, look at the flow $\varphi^t(x,y,\theta) = (e^t x, e^{-t} y, \theta + t)$ studied before. Then resonant states have the form

$$u_{jek} = x^j \otimes \partial_y^\ell \delta_0(y) \otimes e^{-2\pi i k \theta}.$$

Each of these looks like a δ-function on the weak unstable leaf $\{y=0\} \subset \mathbb{R}^2_{x,y} \times \mathbb{S}^1_\theta$, so one can check directly that

$$WF(u_{jk\ell}) \subset \text{conormal bundle of } \{y=0\}$$

$$= \{(x,y,\theta,\xi_x,\xi_y,\xi_\theta) \mid y=0,\ \xi_x=0,\ \xi_\theta=0\}$$

which does lie inside E_u^*

(as $E_0 = \mathbb{R}\partial_\theta$, $E_u = \mathbb{R}\partial_x$)

Cohomological equation

Let's look at the equation

$Xu = f$ where $f \in C^\infty(M)$

and $\int_M f \, d\mu = 0$

What is the space in which we have a unique solution u (modulo $+C$)?

- $C^\infty(M)$? Usually no <u>existence</u>:
  need $\int_\gamma f = 0$ on each closed trajectory γ of φ^t
- $\mathcal{D}'(M)$? No <u>uniqueness</u>:
  e.g. γ closed trajectory, $u = \delta_\gamma \Rightarrow$
  $\Rightarrow Xu = 0$
- Turns out $u \in \mathcal{D}'(M)$, $WF(u) \subset E_u^*$ is the right requirement