What is quantum chaos?

Semyon Dyatlov (UC Berkeley/MIT/CMI)

November 16, 2017

The question: concentration of eigenfunctions

- $M \subset \mathbb{R}^n$ bounded domain
- $-\Delta \geq 0$ Dirichlet Laplacian on M
- A sequence of eigenfunctions:

$$(-\Delta - \lambda_j^2)u_j = 0, \quad \lambda_j \xrightarrow[j\to\infty]{} \infty, \quad \|u_j\|_{L^2(M)} = 1$$

Questions

- How does the mass $|u_j|^2$ concentrate as $j \to \infty$?
- Do $|u_j|^2$ equidistribute, i.e. $\int_\Omega |u_j|^2 \to \mathrm{vol}(\Omega)/\mathrm{vol}(M)$ for all $\Omega \subset M$?
- Can it happen that $\int_\Omega |u_j|^2 \to 0$ for some open nonempty $\Omega \subset M$?

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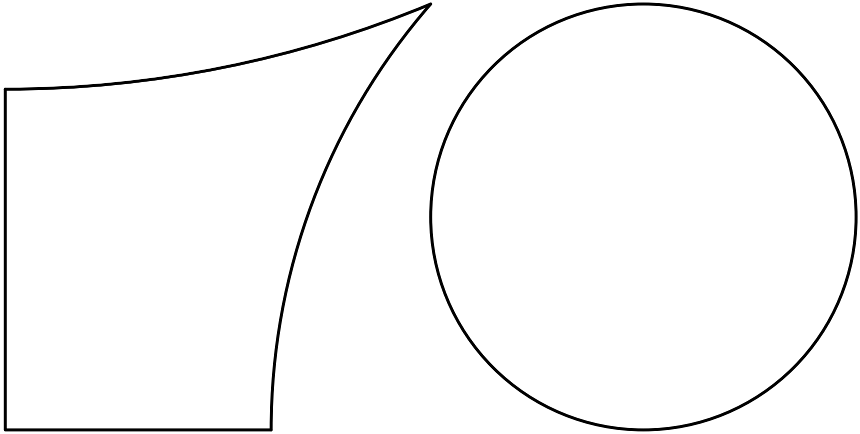
An example: two planar domains



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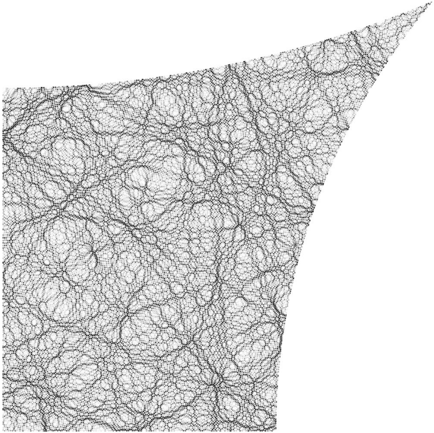
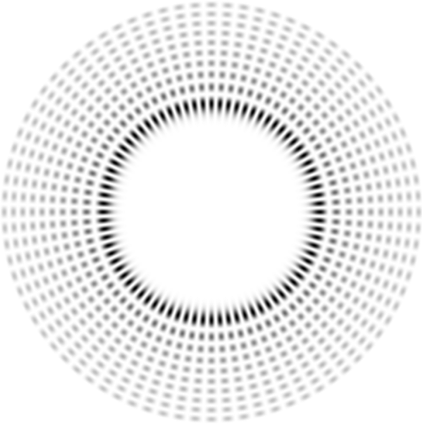
Eigenfunction concentration
(picture on the left by Alex Barnett)





Equidistribution

No equidistribution

An example: two planar domains

Billiard ball dynamics

Chaotic

Completely integrable

A preview of results

(M, g) compact hyperbolic surface (Gauss curvature $= -1$)

The geodesic flow $\varphi_t : S^*M \to S^*M$ is strongly chaotic (hyperbolic)

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Quantum Ergodicity [Shnirelman '74, Zelditch '87, Colin de Verdière '85]

There exists a density 1 sequence of (λ_j, u_j) such that u_j equidistribute:

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Arithmetic Quantum Unique Ergodicity [Lindenstrauss '06]

Assume $M = \Gamma\backslash\mathbb{H}^2$, $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$, is an arithmetic congruence compact hyperbolic surface (in particular $\{\mathrm{tr}\,\gamma : \gamma \in \Gamma\} \subset$ a finite extension of $\mathbb{Q}$). Then the entire sequence of (Hecke) eigenfunctions equidistributes.

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A recent result: lower bound on mass

Theorem [D–Jin '17]

Let M be any hyperbolic surface and $\Omega \subset M$ any open nonempty subset. Then there exist c_Ω depending on Ω such that for all j

$$\int_\Omega |u_j|^2 \, d\,\mathrm{vol}_g \geq c_\Omega > 0.$$

An application is observability for Schrödinger equation:

Theorem [Jin '17]

Let M be a hyperbolic surface, $\Omega \subset M$ a nonempty open set, and $T > 0$. Then there exists $C = C(\Omega, T)$ such that for all $f \in L^2(M)$

$$\int_M |f|^2 \, d\,\mathrm{vol}_g \leq C \int_0^T \int_\Omega |e^{it\Delta_g} f|^2 \, d\,\mathrm{vol}_g \, dt.$$

Previously observability by any open set only known for flat tori
Jaffard '90, Haraux '89, Anantharaman–Macia '14, Bourgain–Burq–Zworski '13

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Semiclassical analysis

Important tool: semiclassical pseudodifferential operators which can localize simultaneously in position and frequency. On $\mathbb{R}^n$:

- Localization in position: $u \mapsto au$, $a \in C^\infty(\mathbb{R}^n)$
- Localization in frequency: $u \mapsto a(D_x)u$, $\quad D_x = \frac{1}{i}\partial_x$
  Defined via the Fourier transform: $\mathcal{F}(a(D_x)u)(\xi) = a(\xi)\mathcal{F}u(\xi)$
- Localization in both: $u \mapsto a(x, D_x)u$, $a \in C^\infty(\mathbb{R}^{2n})$
- $a(x) = \sum_\alpha a_\alpha(x)\xi^\alpha \quad \Longrightarrow \quad a(x, D_x) = \sum_\alpha a_\alpha(x) D_x^\alpha$

On a manifold M, can define $a(x, D_x)$ for $a \in C^\infty(T^*M)$

Semiclassical quantization: introduce a small parameter h (wavelength)

$$a \in C^\infty(T^*M) \quad \mapsto \quad \mathrm{Op}_h(a) := a(x, hD_x) : C^\infty(M) \to C^\infty(M)$$

$(-\Delta_g - \lambda^2)u = 0 \quad \Rightarrow \quad u$ lives at frequencies $\sim \lambda \quad \Rightarrow \quad$ put $h := \lambda^{-1}$

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Classical-quantum correspondence

Classical	$\leftrightarrow$	Quantum
$a \in C^\infty(T^*M)$	$\leftrightarrow$	$\operatorname{Op}_h(a) : C^\infty(M) \to C^\infty(M)$
$\sup \lvert a \rvert < \infty$	$\leftrightarrow$	$\lVert \operatorname{Op}_h(a) \rVert_{L^2(M)\to L^2(M)} \leq C$
Product of symbols	$\leftrightarrow$	Composition of operators
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Semiclassical measures

Rescale $(-\Delta_g - \lambda_j^2)u_j = 0$ using $h_j := \lambda_j^{-1}$

$$(-h_j^2\Delta_g - 1)u_j = 0, \quad h_j \to 0, \quad \|u_j\|_{L^2(M)} = 1$$

Definition

We say that u_j converges (weakly) to a measure μ on T^*M if

$$\langle \mathrm{Op}_{h_j}(a)u_j, u_j\rangle_{L^2(M)} \to \int_{T^*M} a\,d\mu \quad \text{for all } a \in C^\infty(T^*M)$$

Semiclassical measures: weak limits of sequences of eigenfunctions

Note: $|u_j|^2\,d\,\mathrm{vol}_g \to \pi_*\mu$ weakly, $\pi : T^*M \to M$

Properties of semiclassical measures

- μ probability measure
- $\operatorname{supp}\mu$ contained in the cosphere bundle S^*M
- μ invariant under the geodesic flow φ_t

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Examples of φ_t-invariant probability measures

- μ_L the Liouville (volume) measure on S^*M: equidistribution
- δ_γ the delta measure on a closed geodesic: scarring
- and lots of options in between

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Quantum Ergodicity (QE)

Assume the geodesic flow φ_t is ergodic on S^*M, that is all flow-invariant sets $A \subset S^*M$ have Liouville measure $\mu_L(A) = 0$ or $\mu_L(A) = 1$.

Then there exists a density 1 sequence of eigenvalues of $-\Delta_g$ such that the corresponding u_j converge to μ_L.

Shnirelman '74, Zelditch '87, Colin de Verdière '85 ... Zelditch–Zworski '96

Ingredients of the proof

- Egorov's Theorem + equivariance of eigenfunctions under $U(t)$ $\Rightarrow$

  $\langle \mathrm{Op}_{h_j}(a)u_j, u_j\rangle = \langle \mathrm{Op}_{h_j}(\langle a\rangle_T)u_j, u_j\rangle + \mathcal{O}(h), \quad \langle a\rangle_T := \frac{1}{T}\int_0^T a\circ\varphi_t\, dt$
- L^2 ergodic theorem: $\langle a\rangle_T \to \int a\, d\mu_L$ on $L^2(S^*M)$
  $\Rightarrow \quad \langle \mathrm{Op}_{h_j}(\langle a\rangle_T)u_j, u_j\rangle \approx \int a\, d\mu_L$ for most eigenfunctions

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- Egorov's Theorem + equivariance of eigenfunctions under $U(t)$ $\Rightarrow$

$$\langle \mathrm{Op}_{h_j}(a)u_j, u_j\rangle = \langle \mathrm{Op}_{h_j}(\langle a\rangle_T)u_j, u_j\rangle + \mathcal{O}(h), \quad \langle a\rangle_T := \frac{1}{T}\int_0^T a\circ\varphi_t\, dt$$

- L^2 ergodic theorem: $\langle a\rangle_T \to \int a\, d\mu_L$ on $L^2(S^*M)$
  $\Rightarrow \quad \langle \mathrm{Op}_{h_j}(\langle a\rangle_T)u_j, u_j\rangle \approx \int a\, d\mu_L$ for most eigenfunctions

$$(-h_j^2\Delta_g - 1)u_j = 0, \quad h_j = \lambda_j^{-1} \to 0, \quad \|u_j\|_{L^2(M)} = 1$$

$$\langle \mathrm{Op}_{h_j}(a)u_j, u_j\rangle_{L^2(M)} \to \int_{T^*M} a\,d\mu \quad \forall a \in C^\infty(T^*M)$$

Quantum Ergodicity (QE)

Assume the geodesic flow φ_t is ergodic on S^*M, that is all flow-invariant sets $A \subset S^*M$ have Liouville measure $\mu_L(A) = 0$ or $\mu_L(A) = 1$.

Then there exists a density 1 sequence of eigenvalues of $-\Delta_g$ such that the corresponding u_j converge to μ_L.

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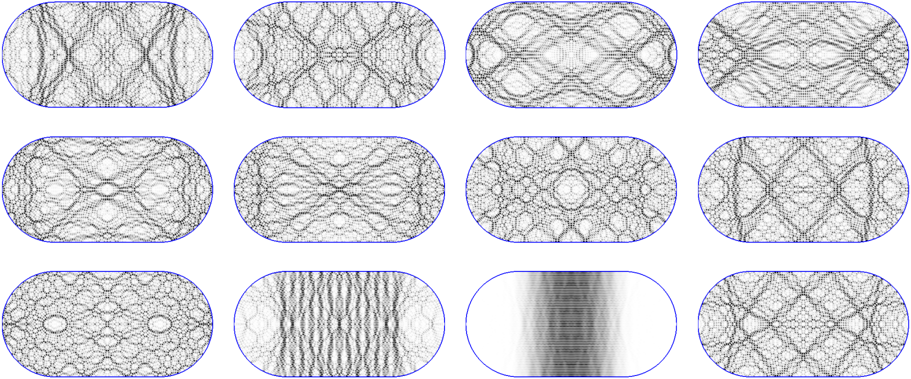
Example with ergodic billiard flow: Bunimovich stadium

Pictures by Alex Barnett



Theorem [Hassell '10]

For almost every choice of the parameter of the stadium there exists a sequence u_j which does not equidistribute

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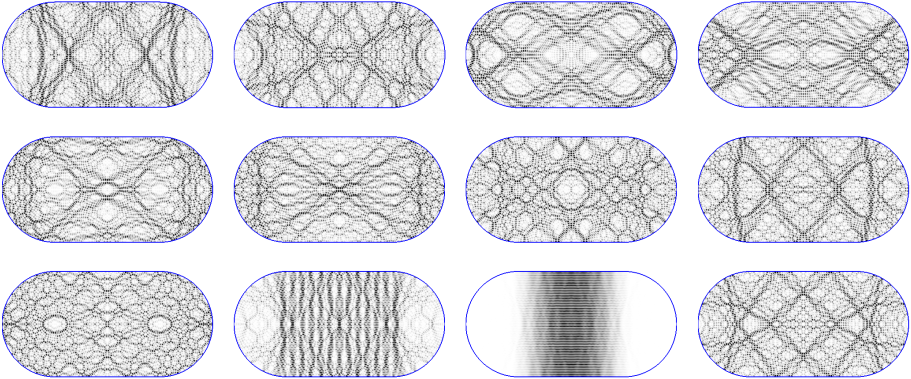

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Strongly chaotic systems

What if φ_t is hyperbolic, i.e. a small perturbation of the initial condition causes exponential divergence from the original trajectory?

Examples: surfaces of negative Gauss curvature and some billiards

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Quantum Unique Ergodicity (QUE) conjecture [Rudnick–Sarnak '94]

If φ_t is hyperbolic, then the whole sequence of eigenfunctions equidistributes, i.e. μ_L is the only semiclassical measure.

True for (some) arithmetic surfaces: Lindenstrauss '06, Soundararajan '10

The general case is still very much open. Counterexamples in toy models: Faure–Nonnenmacher–de Bièvre '03, Anantharaman–Nonnenmacher '07

Between QE and QUE

What can we say about semiclassical measures? E.g. can we get $\mu = \delta_\gamma$? Specialize to the case of (M, g) hyperbolic surface

Entropy bound [Anantharaman–Nonnenmacher '07]

Each semiclassical measure μ has Kolmogorov–Sinai entropy $H_{\mathrm{KS}}(\mu) \geq 1/2$

Note: $H_{\mathrm{KS}}(\mu_L) = 1$, $H_{\mathrm{KS}}(\delta_\gamma) = 0$. Known for more general hyperbolic φ_t with 1/2 replaced by certain number > 0
Anantharaman '08, Rivière '10, Anantharaman–Silberman '13

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Each semiclassical measure μ has $\operatorname{supp} \mu = S^*M$, that is $\mu(A) > 0$ for every open nonempty $A \subset S^*M$

Proof uses fractal uncertainty principle [D–Zahl '15, Bourgain–D '16]

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Fractal uncertainty principle

Definition

A set $X \subset [0,1]$ is ν-porous ($\nu > 0$) on scales h to 1 if for each interval I of size $h \le |I| \le 1$, there is an interval $J \subset I$ with $|J| = \nu|I|$ and $J \cap X = \emptyset$

Example: mid-third Cantor set $\mathcal{C} \subset [0,1]$ is $\frac{1}{18}$-porous on scales 0 to 1

Fractal uncertainty principle [Bourgain–D '16]

Assume that $X, Y \subset [0,1]$ are ν-porous on scales h to 1. Then there exist $\beta > 0, C$ depending on ν but not on X, Y, h such that

$$f \in L^2(\mathbb{R}), \quad \mathrm{supp}(\mathcal{F}_h f) \subset X \quad \Longrightarrow \quad \|f\|_{L^2(Y)} \le Ch^\beta \|f\|_{L^2(\mathbb{R})}.$$

Here $\mathcal{F}_h : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the semiclassical Fourier transform:

$$\mathcal{F}_h f(\xi) = \int_{\mathbb{R}} e^{-ix\xi/h} f(x)\, dx$$

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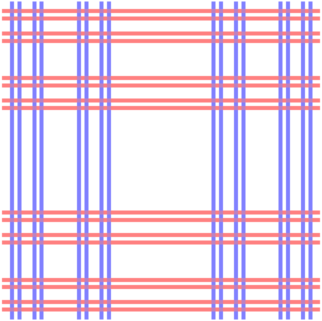
Interpretation: no quantum state can be localized on a porous set in both position and frequency



Thank you for your attention!