Minicourse on fractal uncertainty principle
Lecture 3–4: FUP and transfer operators

Semyon Dyatlov (MIT)

March 22–25, 2021

Review

- $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ Schottky group, $\Lambda_\Gamma \subset \mathbb{R}$ limit set, $\Lambda_\Gamma(h) = \Lambda_\Gamma + [-h, h]$
- $\mathcal{L}_s : \mathcal{H}(D) \to \mathcal{H}(D)$ transfer operator, $Z_M(s) = \det(I - \mathcal{L}_s)$
- $\mathcal{B}_{\chi,h} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by

$$\mathcal{B}_{\chi,h} f(x) = (2\pi h)^{-\frac{1}{2}} \int_{\mathbb{R}} |x-y|^{-\frac{2i}{h}} \chi(x,y) f(y)\, dy$$

where $\chi \in C_c^\infty(\mathbb{R}^2)$, $\operatorname{supp}\chi \cap \{x = y\} = \emptyset$

Theorem

Assume that for some fixed β and all χ

$$\| \mathbb{1}_{\Lambda_\Gamma(h)} \mathcal{B}_{\chi,h} \mathbb{1}_{\Lambda_\Gamma(h)} \|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} = \mathcal{O}(h^\beta) \quad \text{as} \quad h \to 0.$$

Then for each $\alpha > \frac{1}{2} - \beta$, $Z_M(s)$ has finitely many zeroes with $\operatorname{Re} s \geq \alpha$.

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Outline of the proof

- Since resonances form a discrete set and there are none with $\operatorname{Re} s > \delta$, enough to show there are no resonances with $\operatorname{Re} s \geq \alpha$, $|\operatorname{Im} s| \gg 1$
- Take $s = \alpha + \frac{i}{h}$ where $\alpha > \frac{1}{2} - \beta$ and $0 < h \ll 1$
- Recall that s is a resonance iff $I - \mathcal{L}_s$ is not invertible. Assume that $\mathcal{L}_s u = u$ for some $u \in \mathcal{H}(D)$; we will show that $u = 0$
- Step 1: get a rough bound on how fast u oscillates
- Step 2: get finer information on the frequency localization of u and write it in terms of $u|_{\Lambda_\Gamma(h)}$ where $\Lambda_\Gamma(h) = \Lambda_\Gamma + [-h, h]$
- Step 3: use FUP to get $\|u|_{\Lambda_\Gamma(h)}\|_{L^2} \leq C h^{\alpha - \frac{1}{2} + \beta} \|u|_{\Lambda_\Gamma(h)}\|_{L^2}$ which gives $u = 0$
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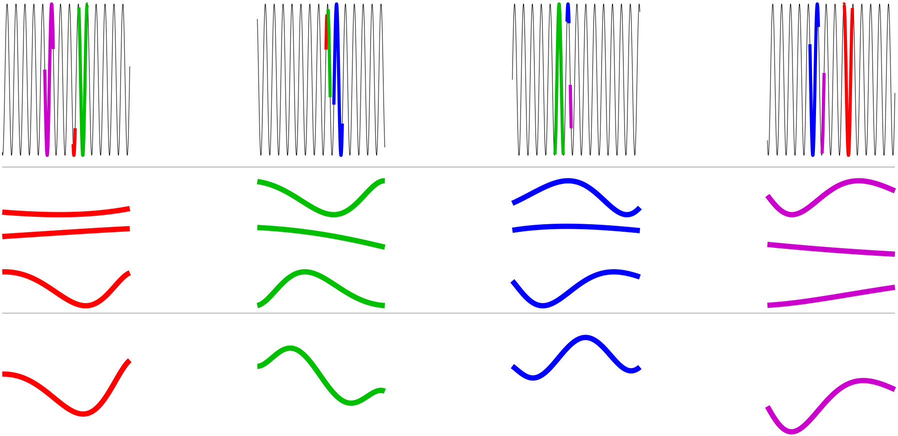
How fast does the solution $u = \mathcal{L}_s u$ oscillate?

Recall from Lecture 1 the picture for $\mathcal{L}_0 f(x) = \sum_{a \neq \bar{b}} f(\gamma_a(x))$, $x \in I_b$:



- $\mathcal{L}_s f$ oscillates less than f when s is bounded
- Thus for $u = \mathcal{L}_s u$ and s bounded, u should be very smooth

Now let us plot $\mathcal{L}_s f$ with $s = \alpha + \frac{i}{h}$, h small:

$$\mathcal{L}_s f(x) = \sum_{a \neq \bar{b}} (\gamma_a'(x))^s f(\gamma_a(x)), \quad x \in I_b$$

$u = \mathcal{L}_s u$ oscillates at frequencies $\lesssim h^{-1}$, owing to the factor $e^{\frac{i}{h} \log \gamma_a'(x)}$

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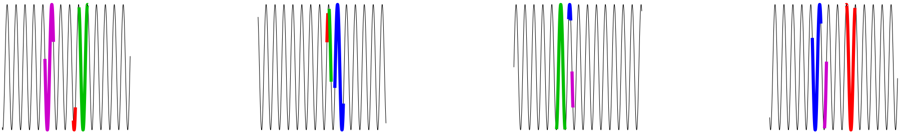

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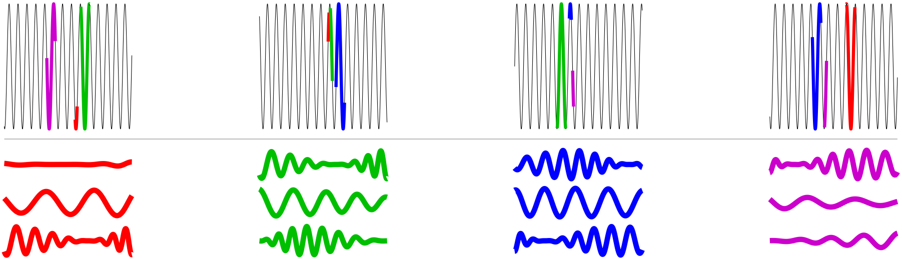

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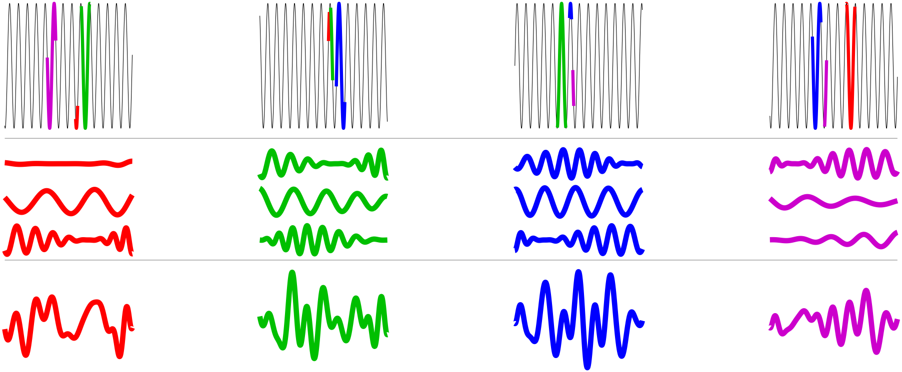

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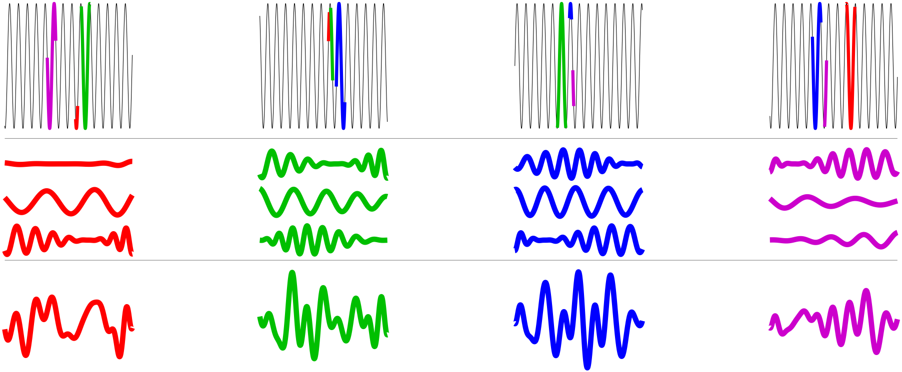

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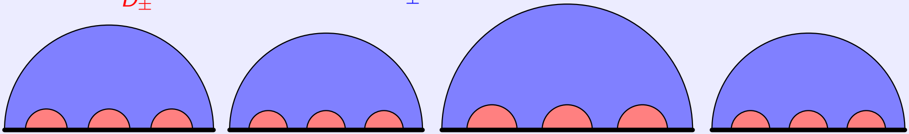
Getting a frequency bound

We prove that u oscillates at frequencies $\lesssim h^{-1}$, starting with

Lemma (Interpolated bound)

Let $D := \bigsqcup_{a\in\mathcal{A}} D_a \subset \mathbb{C}$, $\quad I := D \cap \mathbb{R}$, $\quad \widetilde{D} := \bigsqcup_{\mathbf{a}\in\mathcal{W}^2} D_{\mathbf{a}} \Subset D$,
$D_\pm := D \cap \{\pm \operatorname{Im} z > 0\}$, $\quad \widetilde{D}_\pm := \widetilde{D} \cap \{\pm \operatorname{Im} z > 0\}$. Then $\exists c > 0$:

$$\sup_{\widetilde{D}_\pm} |f| \leq \Big(\sup_I |f|\Big)^c \Big(\sup_{D_\pm} |f|\Big)^{1-c} \quad \text{for all} \quad f \in \mathcal{H}(D_\pm).$$



Proof

- Let $F_\pm : D_\pm \to [0,1]$ be harmonic with $F_\pm|_I \equiv 1$, $F_\pm|_{\partial D_\pm \setminus I} \equiv 0$.
- $\log|f| \leq (\log \sup_I |f|) F_\pm + (\log \sup_{D_\pm} |f|)(1 - F_\pm)$ since this is true on $\partial D_\pm$ and $\log|f|$ is subharmonic. Put $c := \min_{\widetilde{D}_\pm} F_\pm > 0$.

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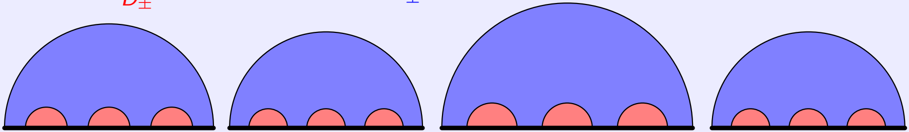

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For holomorphic functions, oscillating at frequencies $\leq L$ on $\mathbb{R}$ is roughly equivalent to being bounded by $e^{L|\operatorname{Im} z|}$ in $\mathbb{C}$. Define the weight

$$w_K(z) := e^{-K|\operatorname{Im} z|/h} \quad \text{where} \quad K = K(\Gamma) \gg 1.$$

Lemma (A priori bound in the complex)

Let $u \in \mathcal{H}(D)$, $u = \mathcal{L}_s u$, $s = \alpha + \frac{i}{h}$. Then $\sup_D |w_K u| \leq C \sup_I |u|$.

Proof

- Assume that $\sup_{D_b} |\gamma_a'| \leq \frac{1}{2}$ for all $a \neq \bar{b}$. (If not, use $\mathcal{L}_s^n u = u$ and that $|\gamma_{\mathbf{a}}'| \leq Ce^{-\theta n}$ for all $\mathbf{a} \in \mathcal{W}^n$.)
- For $z \in D_b$ and $a \neq \bar{b}$ we have $|\operatorname{Im} \gamma_a(z)| \leq \frac{1}{2}|\operatorname{Im} z|$. Now write

$$(w_K u)(z) = \sum_{a \neq \bar{b}} \frac{w_K(z)}{w_K(\gamma_a(z))} (\gamma_a'(z))^s (w_K u)(\gamma_a(z)).$$

For $K \gg 1$ get $\frac{w_K(z)}{w_K(\gamma_a(z))} |(\gamma_a'(z))^s| \leq C e^{-\frac{K|\operatorname{Im} z|}{2h}} e^{-\frac{\arg \gamma_a'(z)}{h}} \leq C$

- So $\sup_D |w_K u| \leq C \sup_{\tilde{D}} |w_K u| \leq C(\sup_I |u|)^c (\sup_D |w_K u|)^{1-c}$ where the second inequality follows from the interpolation bound

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- Recall: $u \in \mathcal{H}(D)$, $u = \mathcal{L}_s u$, and $\sup_D |e^{-K|\operatorname{Im} z|/h} u| \le C \sup_I |u|$
- Semiclassical Fourier transform: $\mathcal{F}_h f(\xi) = (2\pi h)^{-\frac{1}{2}} \widehat{f}(\xi/h)$

Lemma (Fourier localization to frequencies $\le 2K/h$)

Fix $\chi \in C_c^\infty(I)$. Then $\forall N$, $|\mathcal{F}_h(\chi u)(\xi)| \le C_N h^N |\xi|^{-N} \sup_I |u|$ for $|\xi| \ge 2K$.

In particular this implies $\sup |\chi u| \le C h^{-1/2} \|\chi u\|_{L^2} + C_N h^N \sup_I |u|$

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- Let $\widetilde{\chi} \in C_c^\infty(D)$ be an almost analytic extension of χ: $\widetilde{\chi}|_{\mathbb{R}} = \chi$, $|\bar\partial_z \widetilde{\chi}(z)| \le C_N |\operatorname{Im} z|^N$. By Green's Theorem on $D_- = D \cap \{\operatorname{Im} z < 0\}$

$$\widehat{\chi u}(\xi/h) = \int_{\partial D_-} u(z) e^{-\frac{i}{h} z\xi} \widetilde{\chi}(z)\, dz = \int_{\operatorname{Im} z < 0} u(z) e^{-\frac{i}{h} z\xi} \bar\partial_z \widetilde{\chi}(z)\, dz \wedge d\bar z$$

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- Let $\widetilde{\chi} \in C_c^\infty(D)$ be an almost analytic extension of χ: $\widetilde{\chi}|_{\mathbb{R}} = \chi$, $|\bar{\partial}_z \widetilde{\chi}(z)| \le C_N |\operatorname{Im} z|^N$. By Green's Theorem on $D_- = D \cap \{\operatorname{Im} z < 0\}$

$$\widehat{\chi u}(\xi/h) = \int_{\partial D_-} u(z) e^{-\frac{i}{h} z\xi} \widetilde{\chi}(z)\, dz = \int_{\operatorname{Im} z < 0} u(z) e^{-\frac{i}{h} z\xi} \bar{\partial}_z \widetilde{\chi}(z)\, dz \wedge d\bar{z}$$

- For $\xi \ge 2K$, bound $|u(z) e^{-\frac{i}{h} z\xi} \bar{\partial}_z \widetilde{\chi}(z)| \le C_N e^{-\frac{\xi |\operatorname{Im} z|}{h}} |\operatorname{Im} z|^N \sup_I |u|$ and integrate. For $\xi \le -2K$, integrate instead over $\{\operatorname{Im} z > 0\}$.

Large powers of transfer operators

Henceforth we only study u on $I = D \cap \mathbb{R}$.

Since $\mathcal{L}_s u = u$ we also have $\mathcal{L}_s^n u = u$ for all n, where

$$\mathcal{L}_s^n f(x) = \sum_{\mathbf{a} \in \mathcal{W}^n,\ \mathbf{a} \to b} (\gamma'_{\mathbf{a}}(x))^s f(\gamma_{\mathbf{a}}(x)), \quad x \in I_b$$

and $\mathbf{a} \to b$ means $b \neq \overline{a_n}$ where $\mathbf{a} = a_1 \dots a_n$.

Recalling that $s = \alpha + \frac{i}{h}$, rewrite this as

$$\mathcal{L}_s^n f(x) = \sum_{\mathbf{a} \in \mathcal{W}^n,\ \mathbf{a} \to b} (\gamma'_{\mathbf{a}}(x))^\alpha e^{\frac{i}{h}\varphi_{\mathbf{a}}(x)} f(\gamma_{\mathbf{a}}(x)), \quad x \in I_b$$

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Each term in the sum is obtained by the following three operations:

- **Composition** $u \mapsto \gamma_\mathbf{a}^* u$, where $\gamma_\mathbf{a}'(x) \sim |I_\mathbf{a}| \ll 1$ when $n \gg 1$. Since u oscillates at frequencies $\lesssim h^{-1}$, $\gamma_\mathbf{a}^* u$ oscillates at frequencies $\lesssim h^{-1}|I_\mathbf{a}|$.
- **Multiplication by weight** $v \mapsto (\gamma_\mathbf{a}')^\alpha v$ where $(\gamma_\mathbf{a}')^\alpha \sim |I_\mathbf{a}|^\alpha$. Does not change frequency localization much but changes the magnitude.
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We fix $\rho < 1$ close to 1 and choose n so that

$$|I_\mathbf{a}| \sim h^\rho \quad \text{for all} \quad \mathbf{a} \in \mathcal{W}^n$$

(Typically impossible, will discuss how to fix this at the end of the lecture.)

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- An elementary computation shows that up to an additive constant

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End of the proof: applying FUP

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To recap, we started with $u \in \mathcal{H}(D)$, $u = \mathcal{L}_s u = \mathcal{L}_s^n u$, $s = \alpha + \frac{i}{h}$ and got

$$u = h^\alpha \mathcal{B}_{\chi,h} w \text{ on } I, \quad \operatorname{supp} w \subset \Lambda_\Gamma(Ch), \quad \|w\|_{L^2} \sim h^{-\frac{1}{2}} \|u\|_{L^2(\Lambda_\Gamma(Ch))},$$

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Now the Fractal Uncertainty Principle gives

$$\| \mathbb{1}_{\Lambda_\Gamma(Ch)} \mathcal{B}_{\chi,h} \mathbb{1}_{\Lambda_\Gamma(Ch)} \|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \leq C h^\beta$$

so we estimate

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where we use that $\alpha > \frac{1}{2} - \beta$ and $h \ll 1$.
This gives $u|_{\Lambda_\Gamma(Ch)} = 0$ and thus $u = 0$, finishing the proof.

To recap, we started with $u \in \mathcal{H}(D)$, $u = \mathcal{L}_s u = \mathcal{L}_s^n u$, $s = \alpha + \frac{i}{h}$ and got

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Adapted transfer operator

- As remarked above, it is typically impossible to fix n such that $|I_{\mathbf{a}}| \sim h^\rho$ for all words $\mathbf{a}$ of length n
- So we instead consider the adapted partition

$$Z = Z(h^\rho) := \{\mathbf{a} \in \mathcal{W}^\circ : |I_{\mathbf{a}}| \le h^\rho < |I_{\mathbf{a}'}|\}$$

Note that $\Lambda_\Gamma = \bigsqcup_{\mathbf{a} \in Z} (\Lambda_\Gamma \cap I_{\mathbf{a}})$.

- If $\mathcal{L}_s u = u$ then $\mathcal{L}_{\overline{Z},s} u = u$ where $\overline{Z} := \{\overline{\mathbf{a}} \mid \mathbf{a} \in Z\}$,

$$\mathcal{L}_{\overline{Z},s} f(x) = \sum_{\mathbf{a} \in \overline{Z}, \mathbf{a} \rightsquigarrow b} (\gamma'_{\mathbf{a}'}(x))^s f(\gamma_{\mathbf{a}'}(x)), \quad x \in I_b,$$

for $\mathbf{a} = a_1 \dots a_n$, $\mathbf{a} \rightsquigarrow b$ means $a_n = b$, and $\mathbf{a}' := a_1 \dots a_{n-1}$

- Run the previous argument for this $\mathcal{L}_{\overline{Z},s}$, using that $\overline{Z}$ is an approximate partition (bounded overlap of $I_{\mathbf{a}}$, $\mathbf{a} \in \overline{Z}$) and $|I_{\overline{\mathbf{a}}}| \sim |I_{\mathbf{a}}|$

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Thank you for your attention!