Notes on fractal uncertainty principle version 0.5 (October 4, 2017)

Semyon Dyatlov

E-mail address: dyatlov@math.mit.edu

Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Ave, Cambridge, MA 02139

Overview

These notes describe a recently developed approach to several problems in quantum chaos based on *fractal uncertainty principle* (henceforth called FUP). The current version is rough in many places and should be substantially updated before October 2017.

The notes are structured as follows:

- We first present some applications of FUP in quantum chaos:
  - in §1.1, to control of Laplacian eigenfunctions on compact hyperbolic surfaces (we also provide a brief overview of semiclassical analysis and its applications in quantum chaos);
  - in §1.2, to spectral gaps/wave decay for resonances on noncompact hyperbolic surfaces;
  - in §2 we explain informally the strategy of the proofs based on FUP.
- We next describe the FUP itself:
  - in §2.1 we state FUP of [**BD16, DJ17a**] and the needed definitions such as δ-regularity;
  - in §2.2 we introduce quantization of rough symbols and relate it to FUP.
- In §3 we describe how FUP gives control of eigenfunctions:
  - in 3.1 we review local properties of the geodesic flow, in particular introducing horocyclic vector fields $U_\pm$ and explaining how to extend Egorov's Theorem to large times (this section is also used in §4);
  - in §3.2 we sketch a proof of eigenfunction control (Theorem 1.1.3);
  - in §3.3 we sketch a proof of the fractal property needed for the FUP and finish the proof of Theorem 1.1.3.
- In §4 we describe how FUP gives a spectral gap:
  - in §4.1 we use propagation of singularities to obtain microlocalization statements for resonant states;
  - in §4.2 we discuss the fractal nature of the trapped set and finish the proof of Theorem 1.2.1.
- In §5 we present a complete proof of FUP in the special case of discrete Cantor sets:

  - in §5.1 we give the definition and basic properties of uncertainty principle in the discrete setting;
  - in §5.2 we show FUP for Cantor sets.

These notes contain some *Exercises* which (I hope) are solvable. There are also some *Problems* which (as far as I know) are open.

Here are some useful reading materials:

- [**Zw12**] for semiclassical/microlocal analysis, in particular semiclassical quantization, Egorov's Theorem, and semiclassical defect measures;
- [**DZ**[B]] for scattering theory, in particular meromorphic continuation of scattering resolvent in the Euclidean and hyperbolic cases and resonance expansions;
- [**Bo16**] for an introduction to the theory of convex co-compact hyperbolic surfaces, in particular their geometry, meromorphic continuation of the resolvent, and numerics concerning resonances;
- [**Ma06, Sa11, Ze09**] for closed quantum chaos, in particular the QUE conjecture;
- [**No11a, Zw16b**] for open quantum chaos, specifically spectral gaps and resonance counting.

Acknowledgements. This article originated as lecture notes for the minicourse on fractal uncertainty principle at the Third Symposium on Scattering and Spectral Theory in Florianopolis, Brazil, July 2017. These notes were written during the period the author served as a Clay Research Fellow.

CHAPTER 1

Overview

1.1. Control of eigenfunctions

We first present applications of FUP to closed quantum chaotic systems. We start with a simple-to-state theorem which preserves the spirit of more advanced results to follow. For us a *hyperbolic surface* is a connected two-dimensional complete oriented Riemannian manifold of constant Gauss curvature -1. Denote by $-\Delta \geq 0$ the corresponding Laplace–Beltrami operator.

THEOREM 1.1.1. [**DJ17b**] *Assume that (M, g) is a compact hyperbolic surface. Fix a nonempty open set $\Omega \subset M$. Then there exists a constant C_Ω such that for each eigenfunction*

$$u \in C^\infty(M), \quad (-\Delta - \lambda^2)u = 0 \quad \textit{for some } \lambda \geq 0 \tag{1.1.1}$$

we have the control estimate

$$\|u\|_{L^2(\Omega)} \geq \frac{1}{C_\Omega} \|u\|_{L^2(M)}. \tag{1.1.2}$$

Remark. For fixed λ the estimate (1.1.2) follows immediately from the unique continuation principle for elliptic operators [**HöIII**, §17.2]. The novelty of the result is that C_Ω does not depend on λ. Therefore Theorem 1.1.1 is a statement about the *high frequency limit* $\lambda \to \infty$.

EXERCISE 1.1.1. *Show that if $M = \mathbb{S}^2$ is the round sphere, then there exist Ω such that (1.1.2) does **not** hold. (Hint: use spherical harmonics concentrated on closed geodesics.)*

We do not explain the strategy of the proof of Theorem 1.1.1 yet. It will be presented in §**??**, and many ideas come from the spectral gap theorem whose proof is outlined at the end of §1.2.3. In the rest of this section we instead put Theorem 1.1.1 into the context of microlocal analysis and quantum chaos.

1.1.1. Semiclassical analysis of Laplacian eigenfunctions. Theorem 1.1.1 is a result in *quantum chaos*, which aims to understand the behavior of quantum objects (here: eigenfunctions of the Laplacian) in the high frequency limit when the underlying classical system (here: geodesic flow on M) has chaotic behavior. The setting of hyperbolic surfaces appears naturally because they are standard examples of manifolds with strongly chaotic (hyperbolic) geodesic flows. Here we explain some standard tools used in quantum chaos, which will be used throughout these lecture notes.

To explain the classical/quantum correspondence between the Laplacian and the geodesic flow, we introduce a *semiclassical quantization* Op_h (which works on an arbitrary manifold M). It maps each *symbol* $a(x, \xi)$ on the cotangent bundle T^*M to a *pseudodifferential operator* $\operatorname{Op}_h(a)$. The latter depends on the *semiclassical parameter* $h > 0$ which is the effective wavelength, and the high frequency limit corresponds to taking $h \to 0$. We can formally write $\operatorname{Op}_h(a) = a(x, \frac{h}{i}\partial_x)$. We refer the reader to [**Zw12**, §§4 and 14.2] and [**DZ**$^{\mathrm{B}}$, §E.1] for the definition of the quantization procedure, listing here only a few properties:

- if $a \in C_0^\infty(T^*M)$ (or more generally a is a bounded function and satisfies certain derivative bounds), then $\operatorname{Op}_h(a) : L^2(M) \to L^2(M)$ is bounded in norm uniformly in h;
- if $a(x, \xi) = a(x)$, then $\operatorname{Op}_h(a)$ is the multiplication operator by a;
- if $M = \mathbb{R}^n$ and $a(x, \xi) = \xi_j$, then $\operatorname{Op}_h(a) = \frac{h}{i}\partial_{x_j}$;
- $\operatorname{Op}_h(a)$ is independent of the choice of local coordinates on M modulo an $\mathcal{O}(h)$ remainder;
- the *product rule*:
$$\operatorname{Op}_h(a)\operatorname{Op}_h(b) = \operatorname{Op}_h(ab) + \mathcal{O}(h); \tag{1.1.3}$$
- the *adjoint rule*:
$$\operatorname{Op}_h(a)^* = \operatorname{Op}_h(\overline{a}) + \mathcal{O}(h); \tag{1.1.4}$$
- the *commutator rule*:
$$[\operatorname{Op}_h(a), \operatorname{Op}_h(b)] = -ih\operatorname{Op}_h(\{a, b\}) + \mathcal{O}(h^2) \tag{1.1.5}$$
where $\{\bullet, \bullet\}$ is the Poisson bracket: $\{a, b\} = H_a b$ where $H_a = \sum_j (\partial_{\xi_j} a)\partial_{x_j} - (\partial_{x_j} a)\partial_{\xi_j}$ is the *Hamiltonian vector field* of a.
- the *elliptic estimate*: if $a \in C_0^\infty(T^*M)$ and b are symbols and $\operatorname{supp} a \subset \{b \neq 0\}$, then for all $u \in L^2(M)$ and $h \in (0, 1)$
$$\|\operatorname{Op}_h(a)u\|_{L^2} \leq C\|\operatorname{Op}_h(b)u\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2}. \tag{1.1.6}$$
Here $\mathcal{O}(h^\infty)$ denotes a function which decays faster than any power of h. To prove (1.1.6) with $\mathcal{O}(h)$ remainder, we use the symbol $a/b \in C_0^\infty(T^*M)$ and the product rule to write
$$\operatorname{Op}_h(a)u = \operatorname{Op}_h(a/b)\operatorname{Op}_h(b)u + \mathcal{O}(h)\|u\|_{L^2}.$$
The $\mathcal{O}(h^\infty)$ remainder is obtained by iteration, see [**DZ**$^{\mathrm{B}}$, §E.2.2].

We now explore basic applications of semiclassical quantization to analysis of high frequency eigenfunctions. Let M be a compact Riemannian manifold and $-\Delta \geq 0$ be the Laplace–Beltrami operator. We semiclassically rescale the latter:

$$-h^2\Delta = \mathrm{Op}_h(p^2) + \mathcal{O}(h); \quad p(x,\xi) = |\xi|_g.$$

If u solves the eigenfunction equation (1.1.1) and λ is large, then

$$(-h^2\Delta - 1)u = 0 \tag{1.1.7}$$

where $h := \lambda^{-1}$ is small. Then the elliptic estimate (1.1.6) with $b := p^2 - 1$ gives the following statement, roughly saying that solutions to (1.1.7) oscillate at frequency h^{-1}:

PROPOSITION 1.1.2 (Energy localization). *Assume that* $\operatorname{supp} a \cap S^*M = \emptyset$ *where*

$$S^*M := \{(x,\xi) \in T^*M : |\xi|_g = 1\}$$

is the cosphere bundle. Then for any solution u to (1.1.7) we have

$$\|\mathrm{Op}_h(a)u\|_{L^2} = \mathcal{O}(h^\infty)\|u\|_{L^2}. \tag{1.1.8}$$

To relate solutions of (1.1.7) to the geodesic flow we use *Egorov's Theorem.* It relates the homogeneous geodesic flow of g, viewed as the Hamiltonian flow

$$\varphi_t = \exp(tH_p) : T^*M \setminus 0 \to T^*M \setminus 0, \quad T^*M \setminus 0 := \{(x,\xi) \in T^*M \mid \xi \neq 0\} \tag{1.1.9}$$

to the unitary (half-)wave group

$$U(t) = e^{-it\sqrt{-\Delta}} = e^{-it\sqrt{-h^2\Delta}/h} : L^2(M) \to L^2(M).$$

PROPOSITION 1.1.3 (Egorov's Theorem). *Let $a \in C_0^\infty(T^*M \setminus 0)$ and t be bounded independently of h. Then*

$$U(-t)\,\mathrm{Op}_h(a)U(t) = \mathrm{Op}_h(a \circ \varphi_t) + \mathcal{O}(h)_{L^2 \to L^2}. \tag{1.1.10}$$

Remark. One often needs a more advanced version, which can be proved iteratively: there exists an h-dependent symbol $b_t(x,\xi;h)$ having an asymptotic expansion in natural powers of h, such that $b_t = a \circ \varphi_t + \mathcal{O}(h)$, $\operatorname{supp} b \subset \varphi_{-t}(\operatorname{supp} a)$, and (1.1.10) holds with remainder $\mathcal{O}(h^\infty)_{L^2 \to L^2}$ and $a \circ \varphi_t$ replaced by b_t. See [**DG17**, Lemma 2.3].

PROOF. Denote $a_t := a \circ \varphi_t$. Since $U(t)$ is unitary it suffices to show that

$$U(t)\,\mathrm{Op}_h(a_t)U(-t) = \mathrm{Op}_h(a) + \mathcal{O}(h)_{L^2 \to L^2}.$$

This is obviously true for $t = 0$. We then differentiate the left-hand side in t and get by the commutator rule (using that $\sqrt{-h^2\Delta} = \mathrm{Op}_h(p) + \mathcal{O}(h)$ microlocally on $T^*M \setminus 0$)

$$\begin{aligned} \partial_t\big(U(t)\,\mathrm{Op}_h(a_t)U(-t)\big) &= U(t)\big(\mathrm{Op}_h(\partial_t a_t) - i[\sqrt{-\Delta}, \mathrm{Op}_h(a_t)]\big)U(-t) \\ &= U(t)\,\mathrm{Op}_h\big(\partial_t a_t - \{p, a_t\}\big)U(-t) + \mathcal{O}(h)_{L^2 \to L^2} \end{aligned} \tag{1.1.11}$$

which is $\mathcal{O}(h)$ since $\partial_t a_t = H_p a_t = \{p, a_t\}$. □

As an application of the techniques discussed above we give a proof of Theorem 1.1.1 in the simple case when Ω satisfies an additional condition:

THEOREM 1.1.2. *Let (M, g) be any Riemannian manifold and let $\Omega \subset M$ be an open set satisfying the* ***geometric control condition****:*

$$\text{there exists } T > 0 \text{ such that each geodesic segment of length } T \text{ intersects } \Omega. \tag{1.1.12}$$

Then there exists a constant C_Ω such that we have

$$\|u\|_{L^2(M)} \leq C_\Omega \|u\|_{L^2(\Omega)} \quad \textit{when} \quad (-h^2\Delta - 1)u = 0. \tag{1.1.13}$$

PROOF. 1. We henceforth assume that $(-h^2\Delta - 1)u = 0$. We first prove the following microlocal control statement: for each $(x_0, \xi_0) \in S^*M$ there exists a neighborhood $U \subset T^*M$ of (x_0, ξ_0) such that

$$\|\operatorname{Op}_h(a)u\|_{L^2} \leq C\|u\|_{L^2(\Omega)} + \mathcal{O}(h)\|u\|_{L^2(M)} \quad \text{when} \quad \operatorname{supp} a \subset U \tag{1.1.14}$$

where the constant C depends on a but not on h.

By the geometric control condition, there exists t such that $\varphi_{-t}(x_0, \xi_0) \in \Omega$. If U is sufficiently small, then there exists $b(x) \in C^\infty(M)$ such that

- $\operatorname{supp} b \subset \Omega$, and
- $\operatorname{supp}(a \circ \varphi_t) \subset \varphi_{-t}(U) \subset \{b \neq 0\}$.

We then have

$$\begin{aligned} \|\operatorname{Op}_h(a)u\|_{L^2} &= \|U(-t)\operatorname{Op}_h(a)U(t)u\|_{L^2} \\ &= \|\operatorname{Op}_h(a \circ \varphi_t)u\|_{L^2} + \mathcal{O}(h)\|u\|_{L^2} \\ &\leq C\|\operatorname{Op}_h(b)u\|_{L^2} + \mathcal{O}(h)\|u\|_{L^2} \\ &\leq C\|u\|_{L^2(\Omega)} + \mathcal{O}(h)\|u\|_{L^2}. \end{aligned}$$

Here the first equality holds since $U(t)u = e^{-it/h}u$, a consequence of the equation $(-h^2\Delta - 1)u = 0$. The second equality uses Egorov's Theorem (1.1.10), and the next inequality follows from the elliptic estimate (1.1.6). The last inequality uses the fact that $\operatorname{Op}_h(b)u = bu$.

2. Taking a partition of unity subordinate to the covering of S^*M by the sets $U(x_0, \xi_0)$, we see that (1.1.14) holds for some $a \in C_0^\infty(T^*M)$ such that $a \equiv 1$ near S^*M. Recalling that $u = \operatorname{Op}_h(a)u + \operatorname{Op}_h(1 - a)u$ and estimating $\|\operatorname{Op}_h(1 - a)u\|$ by the localization bound (1.1.8), we get

$$\|u\|_{L^2(M)} \leq C\|u\|_{L^2(\Omega)} + \mathcal{O}(h)\|u\|_{L^2(M)}.$$

This gives (1.1.13) for sufficiently small h. The case of bounded h follows from the unique continuation principle. □

Of course the geometric control condition is not satisfied for many choices of Ω (for instance a set which misses one closed geodesic), explaining the need for the more advanced technology developed in the rest of these notes. In particular we will need to take propagation time t which grows as $h \to 0$.

1.1.2. More precise control and application to Schrödinger equation. Armed with semiclassical quantization, we state a more advanced version of Theorem 1.1.1:

THEOREM 1.1.3. [**DJ17b**] *Let M be a compact hyperbolic surface and fix $a \in C_0^\infty(T^*M)$ such that $a|_{S^*M} \not\equiv 0$. Then there exist $C, h_0 > 0$ depending on a such that for all $u \in H^2(M)$ and $0 < h < h_0$*

$$\|u\|_{L^2} \leq C\|\operatorname{Op}_h(a)u\|_{L^2} + \frac{C\log(1/h)}{h}\|(-h^2\Delta - 1)u\|_{L^2}. \tag{1.1.15}$$

Remarks. 1. Theorem 1.1.3 is stronger than Theorem 1.1.1 in two ways: the operator $\operatorname{Op}_h(a)$ allows for localization in frequency in addition to position, and the equation $(-h^2\Delta - 1)u = 0$ only needs to hold approximately.

2. The condition $a|_{S^*M} \not\equiv 0$ is sharp; indeed, if $a|_{S^*M} \equiv 0$ and u solves (1.1.7), then $\|\operatorname{Op}_h(a)u\|_{L^2} = \mathcal{O}(h)\|u\|_{L^2}$ similarly to (1.1.8).

EXERCISE 1.1.4. *Show that Theorem 1.1.3 implies Theorem 1.1.1.*

EXERCISE 1.1.5. *Assume that the set $\{a \neq 0\}$ satisfies the natural generalization of the geometric control condition (1.1.12). Show that Theorem 1.1.3 holds, in fact one can remove the $\log(1/h)$ factor in (1.1.15).*

PROBLEM 1.1.6. *Show that Theorem 1.1.3 holds when M is a Riemannian surface of variable negative curvature.*

Theorem 1.1.3 has a natural application to the time-dependent Schrödinger equation:

THEOREM 1.1.4. [**Ji17**] *Let M be a compact hyperbolic surface and $\Omega \subset M$ be a nonempty open set. Fix $T > 0$. Then there exists a constant C such that the following* ***observability estimate*** *holds for all $u \in L^2(M)$:*

$$\|u\|^2_{L^2(M)} \leq C\int_0^T \|e^{it\Delta}u\|^2_{L^2(\Omega)}\,dt. \tag{1.1.16}$$

The only other manifolds for which observability estimate is known to hold for *any* nonempty open set are flat tori; see [**Ji17**] for the history of the subject.

1.1.3. Semiclassical defect measures. We finally apply Theorem 1.1.3 to semiclassical defect measures, defined as follows:

DEFINITION 1.1.7. *Assume (M, g) is a compact Riemannian manifold and we are given a high frequency sequence of L^2-normalized eigenfunctions:*

$$u_j \in C^\infty(M), \quad (-h_j^2\Delta - 1)u_j = 0, \quad \|u_j\|_{L^2} = 1, \quad h_j \to 0. \tag{1.1.17}$$

We say that u_j converges weakly to a measure μ on T^*M, if

$$\langle \operatorname{Op}_{h_j}(a)u_j, u_j\rangle_{L^2} \to \int_{T^*M} a\, d\mu \quad \text{for all } a \in C_0^\infty(T^*M). \tag{1.1.18}$$

*We call μ a **semiclassical (defect) measure** if it arises as the weak limit of a sequence of eigenfunctions.*

We list a few standard properties of semiclassical measures, see [**Zw12**, Chapter 5]:

- each sequence satisfying (1.1.17) has a subsequence converging weakly to some measure;
- each semiclassical measure μ is a probability measure and $\operatorname{supp}\mu \subset S^*M$;
- each semiclassical measure μ is invariant under the geodesic flow (1.1.9), i.e. $\mu(\varphi_t(\mathcal{A})) = \mu(\mathcal{A})$ for all $\mathcal{A} \subset S^*M$.

There are plenty of φ_t-invariant measures on S^*M. The extreme possibilities are:

- $\mu = \mu_L$, the Liouville measure, which has a smooth density and is naturally induced by the metric g [**Zw12**, §15.1]. If a sequence u_j converges to μ_L, then we say that it *equidistributes*;
- $\mu = \delta_\gamma$, the delta measure supported on a closed geodesic γ. If u_j converges to δ_γ, then we say that it *scars*.

The application of Theorem 1.1.3 to semiclassical measures is

THEOREM 1.1.5. [**DJ17b**] *Let μ be a semiclassical defect measure on a hyperbolic surface M. Then the support of μ is equal to S^*M, that is $\mu(\mathcal{A}) > 0$ for any nonempty open subset $\mathcal{A} \subset S^*M$.*

EXERCISE 1.1.8. *Show that Theorem 1.1.5 follows from Theorem 1.1.3. (Hint: use that $\|\operatorname{Op}_{h_j}(a)u_j\|_{L^2}^2$ converges to $\int |a|^2\, d\mu$ for all $a \in C_0^\infty(T^*M)$.)*

EXERCISE 1.1.9. *Construct examples of eigenfunctions convering to measures supported on proper submanifolds of S^*M in the case when M is the round sphere or the flat torus.*

The study of semiclassical measures on Riemannian manifolds with chaotic geodesic flows has a rich history:

- *Quantum ergodicity* [**Sh74, Ze87, CdV85, HMR87, ZZ96**]: if φ_t is ergodic (mildly chaotic) with respect to μ_L, then there is a *density 1 sequence* of u_j converging to μ_L (i.e. most eigenfunctions equidistribute). The proof is not long and based on Egorov's Theorem, the L^2 ergodic theorem, and a trace formula (relating averaged behavior of the expressions $\langle \mathrm{Op}_h(a)u_j, u_j\rangle$ over many eigenfunctions to the integral of a), see [**Zw12**, Chapter 15].
- [**Ha10**] (see also [**Do03**]) gives examples of manifolds with ergodic geodesic flows which have sequences of eigenfunctions that do not equidistribute.
- *Quantum unique ergodicity (QUE)* conjecture [**RS94**]: if M is strongly chaotic (e.g. has negative curvature) then μ_L is the only semiclassical defect measure, i.e. the entire sequence of eigenfunctions equidistributes.
- QUE has been proved for the special case of Hecke forms on arithmetic hyperbolic surfaces in [**Li06**], see also [**So10**]. The proofs use strongly the additional infinite family of symmetries given by the Hecke operators.
- *Entropy bounds* [**An08, AN07, Ri10a, Ri10b, AS13**]: if φ_t is Anosov (e.g. M is negatively curved), then there is a lower bound on the *Kolmogorov–Sinai entropy* H_{KS} of every semiclassical measure. In particular if M is a hyperbolic surface as in Theorem 1.1.5, then the entropy bound is [**AN07**]
$$H_{\mathrm{KS}}(\mu) \geq 1/2. \tag{1.1.19}$$
Here we have $H_{\mathrm{KS}}(\mu_L) = 1$ and $H_{\mathrm{KS}}(\delta_\gamma) = 0$.
- In (1.1.17) we may replace the equality $(-h_j^2\Delta - 1)u_j = 0$ with the bound $\|(-h_j^2\Delta - 1)u_j\|_{L^2} = o(h_j/\log(1/h_j))$, i.e. u_j needs to be an $o(h/\log(1/h))$ *quasimode*. The entropy bounds discussed above also apply to quasimodes of this strength. On the other hand [**Br15, EN15, ES16**] construct examples of $\mathcal{O}(h/\log(1/h))$ quasimodes with anomalous concentration. In particular [**EN15**, Proposition 1.9] gives such quasimodes with the limiting measure equal to δ_γ for any given closed geodesic γ.

The support property of Theorem 1.1.5 is in some sense orthogonal to the entropy bound (1.1.19):

- Both results exclude the case $\mu = \delta_\gamma$;
- The bound (1.1.19) excludes convex combinations $\mu = \alpha\mu_L + (1-\alpha)\delta_\gamma$ with $0 < \alpha < 1/2$, which however have full support;
- On the other hand, one can construct invariant probability measures μ on S^*M such that $\operatorname{supp}\mu$ is a (typically fractal) proper closed subset of S^*M and the entropy of μ is arbitrarily close to 1. For instance, if we fix a geodesic γ_0 on M, then the set of geodesics on M which never cross γ_0 has a natural invariant measure μ of entropy $\delta \in (0,1)$. Indeed, this set is the trapped set of the convex co-compact surface obtained from M by cutting along γ_0

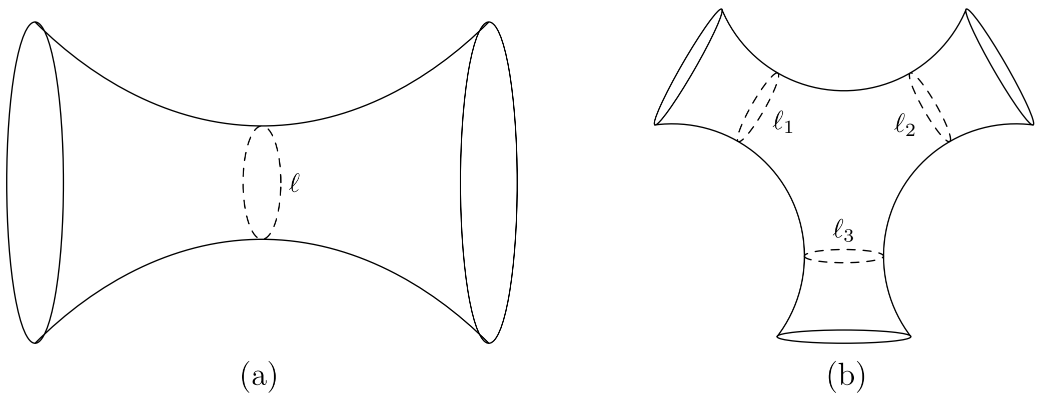


FIGURE 1. Examples of convex co-compact hyperbolic surfaces: (a) hyperbolic cylinder with neck length ℓ, obtained by gluing together two funnels, (b) three-funnel surface with neck lengths ℓ_1, ℓ_2, ℓ_3, obtained by gluing funnels to a pair of pants.

and attaching two funnel ends, and the Patterson–Sullivan measure induces a natural φ_t-invariant measure on this trapped set, see [**Bo16**, §14.2]. We may choose M and γ_0 so that the entropy δ is arbitrarily close to 1.

1.2. Spectral gaps

We next present applications of fractal uncertainty principle to the spectral gap problem in open quantum chaos. We still work on hyperbolic surfaces, but now they will be *noncompact*, which allows for energy of solutions to the wave equation to escape to spatial infinity. Resonances, defined below, in some sense describe the part of the wave that does not escape.

1.2.1. Resonances. Let (M, g) be a *convex co-compact hyperbolic surface*, that is a noncompact hyperbolic surface with infinite ends which are *funnels* of the form

$$[0,\infty)_r \times \mathbb{S}^1_\theta, \quad \mathbb{S}^1_\theta = \mathbb{R}/(\ell\mathbb{Z}), \quad \ell > 0; \quad g = dr^2 + \cosh^2 r d\theta^2.$$

The complement of all infinite ends, called the *convex core*, is a compact hyperbolic surface with geodesic boundary. See the book [**Bo16**] for the geometry and scattering theory on such surfaces. A couple of examples are shown on Figure 1. As before, denote by $-\Delta \geq 0$ the Laplace–Beltrami operator on M, which has a natural extension to a self-adjoint operator on $L^2(M)$. This operator has essential spectrum $[\frac{1}{4}, \infty)$ as opposed to the Euclidean Laplacian which has essential spectrum $[0, \infty)$. This shift by $\frac{1}{4}$, due to the exponential growth of volumes of large balls, explains the shift of the spectral parameter below.

The *scattering resolvent* $R(\lambda)$ is a left inverse to $-\Delta - \frac{1}{4} - \lambda^2$ defined as follows:

- for $\operatorname{Im}\lambda > 0$, $R(\lambda)$ is the L^2 resolvent given by spectral theory;
- for general λ, $R(\lambda)$ is obtained by meromorphic continuation and maps L^2_{comp} (the space of compactly supported L^2 functions) to H^2_{loc} (the space of locally H^2 functions). Such continuations were proved in [**MM87, GuZw95a, Gu05, Va13a, Va13b**]. See [**Bo16**, Theorem 6.8] for a direct proof of meromorphic continuation in the constant curvature case and [**Zw16a**] and [**DZ**$^{\mathrm{B}}$, Chapter 5] for the more general case of even asymptotically hyperbolic manifolds using the recent approach of [**Va13a, Va13b**].

Resonances are defined as the poles of $R(\lambda)$. They are natural generalizations of the discrete spectrum of the Laplacian to noncompact manifolds. Resonances are intimately related to decay properties of solutions to the wave equation

$$\begin{gathered}\Big(\partial_t^2-\Delta-\frac{1}{4}\Big)u(t,x)=0,\quad t\geq 0,\ x\in M;\\ u|_{t=0}=f_0\in C_0^\infty(M),\quad \partial_t u|_{t=0}=f_1\in C_0^\infty(M)\end{gathered}\tag{1.2.1}$$

in particular to *resonance expansions* (presented here in the case when there is no algebraic multiplicity)

$$u(t,x)=\sum_{\substack{\lambda_j\text{ resonance}\\ \operatorname{Im}\lambda_j\geq-\nu}} e^{-it\lambda_j}v_j(x)+\mathcal{O}(e^{-\nu t}).\tag{1.2.2}$$

Here $\nu>0$ is some number and the remainder is in the space $H^s_{\mathrm{loc}}(M)$ for all s (i.e. the remainder bound is only valid on compact subsets of M). The functions $v_j(x)\in C^\infty(M)$ depend on f_0,f_1 but the resonances λ_j do not. We refer the reader to [**DZ**$^{\mathrm{B}}$, §3.2] for a detailed presentation of meromorphic continuation and resonance expansion in the simpler case of Euclidean potential scattering.

The real part of a resonance λ_j corresponds to the rate of oscillation of the corresponding term $e^{-it\lambda_j}v_j(x)$, and its (negative) imaginary part gives the rate of exponential decay. In particular, the leading part of (1.2.2) is generically given by resonances with the largest imaginary parts.

To explain why resonances, defined as the poles of the scattering resolvent, appear in the resonance expansion (1.2.2), we Fourier transform the solution u:

$$\widehat{u}(\lambda):=\int_0^\infty e^{i\lambda t}u(t)\,dt\in H^2(M),\quad \operatorname{Im}\lambda\gg 1.\tag{1.2.3}$$

Here the integral converges exponentially. The wave equation (1.2.1) gives

$$\Big(-\Delta-\frac{1}{4}-\lambda^2\Big)\widehat{u}(\lambda)=f_1-i\lambda f_0.\tag{1.2.4}$$

This gives the formula for $\widehat{u}$,

$$\widehat{u}(\lambda)=R(\lambda)(f_1-i\lambda f_0)\tag{1.2.5}$$

where the right-hand side makes sense when $\lambda \in \mathbb{C}$ as a meromorphic family of functions in $H^2_{\mathrm{loc}}(M)$, even though the integral (1.2.3) may diverge when $\operatorname{Im}\lambda \le 0$.

Imagine that u did satisfy the resonance expansion (1.2.2) for some discrete set of frequencies λ_j, and there were only finitely many terms in the expansion. If $u = \mathcal{O}(e^{-\nu t})$, then the integral (1.2.3) converges when $\operatorname{Im}\lambda > -\nu$ and gives a holomorphic function of λ. Using the formula

$$\int_0^\infty e^{-i\lambda_j t} e^{i\lambda t}\, dt = \frac{i}{\lambda - \lambda_j}$$

we see that $\widehat{u}$ extends meromorphically to $\{\operatorname{Im}\lambda > -\nu\}$ with poles which are exactly given by λ_j. Thus by (1.2.5) we see that the frequencies λ_j in the resonance expansion have to be poles of the scattering resolvent.

We next indicate how to prove the resonance expansion (1.2.2) from meromorphic continuation of $R(\lambda)$. By Fourier inversion formula (applied to $e^{-\nu_0 t}u(t)$) we have for some large constant ν_0

$$u(t,x) = \frac{1}{2\pi}\int_{\operatorname{Im}\lambda=\nu_0} e^{-it\lambda} R(\lambda)(f_1 - i\lambda f_0)(x)\, d\lambda. \tag{1.2.6}$$

We then deform the contour of integration to $\{\operatorname{Im}\lambda = -\nu\}$ using the residue theorem in the strip $\{-\nu \le \operatorname{Im}\lambda \le \nu_0\}$. The residues give the terms in the resonance expansion and the integral over $\{\operatorname{Im}\lambda = -\nu\}$ is $\mathcal{O}(e^{-\nu t})$.

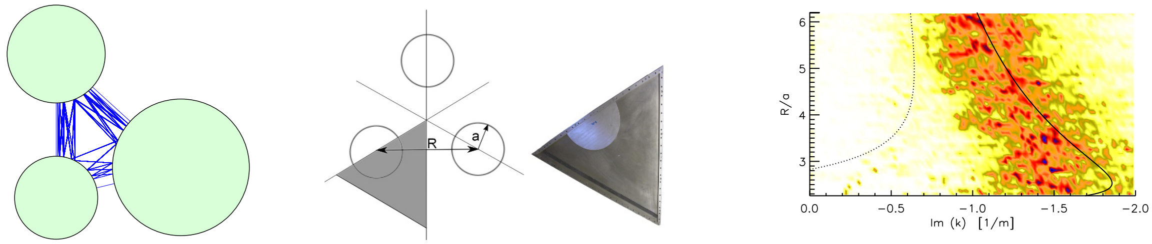
1.2.2. Essential spectral gaps. There is a very important caveat in the above 'proof' of the resonance expansion: the strip $\{-\nu \le \operatorname{Im}\lambda \le \nu_0\}$ is not compact. Therefore to justify the residue theorem and to show decay of the remainder, we need to bound $\widehat{u}(\lambda)$ in this strip when $|\operatorname{Re}\lambda| \to \infty$. Such bound is given by

DEFINITION 1.2.1. *We say that M has an* ***essential spectral gap*** *of size $\beta \ge 0$ if:*

1. *the half-plane $\{\operatorname{Im}\lambda \ge -\beta\}$ contains only finitely many resonances, and*
2. *the cutoff resolvent $\chi R(\lambda)\chi$, where $\chi \in C_0^\infty(M)$ is arbitrary, is bounded in $L^2 \to L^2$ norm in this half-plane away from resonances by some power of $|\lambda|$.*

If M has an essential spectral gap of size β, then the contour deformation argument sketched following (1.2.6) gives the resonance expansion (1.2.2) for any $\nu \le \beta$. In particular if $\nu > 0$ and there are no resonances with $\operatorname{Im}\lambda \ge -\nu$, this gives *exponential local energy decay of waves* $\mathcal{O}(e^{-\nu t})$. Such local energy decay statements (on more general manifolds) have plenty of applications to linear and nonlinear differential equations, in particular

- local smoothing estimates [**Da09**];
- Strichartz estimates [**BGH10, Wa17**];



(a) (b) (c)

Figure 2. (a) An example of an open classical chaotic system, the billiard ball flow on the exterior of three disks. The trapped trajectories form a fractal set. (b) Experimental setup for microwave scattering on three disks [**B+13**]. (c) Experimental data [**B+13**, Figure 4(b)] for the latter system. The vertical direction is the parameter of the system and the horizontal direction is the density plot of decay rates of resonances. The dotted line is the pressure gap. The solid line is half the classical escape rate, see [**TODO**].

- asymptotic stability of nonlinear wave equations, see in particular the recent proof of the full nonlinear stability of the Kerr–de Sitter black hole spacetime [**HV16**] which uses meromorphic continuation of the resolvent and an essential spectral gap.

At this point the reader may wonder why we care about waves and resonances in the particular setting of convex co-compact hyperbolic surfaces. Here are several reasons:

- Hyperbolic surfaces (compact and noncompact) are classical examples of manifolds with strongly chaotic geodesic flows. Thus eigenvalues (in the compact case) and resonances (in the noncompact case) of these surfaces are an important model the more general closed and open quantum chaotic systems. An example of a 'real-world' open quantum chaotic system, scattering by several obstacles, is presented on Figure 2.

  Compared to more general chaotic systems, hyperbolic surfaces have several features that make the dynamics less technically complicated to handle. In particular, they have no boundary, the stable/unstable spaces vary smoothly with the base point, and the expansion rate of the flow is constant.
- Resonances for hyperbolic surfaces are related to zeroes of the *Selberg zeta function* [**Bo16**, Chapter 10]. As such they give information about the asymptotics of the counting function of lengths of closed geodesics on M (similarly to how zeroes of the Riemann zeta function tell us about distribution of prime

numbers). A modified version of the essential spectral gap implies exponential remainders in the prime geodesic theorem, see [**Na05**] and [**Bo16**, §14.6].

- Any hyperbolic surface M is the quotient of the hyperbolic plane by a discrete group of Möbius transformations $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$. For specific choices of Γ (typically as a subgroup of $\mathrm{SL}(2,\mathbb{Z})$), resonances give information on counting solutions to diophantine equations, with deep applications in number theory. See the review [**Sa13**].

To each hyperbolic surface is associated a parameter $\delta \in [0,1]$. This parameter has many interpretations, in particular the set of all trapped geodesics in S^*M has Hausdorff dimension $2\delta+1$. That is, surfaces with smaller δ are more open and have less trapping, and surfaces with larger δ are more closed and have more trapping. Compact hyperbolic surfaces have $\delta = 1$. For all (noncompact) convex co-compact surfaces which are nonelementary (i.e. M is not the hyperbolic plane or a hyperbolic cylinder) we have $0 < \delta < 1$.

The parameter δ is related to the spectral gap question by *Patterson–Sullivan theory*: the topmost resonance is given by $i(\delta - \frac{1}{2})$, see [**Bo16**, Chapter 14]. On the other hand the self-adjointness of the Laplacian implies that the upper half-plane has only finitely many resonances, corresponding to L^2 eigenvalues in $[0, 1/4)$. Therefore M has an essential spectral gap of size

$$\beta = \max\Big(0, \frac{1}{2} - \delta\Big) \tag{1.2.7}$$

which we call the *standard gap*.

We now present an application of the fractal uncertainty principle to the spectral gap question:

Theorem 1.2.1. *Let M be a convex co-compact hyperbolic surface with $0 < \delta < 1$. Then M has an essential spectral gap of size:*

[**DZ16**, **BD16**] $\beta = \beta(M) > 0$*;*

[**DZ16**, **BD17**] $\beta > \frac{1}{2} - \delta$ *which depends only on δ.*

Remarks. 1. The first statement of Theorem 1.2.1 is only interesting (i.e. improves over the standard gap (1.2.7)) when $\delta \geq 1/2$ and the second one, when $\delta \leq 1/2$. A gap of size $\beta > \frac{1}{2} - \delta$ with β which depends on the surface (not just on δ) was proved in [**Na05**] and recently quantitatively revisited from the fractal uncertainty principle point of view in [**DJ17a**]. Both of these papers use the method developed in [**Do98**].
2. For more general hyperbolic systems, the Patterson–Sullivan gap $\beta = \frac{1}{2} - \delta$ is replaced by the *pressure gap* $\beta = -P(\frac{1}{2})$, which is nontrivial under the *pressure condition* $P(\frac{1}{2}) < 0$. The pressure gap was established for various hyperbolic systems in [**Ik88**, **GR89**, **NZ09**], and an improved gap $\beta > -P(\frac{1}{2})$ is proved under additional

assumptions in [**Na05, PS10, St11, St12**]. See [**No11a**, §1.2] and [**Zw16b**, §3.2] for more information. The papers [**DJ16, BD16**] explained in this note give the first cases where the dynamics is chaotic, the trapped set is fractal, and the gap holds without any pressure condition. This provides evidence in favor of the conjecture that essential spectral gaps are present for *any* open hyperbolic system [**Zw16b**, Conjecture 3].

1.2.3. Discussion. We now explain the intuition behind the Patterson–Sullivan gap (which also applies to the pressure gap mentioned below) and behind the gap proved in [**DZ16, BD16**]:

(1) An essential spectral gap of size β can be interpreted as exponential decay $\mathcal{O}(e^{-\beta t})$ of local L^2 norm of solutions to the wave equation for initial data which is compactly supported and localized to high frequency $\sim h^{-1}$.

(2) Geometric optics approximation implies that at high frequency, solutions to the wave equation propagate along geodesics of M (just like light in vacuum propagates along straight lines). A mathematical formulation of this uses microlocal analysis, for instance it can be deduced from Egorov's Theorem (Proposition 1.1.3). Note that we need to define localization of a function in both position and momentum (namely, the microsupport/semiclassical wave-front set – see [**Zw12**, §8.4.2] or [**DZ**$^{\mathrm{B}}$, §E.2.3]), which can be done using semiclassical pseudodifferential operators.

(3) Localization in phase space is limited by the (standard) uncertainty principle which implies that no function can localize precisely on a single trajectory. The geodesic flow on M is *hyperbolic*, i.e. it has a stable/unstable decomposition, see §**??**. Therefore, even if a wave is initially localized very close to one trapped geodesic, after time t its microsupport will spread out by a factor of e^t in the unstable direction. If M had only one trapped geodesic (as for the hyperbolic cylinder), then after a long time most of the energy of the wave would spread out to nontrapped geodesics and eventually escape. More precisely, the portion of the energy that stays in a compact set would be bounded by $e^{-t/2}$, just like the constant function on an interval of size e^t has only $e^{-t/2}$ of its L^2 norm on any interval of size 1, and we obtain a gap of size $\beta = \frac{1}{2}$.

(4) The situation becomes more difficult if there are multiple trapped geodesics, since a wave initially localized near a single geodesic can over time spread to neighboring trapped geodesics. One way to handle this problem is to split the wave into a linear combination of many pieces using a dynamical partition of unity. Each piece corresponds to a single combinatorial way of being trapped; say, for the three-funnel surface on Figure 1(b) this fixes the order in which the geodesic winds around the three funnels, and for the three-obstacle scattering on Figure 2(a) this fixes the order in which the billiard ball trajectory bounces

off the obstacles. Each piece of the wave lives near a single trapped trajectory, and similarly to the previous paragraph its local L^2 norm decays like $e^{-t/2}$. However, the number of pieces grows exponentially like $e^{\delta t}$. The triangle inequality then gives the bound $e^{(\delta-1/2)t}$ on the local L^2 norm of the whole wave which explains the Patterson–Sullivan gap.

(5) The above use of triangle inequality can give an answer which is very far from the real size of the gap. For instance, when $\delta > 1/2$ it gives a growing exponential bound on the local L^2 norm of the wave, while in fact the global L^2 norm is bounded as $t \to \infty$. One can get a gap of size $\beta > \frac{1}{2} - \delta$ by exploiting on many intermediate scales the fact that triangle inequality in L^2 is rarely sharp; this is the strategy taken in [**Na05, DJ17a**]. In other terms, one takes advantage of the cancellations (or as a physicist would say, negative interferences) between different components of the wave.

(6) To show there is a gap for any $\delta > 1/2$, [**DZ16, BD16**] avoids the triangle inequality altogether. Instead we treat the wave as a whole and study its localization. Denote by $U(t)$ the wave propagator and fix a compactly supported cutoff $\chi \in C_0^\infty(M)$. We make the following two observations which take the geometric optics approximation to the limit of its validity:
   (a) $U(T)\chi = \operatorname{Op}_h(\chi_+)U(T)\chi + \mathcal{O}(h^\infty)$ where $T = \log(1/h)$, Op_h is a quantization procedure, and the symbol χ_+ lives distance $h = e^{-T}$ close to the outgoing tail Γ_+, consisting of geodesics trapped backwards in time. In other words, if the initial data is in a compact set, then after time T the solution lives very close to the outgoing tail.
   (b) similarly $\chi U(T) = \chi U(T)\operatorname{Op}_h(\chi_-) + \mathcal{O}(h^\infty)$ where χ_- lives distance h close to the incoming tail Γ_-, consisting of geodesics trapped forward in time. In other words, if we only need to know $U(T)f$ in a compact set, then it suffices to know f very close to the incoming tail.

   Combining these observations, we get

$$\chi U(2T)\chi = \chi U(T)\big(\operatorname{Op}_h(\chi_-)\operatorname{Op}_h(\chi_+)\big)U(-T)\chi + \mathcal{O}(h^\infty). \tag{1.2.8}$$

   This is where the *fractal uncertainty principle* comes in, giving the estimate

$$\|\operatorname{Op}_h(\chi_-)\operatorname{Op}_h(\chi_+)\|_{L^2\to L^2} = \mathcal{O}(h^\beta) \quad \text{for some } \beta > 0. \tag{1.2.9}$$

?

   The bound (1.2.9), discussed in [**TODO**] below, exploits the fractal nature of the supports of $\chi_\pm$ and is proved using tools from harmonic analysis which are very different from the microlocal ideas presented so far. Combining (1.2.8) and (1.2.9), we see that $\|\chi U(2T)\chi\|_{L^2\to L^2} = \mathcal{O}(h^\beta)$ decays exponentially in T, which implies existence of an essential spectral gap.

There are many holes in the above rough presentation of [**DZ16, BD16**], which we need to fill in later. For instance, one needs to understand why a bound on $\chi U(2T)\chi$

implies the bound on $\chi U(t)\chi$ for $t \geq 2T$, i.e. why the wave cannot 'bounce back from infinity' after a long time. For that we use [**Va13a**]; in fact it is convenient on a technical level to bypass the wave propagator and use the scattering resolvent directly. Another issue is how to quantize $\chi_\pm$ since they have a very rough behavior; in fact to make this quantization work we take $T = \rho \log(1/h)$ where $\rho < 1$ is very close to 1.

CHAPTER 2

Fractal uncertainty principle

We now introduce the central component of these notes, the *fractal uncertainty principle (FUP)*. Roughly speaking it states that *no function can be localized close to a fractal set in both position and frequency*, with a precise statement given in Definition 2.1.2 below and in the question following Exercise 2.1.4. The currently known results are in dimension 1, with extension to higher dimensions an important open problem.

2.1. Statement and basic properties

2.1.1. Uncertainty principle. Before going fractal, we briefly review the standard uncertainty principle. Let $0 < h \ll 1$ be the semiclassical parameter and consider the unitary semiclassical Fourier transform

$$\mathcal{F}_h : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad \mathcal{F}_h f(\xi) = (2\pi h)^{-1/2} \int_{\mathbb{R}} e^{-ix\xi/h} f(x)\, dx.$$

The version of the uncertainty principle we use is the following: for any $f \in L^2(\mathbb{R})$, either f or its Fourier transform $\mathcal{F}_h f$ have little mass on the interval $[0, h]$.[1] Specifically we have

$$\| \mathbb{1}_{[0,h]} \mathcal{F}_h \mathbb{1}_{[0,h]} \|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \leq h^{1/2}. \tag{2.1.1}$$

Here for $X \subset \mathbb{R}$, the operator $\mathbb{1}_X : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is multiplication by the indicator function of X. One way to prove (2.1.1) is via Hölder's inequality:

$$\begin{aligned} \| \mathbb{1}_{[0,h]} \mathcal{F}_h \mathbb{1}_{[0,h]} \|_{L^2 \to L^2} &\leq \| \mathbb{1}_{[0,h]} \|_{L^\infty \to L^2} \cdot \|\mathcal{F}_h\|_{L^1 \to L^\infty} \cdot \| \mathbb{1}_{[0,h]} \|_{L^2 \to L^1} \\ &= h^{1/2} \cdot (2\pi h)^{-1/2} \cdot h^{1/2} \leq h^{1/2}. \end{aligned} \tag{2.1.2}$$

A useful way to think about the norm bound (2.1.1) is as follows: if a function f is supported in $[0, h]$, then the interval $[0, h]$ contains at most $h^{1/2}$ of the L^2 mass of $\mathcal{F}_h f$.

Exercise 2.1.1. *Show that the norm bound (2.1.1) is sharp up to constants, by finding a function f supported in $[0, h]$ such that $\|f\|_{L^2} = 1$ and $\|\mathcal{F}_h f\|_{L^2([0,h])} \sim h^{1/2}$.*

---

[1] This is consistent with the uncertainty principle in quantum mechanics. Indeed, if both f and $\mathcal{F}_h f$ are large on $[0, h]$ then we know the wave function f is at position and momentum 0 with certainty h, but $h \cdot h \ll h$.

The fractal uncertainty principle studied below concerns localization in position and frequency on more general sets:

DEFINITION 2.1.2. *Let $X, Y \subset \mathbb{R}$ be h-dependent families of sets. We say that X, Y satisfy* ***uncertainty principle*** *with exponent $\beta \geq 0$, if*

$$\| \mathbb{1}_X \mathcal{F}_h \mathbb{1}_Y \|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} = \mathcal{O}(h^\beta) \quad \text{as } h \to 0. \tag{2.1.3}$$

2.1.2. Ahlfors–David regular sets. To make the uncertainty principle a fractal one, we apply it when X, Y are fractal sets. In these lectures, the word 'fractal' will usually mean Ahlfors–David regularity, defined as follows:

DEFINITION 2.1.3. *Assume that $X \subset \mathbb{R}$ is a nonempty closed set and $0 \leq \delta \leq 1$, $C_R \geq 1$, $0 \leq \alpha_{\min} \leq \alpha_{\max} \leq \infty$. We say that X is* ***δ-regular with constant C_R on scales $\alpha_{\min}$ to $\alpha_{\max}$*** *if there exists a finite measure μ_X supported on X such that:*

- *for each interval $I \subset \mathbb{R}$ of size $|I|$ satisfying $\alpha_{\min} \leq |I| \leq \alpha_{\max}$, we have the upper bound*
$$\mu_X(I) \leq C_R |I|^\delta; \tag{2.1.4}$$
- *if the interval I is as above and additionally the center of I lies in X, then we have the lower bound*
$$\mu_X(I) \geq C_R^{-1} |I|^\delta. \tag{2.1.5}$$

Remark. We will typically consider an h-dependent family of sets X, taking $\alpha_{\min} = h$, $\alpha_{\max} = 1$, and requiring that the regularity constant C_R be independent of h.

One should think of a δ-regular set on scales $\alpha_{\min}$ to $\alpha_{\max}$ as an $\alpha_{\min}$-neighborhood of a fractal set of dimension δ. The dimension is required to be the same at each point and each scale. Some examples are given by

EXERCISE 2.1.4. *Show that for some h-independent regularity constant C_R,*

(1) *$\{0\}$ is 0-regular on scales 0 to ∞;*
(2) *$[0, h]$ is 0-regular on scales h to ∞;*
(3) *$[0, 1]$ is 1-regular on scales 0 to 1;*
(4) *the mid-third Cantor set $\mathcal{C} \subset [0, 1]$ is $\frac{\log 2}{\log 3}$-regular on scales 0 to 1 (see also Exercise 5.2.1);*
(5) *the h-neighborhood of the Cantor set $\mathcal{C}$ is $\frac{\log 2}{\log 3}$-regular on scales h to 1.*

On the other hand, show that no matter what h-independent C_R and δ we take,

(6) *$[0, 1] \cup \{2\}$ cannot be δ-regular on scales 0 to 1;*
(7) *$[0, h^{1/2}]$ cannot be δ-regular on scales h to 1.*

2.1.3. Statement of fractal uncertainty principle. We now state the main question of this section:

Fix $\delta \in [0,1]$ and $C_R \geq 1$. What is the best value of β such that (2.1.3) holds for all h-dependent families of sets $X, Y \subset [0,1]$ which are δ-regular with constant C_R on scales h to 1?

One way to establish (2.1.3) is to use the following volume (Lebesgue measure) bound: if $X \subset [0,1]$ is δ-regular on scales h to 1 with some constant C_R, then for some $C = C(\delta, C_R)$

$$\mathrm{vol}(X) \leq Ch^{1-\delta}. \tag{2.1.6}$$

EXERCISE 2.1.5. *Show* (2.1.6). *(Hint: cover X by some number N of h-sized intervals centered on X and with little overlap. Then use regularity to show that $N = \mathcal{O}(h^{-\delta})$.)*

Using (2.1.6) and arguing as in (2.1.2), we see that (2.1.3) holds for $\beta = \frac{1}{2} - \delta$. It also holds for $\beta = 0$ since $\mathcal{F}_h$ is unitary. Therefore, we get the *basic FUP exponent* (which is the same as the standard size of the spectral gap (1.2.7) – this is not a coincidence)

$$\beta_0 = \max\Big(0, \frac{1}{2} - \delta\Big). \tag{2.1.7}$$

EXERCISE 2.1.6 (Brick). *Show that for $0 \leq \delta \leq 1$, we have as $h \to 0$*

$$\| \mathbb{1}_{[0,h^{1-\delta}]} \mathcal{F}_h \mathbb{1}_{[0,h^{1-\delta}]} \|_{L^2 \to L^2} \sim h^{\max(0,1/2-\delta)}.$$

This example shows that (2.1.7) *cannot be improved if we only use the volumes of X, Y.*

Using examples (2) and (3) from Exercise 2.1.4, we see that (2.1.7) cannot be improved if $\delta = 0$ or $\delta = 1$. It turns out that in every other case there is an improvement, which is the central statement of these notes:

THEOREM 2.1.1. [**BD16, DJ17a**] *Fix $\delta \in (0,1)$ and $C_R \geq 1$. Then there exists*

$$\beta = \beta(\delta, C_R) > \max\Big(0, \frac{1}{2} - \delta\Big) \tag{2.1.8}$$

such that (2.1.3) *holds for all h-dependent families of sets $X, Y \subset [0,1]$ which are δ-regular with constant C_R on scales h to 1.*

Remark. There exist estimates on the size of the improvement $\beta - \max(0, \frac{1}{2} - \delta)$ in terms of δ, C_R, see [**DJ17a**] for improvement over $\frac{1}{2} - \delta$ and [**JZ17**] for improvement over 0.

The proof of Theorem 2.1.1 is not given in the notes (at least not yet). However, a complete proof in the special case of Cantor sets is given in §5.3 below.

2.1.4. Relation to Fourier decay and additive energy. This section explains how a fractal uncertainty principle can be proved if we have a Fourier decay bound or an additive energy bound on one of the sets X, Y. It not used in the rest of this notes and thus is strictly optional. Instead it provides some perspective from the point of view of fractal harmonic analysis and combinatorics.

Let $X, Y \subset [0,1]$ be two h-dependent closed sets. We assume that they are both δ-regular on scales h to 1 with some h-independent regularity constant C_R. In particular, the volumes of X and Y are $\mathcal{O}(h^{1-\delta})$ by (2.1.6).

To estimate the norm on the left-hand side of the uncertainty principle (2.1.3), we use the T^*T argument:

$$\| \mathbb{1}_X \mathcal{F}_h \mathbb{1}_Y \|^2_{L^2\to L^2} = \| \mathbb{1}_Y \mathcal{F}_h^* \mathbb{1}_X \mathcal{F}_h \mathbb{1}_Y \|_{L^2\to L^2}.$$

We write $\mathcal{F}_h^* \mathbb{1}_X \mathcal{F}_h$ as an integral operator:

$$\mathcal{F}_h^* \mathbb{1}_X \mathcal{F}_h f(y) = \int_{\mathbb{R}} \mathcal{K}(y-y') f(y')\,dy'$$

where

$$\mathcal{K}(y) = (2\pi h)^{-1} \int_X e^{ixy/h}\,dx.$$

Note that $\mathcal{K}(y)$ is just the rescaled Fourier transform of the indicator function of X.

By Schur's inequality applied to $\mathcal{K}(y-y')$ we see that

$$\| \mathbb{1}_X \mathcal{F}_h \mathbb{1}_Y \|^2_{L^2\to L^2} \leq \sup_{y'\in Y} \int_Y |\mathcal{K}(y-y')|\,dy. \tag{2.1.9}$$

In particular by Hölder inequality using the volume bound on Y we get for all $p \in [1,\infty]$

$$\| \mathbb{1}_X \mathcal{F}_h \mathbb{1}_Y \|_{L^2\to L^2} \leq C h^{\frac{(1-\delta)(1-1/p)}{2}} \|\mathcal{K}\|^{1/2}_{L^p([-2,2])}. \tag{2.1.10}$$

The estimates (2.1.9),(2.1.10) can give the uncertainty principle (2.1.3) with exponent $\beta > \max(0, \frac{1}{2} - \delta)$ if we make additional assumptions about X.

EXERCISE 2.1.7. *Assume that X is a union of $\sim h^{-\delta}$ intervals of size h each. Show that $\|\mathcal{K}\|_{L^2([-2,2])} \sim h^{-\delta/2}$. (Hint: for the lower bound, estimate instead $\|\chi\mathcal{K}\|_{L^2(\mathbb{R})}$ where $\chi \in C_0^\infty((-2,2))$ has nonnegative Fourier transform.) Deduce that* (2.1.10) *never improves over the standard bound* (2.1.7) *if we take $p = 2$.*

We now explore two possible assumptions on X where (2.1.9) improves over (2.1.7).

Fourier decay. From the volume bound on X we know that $\sup|\mathcal{K}| = \mathcal{O}(h^{-\delta})$. Assume that we have a better Fourier decay bound, for some $\beta_F > 0$:

$$|\mathcal{K}(y)| = \mathcal{O}(h^{\beta_F-\delta}|y|^{-\beta_F}) \quad \text{for } h \leq |y| \leq 2. \tag{2.1.11}$$

From δ-regularity of Y (similarly to Exercise 2.1.5) we see that for any interval I with $h \leq |I| \leq 2$

$$\mathrm{vol}(Y \cap I) \leq Ch^{1-\delta}|I|^{\delta}.$$

Breaking the integral below into dyadic pieces centered at y', we see that (2.1.11) implies (assuming $\beta_F < \delta$)

$$\sup_{y' \in Y} \int_Y |\mathcal{K}(y - y')|\, dy \leq Ch^{1-2\delta+\beta_F},$$

thus by (2.1.9) the uncertainty principle (2.1.3) holds with

$$\beta = \frac{1}{2} - \delta + \frac{\beta_F}{2}. \tag{2.1.12}$$

EXERCISE 2.1.8. *Under the assumptions of Exercise 2.1.7 show that if* (2.1.11) *holds and* $\delta < 1$, *then* $\beta_F \leq \delta/2$. *Deduce that the exponent* β *in* (2.1.12) *always satisfies* $\beta \leq \frac{1}{2} - \frac{3\delta}{4}$, *in particular it can only improve over* (2.1.7) *when* $0 < \delta < 2/3$.

EXERCISE 2.1.9. *Assume that* $h = 3^{-k}$ *for some* $k \in \mathbb{N}$ *and let* X *be the* h-*neighborhood of the mid-third Cantor set. Show that* (2.1.11) *cannot hold with any* $\beta_F > 0$.

Exercise 2.1.9 is discouraging as it shows that even for simple fractal sets there may be no Fourier decay. However, for more complicated (in some sense, nonlinear) fractal sets a Fourier decay bound holds. In particular, [**BD17**] establishes such bound for limit sets of convex co-compact hyperbolic surfaces.

Additive energy. Instead of a pointwise decay bound on $\mathcal{K}(y)$, we can try to use a bound on its L^p norm together with (2.1.10). The case $p = 4$ is related to the *additive energy*

$$\mathcal{E}_A(X) = \mathrm{vol}\{(a, b, c, d) \in X^4 \mid a + b = c + d\}$$

where we use the volume form on the hypersurface $\{a + b = c + d\} \subset \mathbb{R}^4$ induced by the standard volume form in the (a, b, c) variables. More precisely we have

$$\|\mathcal{K}\|_{L^4(\mathbb{R})} = (2\pi h)^{-3/4}\mathcal{E}_A(X)^{1/4}.$$

It follows immediately from the volume bound on X that

$$\mathcal{E}_A(X) \leq \mathrm{vol}(X)^3 \leq Ch^{3(1-\delta)}.$$

Using δ-regularity of X, one can show that this bound can always be improved:

THEOREM 2.1.2. [**DZ16**, Theorem 6] *Assume that* $X \subset [0, 1]$ *is* δ-*regular on scales* h *to* 1 *with constant* C_R, *and* $0 < \delta < 1$. *Then there exists* $\beta_A = \beta_A(\delta, C_R) > 0$ *such that*

$$\mathcal{E}_A(X) \leq Ch^{3(1-\delta)+\beta_A}. \tag{2.1.13}$$

Exercise 2.1.10. *Show* (2.1.13) *when* X *is the* h*-neighborhood of the mid-third Cantor set. (Hint: look at the equation* $a + b = c + d$ *written base 3.)*

From (2.1.13) and (2.1.10) we see that the fractal uncertainty principle (2.1.3) holds with

$$\beta = \frac{3}{4}\Big(\frac{1}{2} - \delta\Big) + \frac{\beta_A}{8}. \tag{2.1.14}$$

This improves over the standard bound (2.1.7) when $\delta = 1/2$.

Exercise 2.1.11. *Under the assumptions of Exercise 2.1.7 show that if* (2.1.13) *holds, then* $\beta_A \le \min(\delta, 1-\delta)$*. Deduce that* (2.1.14) *can possibly improve over the standard bound* (2.1.7) *only when* $1/3 < \delta < 4/7$*.*

Exercise 2.1.12. *Obtain the bound* (2.1.14) *for more general additive energies corresponding to the cases* $p = 2k$, $k \in \mathbb{N}$, $k > 2$*. For which values of* δ *does improvement in these generalized additive energies guarantee an FUP exponent improving over* (2.1.7)*?*

We remark that the bound (2.1.14) uses structure of X and the volume bound on Y (so for instance it would work for $Y = [0, h^{1-\delta}]$). This is in contrast to the proof of Theorem 2.1.1 which uses the structure of both X and Y.

2.2. A semiclassical interpretation of FUP

We now give an interpretation of fractal uncertainty principle in terms of semiclassical quantization.

2.2.1. Quantization of rough symbols. We first discuss quantization of symbols which depend on h, in particular their derivatives may grow as $h \to 0$. This will be important later as it shows the limit to which the classical/quantum correspondence still applies, and determines the maximal time for which Egorov's theorem still holds.

We use the standard quantization procedure on $\mathbb{R}$ defined by (see [**Zw12**, (4.1.2)])

$$\mathrm{Op}_h(a)f(x) = (2\pi h)^{-1} \int_{\mathbb{R}^2} e^{\frac{i}{h}(x-y)\xi} a(x,\xi) f(y)\, dy d\xi. \tag{2.2.1}$$

For this quantization procedure to be useful, we in particular need to have an asymptotic expansion for products of operators, which for the standard quantization is as follows [**Zw12**, Theorem 4.14]: $\mathrm{Op}_h(a)\,\mathrm{Op}_h(b) = \mathrm{Op}_h(a\#b)$ where

$$a\#b(x,\xi;h) \sim \sum_{j=0}^{\infty} (-ih)^j \partial_\xi^j a(x,\xi;h) \partial_x^j b(x,\xi;h) \quad \text{as } h \to 0. \tag{2.2.2}$$

If a, b are smooth and h-independent, then the j-th term in the asymptotic sum is $\mathcal{O}(h^j)$. (Note that by looking at terms with $j = 0, 1$ we get the product rule (1.1.3) and the commutator rule (1.1.5).)

For h-dependent a, b a bound on growth of derivatives is needed to make sure each next term decays faster than the previous one and the expansion (2.2.1) is still valid. A typical condition to impose is $\partial^\alpha a = \mathcal{O}(h^{-\rho|\alpha|})$ for some $\rho \in [0, 1/2)$, see [**Zw12**, §4.4.1]; then the j-th term in (2.2.2) is $\mathcal{O}(h^{(1-2\rho)j})$. However, in our case it is convenient to impose one of the following *anisotropic bounds* for some $\rho \in [0, 1)$ (which in practice is taken very close to 1):

$$|\partial_x^j \partial_\xi^k a(x, \xi; h)| = \mathcal{O}(h^{-\rho j}), \quad \text{denoted } a \in S_{L_0,\rho}; \tag{2.2.3}$$

$$|\partial_x^j \partial_\xi^k a(x, \xi; h)| = \mathcal{O}(h^{-\rho k}), \quad \text{denoted } a \in S_{L_1,\rho}. \tag{2.2.4}$$

That is, symbols in $S_{L_0,\rho}$ are smooth in the ξ direction but rough in the x direction; same is true for $S_{L_1,\rho}$ but with the roles of x, ξ reversed.

If $a, b \in S_{L_0,\rho}$ or $a, b \in S_{L_1,\rho}$, then the j-th term in (2.2.2) is $\mathcal{O}(h^{(1-\rho)j})$, so the asymptotic expansion still converges. In addition the $S_{L_0,\rho}$ calculus and $S_{L_1,\rho}$ calculus have asymptotic expansions for adjoints. This makes it possible to prove the norm bounds

$$\| \operatorname{Op}_h(a)\|_{L^2\to L^2} = \sup |a| + o(1) \quad \text{as } h \to 0 \quad \text{if } a \in S_{L_0,\rho} \text{ or } a \in S_{L_1,\rho}. \tag{2.2.5}$$

2.2.2. FUP via products of pseudodifferential operators. We now study the product $\operatorname{Op}_h(b)\operatorname{Op}_h(a)$ when $a \in S_{L_0,\rho}$ and $b \in S_{L_1,\rho}$. The j-th term in (2.2.2) is $\mathcal{O}(h^{(1-2\rho)j})$; when $\rho > 1/2$ the asymptotic expansion no longer makes sense. Thus the product $\operatorname{Op}_h(b)\operatorname{Op}_h(a)$ can no longer be described by semiclassical analysis. However the fractal uncertainty principle (Theorem 2.1.1) can be interpreted as an estimate on its norm when the supports of a, b have fractal structure:

PROPOSITION 2.2.1. *Assume that $\delta, \rho \in (0, 1)$ and (with $X(h^\rho)$ denoting the h^ρ-neighborhood of X)*

- *$X, Y \subset [0, 1]$ are δ-regular with constant C_R on scales 0 to 1;*
- *$a_0 \in S_{L_0,\rho}$ and $\operatorname{supp} a_0 \subset \{(x, \xi) : x \in Y(h^\rho)\}$;*
- *$a_1 \in S_{L_1,\rho}$ and $\operatorname{supp} a_1 \subset \{(x, \xi) : \xi \in X(h^\rho)\}$.*

Then there exists $\beta = \beta(\delta, C_R) > \max(0, \frac{1}{2} - \delta)$ such that we have the norm bound

$$\| \operatorname{Op}_h(a_1)\operatorname{Op}_h(a_0)\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} = \mathcal{O}(h^{\beta-2(1-\rho)}) \quad \text{as } h \to 0. \tag{2.2.6}$$

PROOF. Assume first that a_0 depends only on x, while a_1 depends only on ξ. Then $\operatorname{Op}_h(a_0) = a_0$ is a multiplication operator, while $\operatorname{Op}_h(a_1) = \mathcal{F}_h^* a_1 \mathcal{F}_h$ is a Fourier multiplier. Since $\sup |a_j|$ are bounded uniformly in h, we have

$$\| \operatorname{Op}_h(a_1)\operatorname{Op}_h(a_0)\|_{L^2\to L^2} = \|a_1 \mathcal{F}_h a_0\|_{L^2\to L^2} \le C\| \mathbb{1}_{X(h^\rho)} \mathcal{F}_h \mathbb{1}_{Y(h^\rho)} \|.$$

Writing $X(h^\rho)$ as a union of no more than $10h^{\rho-1}$ shifts of $X(h)$ and same for $Y(h^\rho)$, using triangle inequality and the fact that the norm $\| \mathbb{1}_X \mathcal{F}_h \mathbb{1}_Y \|_{L^2\to L^2}$ does not change when shifting X, Y, we bound

$$\| \mathbb{1}_{X(h^\rho)} \mathcal{F}_h \mathbb{1}_{Y(h^\rho)} \| \leq 100 h^{2(\rho-1)} \| \mathbb{1}_{X(h)} \mathcal{F}_h \mathbb{1}_{Y(h)} \|_{L^2 \to L^2}.$$

The sets $X(h), Y(h)$ are δ-regular with constant $100 C_R$ on scales h to 1, so Theorem 2.1.1 gives the required bound.

We now consider the general case. From the support property of a_0 we see that there exists $\tilde{a}_0(x) \in S_{L_0,\rho}$ such that

$$\operatorname{supp} a_0 \cap \operatorname{supp}(1 - \tilde{a}_0) = \emptyset, \quad \operatorname{supp} \tilde{a}_0 \subset X(2h^\rho).$$

Similarly we can find $\tilde{a}_1(\xi) \in S_{L_1,\rho}$ such that

$$\operatorname{supp} a_1 \cap \operatorname{supp}(1 - \tilde{a}_1) = \emptyset, \quad \operatorname{supp} \tilde{a}_1 \subset Y(2h^\rho).$$

By the product formula (2.2.2) we have

$$\operatorname{Op}_h(a_0) = \operatorname{Op}_h(\tilde{a}_0) \operatorname{Op}_h(a_0) + \mathcal{O}(h^\infty)_{L^2 \to L^2}, \quad \operatorname{Op}_h(a_1) = \operatorname{Op}_h(a_1) \operatorname{Op}_h(\tilde{a}_1) + \mathcal{O}(h^\infty)_{L^2 \to L^2}.$$

Since $\operatorname{Op}_h(a_0), \operatorname{Op}_h(a_1)$ are bounded $L^2 \to L^2$ uniformly in h, we have

$$\| \operatorname{Op}_h(a_1) \operatorname{Op}_h(a_0) \|_{L^2 \to L^2} \leq C \| \operatorname{Op}_h(\tilde{a}_1) \operatorname{Op}_h(\tilde{a}_0) \|_{L^2 \to L^2} + \mathcal{O}(h^\infty),$$

reducing to the previously considered special case. □

2.2.3. FUP with general amplitude. As an application of Proposition 2.2.1 we give a version of the fractal uncertainty principle (Theorem 2.1.1) with the semiclassical Fourier transform $\mathcal{F}_h$ replaced by an integral operator

$$\mathcal{B}_h f(x) = (2\pi h)^{-1/2} \int_{\mathbb{R}} e^{-ix\xi/h} b(x, \xi; h) f(\xi) \, d\xi, \quad f \in L^2(\mathbb{R}). \tag{2.2.7}$$

The following statement (more precisely, its weaker version with $\beta > 0$ rather than $\beta > \max(0, \frac{1}{2} - \delta)$) comes from [**BD16**, Proposition 4.1].

PROPOSITION 2.2.2. *Assume that $\delta, \rho \in (0,1)$ and*

- *$X, Y \subset \mathbb{R}$ are δ-regular with constant C_R on scales 0 to 1;*
- *the h-dependent family of functions $b(x, \xi; h)$ has all derivatives bounded uniformly in h and is supported in an h-indepedent compact subset of $\mathbb{R}^2$.*

Let $\mathcal{B}_h$ be defined by (2.2.7). Then there exists $\beta = \beta(\delta, C_R) > \max(0, \frac{1}{2} - \delta)$ such that we have the norm bound

$$\| \mathbb{1}_{X(h^\rho)} \mathcal{B}_h \mathbb{1}_{Y(h^\rho)} \|_{L^2 \to L^2} = \mathcal{O}(h^{\beta - 2(1-\rho)}) \quad \text{as } h \to 0. \tag{2.2.8}$$

PROOF. First of all, we may assume that $X, Y \subset [0,1]$. Indeed, using a partition of unity we reduce to the case when $\operatorname{supp} b \subset I \times J$ where $I, J \subset \mathbb{R}$ are small h-independent intervals. We choose I, J such that $|I|, |J| \leq 1$ and $X \cap I$, $Y \cap J$ are

δ-regular with some constant $C'_R(\delta, C_R)$.[2] Then it is enough to establish (2.2.8) with X, Y replaced by $X \cap I, Y \cap J$. (Alternatively, the proof of Theorem 2.1.1 applies to an intersection of a δ-regular set with an interval.)

Take cutoffs $a_0(x), a_1(\xi)$ with $a_j \in S_{L_j,\rho}$ and such that $\operatorname{supp} a_0 \subset Y(2h^\rho)$, $a_0 = 1$ on $Y(h^\rho)$, and similarly for a_1 and X. Recall that $\operatorname{Op}_h(a_1)$ is the multiplication operator by a_0 and $\mathcal{F}_h \operatorname{Op}_h(a_1)\mathcal{F}_h^*$ is the multiplication operator by a_1. Then it suffices to prove the estimate

$$\|\mathcal{F}_h \operatorname{Op}_h(a_1)\mathcal{F}_h^* \mathcal{B}_h \operatorname{Op}_h(a_0)\|_{L^2\to L^2} = \mathcal{O}(h^{\beta-2(1-\rho)}).$$

We have

$$\mathcal{F}_h^*\mathcal{B}_h = \operatorname{Op}_h(b_1)^* \quad \text{where} \quad b_1(x,\xi;h) = \overline{b(\xi,x;h)}.$$

Thus it is enough to show that

$$\|\operatorname{Op}_h(a_1)\operatorname{Op}_h(b_1)^*\operatorname{Op}_h(a_0)\|_{L^2\to L^2} = \mathcal{O}(h^{\beta-2(1-\rho)}). \tag{2.2.9}$$

By the product and adjoint rules in the $S_{L_1,\rho}$ calculus we have $\operatorname{Op}_h(a_1)\operatorname{Op}_h(b_1)^* = \operatorname{Op}_h(\tilde a_1) + \mathcal{O}(h^\infty)_{L^2\to L^2}$ where $\tilde a_1 \in S_{L_1,\rho}$ satisfies $\operatorname{supp} \tilde a_1 \subset \operatorname{supp} a_1$. Now Proposition 2.2.1 gives a bound on the norm of $\operatorname{Op}_h(\tilde a_1)\operatorname{Op}_h(a_0)$ which implies (2.2.9). $\square$

2.3. Generalized FUP

In this section we further generalize the fractal uncertainty principle to obtain a microlocal statement which is the one used in applications.

2.3.1. FUP with general phase. In §2.2.3 we generalized the fractal uncertainty principle (Theorem 2.1.1) replacing the Fourier transform $\mathcal{F}_h$ by an integral operator $\mathcal{B}_h$ with a general amplitude. We now further generalize it by allowing a general phase. More precisely we consider an operator $\mathcal{B}_h$ of the form

$$\mathcal{B}_h f(x) = (2\pi h)^{-1/2}\int_{\mathbb{R}} e^{i\Phi(x,y)/h} b(x,y;h) f(y)\,dy, \quad f \in L^2(\mathbb{R}), \tag{2.3.1}$$

where:

(1) $\Phi \in C^\infty(U;\mathbb{R})$ where $U \subset \mathbb{R}^2$ is an open set;
(2) Φ satisfies the nondegeneracy condition on mixed derivative

$$\partial^2_{xy}\Phi \neq 0 \quad \text{on } U; \tag{2.3.2}$$

(3) the h-dependent family of functions $b(x,y;h)$ has all derivatives bounded uniformly in h and is supported in an h-independent compact subset of U.

The operator $\mathcal{B}_h$ previously defined in (2.2.7) is a special case with $\Phi(x,y) = -xy$.

[2]One has to take care in choosing I, J since it can happen that $X \cap I$ is not δ-regular – for example, consider the intersection of the mid-third Cantor set with the interval $[0, 2/3]$.

EXERCISE 2.3.1. *Show that $\|\mathcal{B}_h\|_{L^2\to L^2}$ is bounded uniformly as $h\to 0$. (Hint: apply Schur's inequality to the operator $\mathcal{B}_h^*\mathcal{B}_h$, using that its integral kernel $\mathcal{K}(x,x')$ satisfies $|\mathcal{K}(x,x')| = \mathcal{O}(h^{-1}(1+h^{-1}|x-x'|)^{-N})$ for all N. The nondegeneracy condition (2.3.2) is crucial for the latter point.)*

The generalization of Theorem 2.1.1 and Proposition 2.2.2 is the following

PROPOSITION 2.3.2. *Assume that $\delta,\rho\in(0,1)$ and $X,Y\subset\mathbb{R}$ are δ-regular with constant C_R on scales 0 to 1. Let $\mathcal{B}_h$ be defined by (2.3.1) where Φ, b satisfy properties (1)–(3) above. Then there exists $\beta=\beta(\delta,C_R)>\max(0,\frac{1}{2}-\delta)$ such that we have the norm bound*

$$\|\mathbb{1}_{X(h^\rho)}\mathcal{B}_h\mathbb{1}_{Y(h^\rho)}\|_{L^2\to L^2} = \mathcal{O}(h^{\beta-2(1-\rho)}) \quad \text{as } h\to 0. \tag{2.3.3}$$

Remark. We only give here the proof of $\beta>0$, reducing to FUP for the Fourier transform (specifically to Proposition 2.2.2) following [**BD16**, Proposition 4.3]. This reduction is useful because the case $\beta>0$ in Theorem 2.1.1 is proved in [**BD16**] using harmonic analysis methods which are very specific to the Fourier transform and do not apply directly to the case of a general phase. On the other hand, the argument below effectively replaces β by $\beta/2$, which makes it useless for showing the inequality $\beta>\frac{1}{2}-\delta$. Luckily, the proof of the case $\beta>\frac{1}{2}-\delta$ in Theorem 2.1.1, given in [**DJ17a**], applies to general phases, so no reduction to the Fourier transform is needed there.

PROOF. As remarked above we only show that $\beta>0$. We moreover allow β to depend on Φ and the support of b; see the first step of the proof of [**BD16**, Proposition 4.3] for how to get rid of this dependence. Similarly to the proof of Proposition 2.2.1 we could reduce to the case $\rho=1$ but it would be useful for the argument below to keep $\rho<1$. We henceforth assume that ρ is very close to 1. We also assume that U is a rectangle, the general case following by a partition of unity.

We first fatten the set X by $h^{\rho/2}$. This seems like a huge loss of information in (2.3.3) but will ultimately be crucial in making the linearization argument work. More precisely, take a smooth h-dependent function ψ such that

$$0\le\psi\le 1,\quad \operatorname{supp}(1-\psi)\cap X(h^\rho)=\emptyset,\quad \operatorname{supp}\psi\subset X(h^{\rho/2}),$$

and ψ satisfies the derivative bounds

$$\sup|\partial_x^k\psi|\le C_k h^{-\rho k/2}. \tag{2.3.4}$$

Then it suffices to show the bound

$$\|\psi\mathcal{B}_h\mathbb{1}_{Y(h^\rho)}\|_{L^2\to L^2} = \mathcal{O}(h^{\beta-2(1-\rho)}) \quad \text{as } h\to 0. \tag{2.3.5}$$

Let $J_1, \dots, J_N$, $N = N(h) = \mathcal{O}(h^{-1})$, be the ordered list of all the intervals of the form $h^{1/2}[j, j+1]$ which intersect $Y(h^\rho)$. We split the operator in (2.3.5) into pieces:

$$\psi \mathcal{B}_h \mathbb{1}_{Y(h^\rho)} = \sum_{n=1}^{N} A_n, \quad A_n := \psi \mathcal{B}_h \mathbb{1}_{Y(h^\rho)\cap J_n}.$$

We will establish almost orthogonality of the operators A_n. We immediately have

$$A_n A_m^* = 0 \quad \text{if } n \neq m. \tag{2.3.6}$$

We next write

$$A_n^* A_m = \mathbb{1}_{Y(h^\rho)\cap J_n} \mathcal{B}_h^* \psi^2 \mathcal{B}_h \mathbb{1}_{Y(h^\rho)\cap J_m}$$

and the operator $\mathcal{B}_h^* \psi^2 \mathcal{B}_h$ has the integral kernel

$$\mathcal{K}(y, y') = (2\pi h)^{-1} \int_{\mathbb{R}} e^{\frac{i}{h}(\Phi(x,y') - \Phi(x,y))} \overline{b(x, y; h)} b(x, y'; h) \psi(x)^2 \, dx. \tag{2.3.7}$$

Due to the nondegeneracy condition (2.3.2) we have

$$|\partial_x \big(\Phi(x, y') - \Phi(x, y)\big)| \sim |y - y'|.$$

We integrate (2.3.7) by parts L times. Each time the phase produces a factor of $h/|y - y'|$ and the amplitude gives $h^{-\rho/2}$ due to (2.3.4). Thus we get

$$|\mathcal{K}(y, y')| \leq C_L \Big(\frac{h^{1-\rho/2}}{|y - y'|}\Big)^L \quad \text{for all } L.$$

Since the intervals J_n have size $h^{1/2}$ and are nonoverlapping and $\rho < 1$, this implies

$$\|A_n^* A_m\|_{L^2 \to L^2} = \mathcal{O}(h^\infty) \quad \text{if } |n - m| > 1. \tag{2.3.8}$$

By Cotlar–Stein Theorem [**Zw12**, Theorem C.5] and (2.3.6), (2.3.8) we reduce (2.3.5) to the bound on each individual A_n, which in light of the support property of ψ reduces to

$$\sup_n \| \mathbb{1}_{X(h^{\rho/2})} \mathcal{B}_h \mathbb{1}_{Y(h^\rho)\cap J_n} \|_{L^2 \to L^2} = \mathcal{O}(h^{\beta - 2(1-\rho)}) \quad \text{as } h \to 0. \tag{2.3.9}$$

Without loss of generality we assume that $J_n = [0, h^{1/2}]$. Subtracting $\Phi(x, 0)$ from $\Phi(x, y)$, we may assume that $\Phi(x, 0) = 0$. For $y = h^{1/2}\tilde{y} \in [0, h^{1/2}]$, we take the Taylor expansion of the phase $\Phi(x, y)$ at the point $(x, 0)$:

$$\Phi(x, h^{1/2}\tilde{y}) = -h^{1/2}\tilde{y}\varphi(x) + h\Psi(x, \tilde{y}; h), \quad \varphi(x) := -\partial_y \Phi(x, 0)$$

where $\Psi(x, \tilde{y}; h)$ is smooth in x and $\tilde{y} \in [0, 1]$, uniformly in h. In the integral formula (2.3.1) for $\mathcal{B}_h$, the term $h\Psi(x, \tilde{y})$ becomes $e^{i\Psi(x,\tilde{y})}$, thus it can be put into the amplitude. The term $h^{1/2}\tilde{y}\varphi(x)$ is linear in $\tilde{y}$, thus after a change of variables in x and appropriate rescaling we reduce the operator $\mathcal{B}_h$ to Fourier transform with a general amplitude, in the form (2.2.7).

More precisely, composing $\mathcal{B}_h$ on the right with the unitary rescaling operator $T_1 f(y) = h^{-1/4} f(h^{-1/2} y)$, we get the operator

$$\mathcal{B}'_h f(x) = (2\pi\tilde h)^{-1/2} \int_{\mathbb{R}} e^{-i\tilde y\varphi(x)/\tilde h} e^{i\Psi(x,\tilde y;h)} b(x, \tilde h\tilde y; h) f(\tilde y)\, d\tilde y, \quad \tilde h := h^{1/2}.$$

Note that $\partial_x \varphi \neq 0$ by the nondegeneracy condition (2.3.2). Composing $\mathcal{B}'_h$ on the left with the change of variable operator $T_2 f(\tilde x) = f(\varphi^{-1}(\tilde x))$ we get the operator

$$\mathcal{B}''_h f(\tilde x) = (2\pi\tilde h)^{-1/2} \int_{\mathbb{R}} e^{-i\tilde x\tilde y/\tilde h} e^{i\Psi(\varphi^{-1}(\tilde x),\tilde y;h)} b(\varphi^{-1}(\tilde x), \tilde h\tilde y; h) f(\tilde y)\, d\tilde y.$$

The operator $\mathcal{B}''_h$ is of the form (2.2.7) which is covered by Proposition 2.2.2. The needed estimate (2.3.9) reduces to

$$\| \mathbb{1}_{\widetilde X(C\tilde h^\rho)} \mathcal{B}''_h \mathbb{1}_{\widetilde Y(\tilde h^{2\rho-1})\cap[0,1]} \|_{L^2\to L^2} \leq C h^{\beta-2(1-\rho)}, \quad \widetilde X := \varphi(X),\ \widetilde Y := h^{-1/2} Y. \quad (2.3.10)$$

The sets $\widetilde X, \widetilde Y$ are δ-regular on scales 0 to 1. Applying FUP with general amplitude, Proposition 2.2.2, with h replaced by $\tilde h$, we see that the left-hand side of (2.3.10) is $\mathcal{O}(\tilde h^{\tilde\beta-3(1-\rho)})$ for some $\tilde\beta > 0$. Putting $\beta := \tilde\beta/2$ we get (2.3.10) which finishes the proof.

The above argument explains why the intervals J_n were chosen of size $h^{1/2}$: we needed the second derivative term in the Taylor expansion of the phase to be $\mathcal{O}(h)$ so that it could be put into the amplitude. On the other hand, in order for the almost orthogonality estimate (2.3.8) to hold the function ψ has to oscillate at a scale $\gg h^{1/2}$, explaining the need for fattening X in the beginning of the proof. □

2.3.2. Fourier integral operators. Semiclassical pseudodifferential operators, introduced in §1.1.1, are a microlocal generalization of differentiation operators. In particular they preserve the microsupport of a function, just like differential operators preserve support. In this section we introduce *Fourier integral operators (FIOs)* which instead move microsupport by an exact symplectomorphism. They can be viewed as generalization of pullback operators. Here we only briefly review the theory of FIOs; for details the reader is referred to [**DZ16**, §2.2] and the references there.

We first give the definition of an exact symplectomorphism. Let M_1, M_2 be manifolds of the same dimension and $U_j \subset T^*M_j$ be open sets. Denote by $\xi\, dx$ and $\eta\, dy$ the canonical 1-forms on T^*M_1 and T^*M_2, then $\omega_1 = d(\xi\, dx)$ and $\omega_2 = d(\eta\, dy)$ are the standard symplectic forms.

Definition 2.3.3. *An* ***exact symplectomorphism*** *is a pair* $(\varkappa, F)$ *where:*

- $\varkappa : U_2 \to U_1$ *is a symplectomorphism, i.e.* $\varkappa^*(\omega_1) = \omega_2$*;*
- $F \in C^\infty(\mathrm{Gr}(\varkappa))$ *where* $\mathrm{Gr}(\varkappa)$ *denotes the graph of* $\varkappa$*,*

$$\mathrm{Gr}(\varkappa) := \big\{(x,\xi,y,\eta) \mid (y,\eta)\in U_2,\ (x,\xi) = \varkappa(y,\eta)\big\} \subset T^*(M_1\times M_2);$$

- *dF is equal to the restriction $(\xi\,dx-\eta\,dy)|_{\mathrm{Gr}(\varkappa)}$. Note that the latter 1-form is always closed when $\varkappa$ is a symplectomorphism.*

We will often suppress the antiderivative F below, just denoting an exact symplectomorphism by $\varkappa$. Natural operations with exact symplectomorphisms (e.g. composition) induce a natural choice of antiderivative.

A *Fourier integral operator* is an operator $B:\mathcal{D}'(M_2)\to C_0^\infty(M_1)$ which can be locally written in an oscillatory integral form

$$Bf(x)=(2\pi h)^{-\frac{n+m}{2}}\int_{M_2\times\mathbb{R}^m} e^{\frac{i}{h}\Phi(x,y,\zeta)}b(x,y,\zeta;h)f(y)\,dyd\zeta \tag{2.3.11}$$

where $n=\dim M_j$ and

- $\Phi\in C^\infty(U_\Phi;\mathbb{R})$ where $U_\Phi\subset M_1\times M_2\times\mathbb{R}^m$ is an open set, for some choice of m;
- the phase function Φ is *nondegenerate* in the sense that the differentials $d(\partial_{\zeta_1}\Phi),\dots,d(\partial_{\zeta_m}\Phi)$ are linearly independent on the *critical set*

  $$\mathcal{C}_\Phi:=\{(x,y,\zeta)\in\partial_\zeta\Phi(x,y,\zeta)=0\}.$$

  It follows that $\mathcal{C}_\Phi$ is a smooth $2n$-dimensional submanifold of U_Φ;
- the symbol b is smooth in $(x,y,\zeta,h)\in U_\Phi\times[0,h_0)$ for some $h_0>0$, and its support in the (x,y,ζ) variables is contained in an h-independent compact subset of U_Φ. In particular this implies that b has an asymptotic expansion in nonnegative integer powers of h as $h\to 0$.

The symplectomorphism quantized by B is determined from the phase function using

Definition 2.3.4. *We say that a nondegenerate phase function Φ **generates** an exact symplectomorphism $(\varkappa,F)$ if the map*

$$j_\Phi:\mathcal{C}_\Phi\to T^*(M_1\times M_2),\quad j_\Phi(x,y,\zeta)=(x,\partial_x\Phi(x,y,\zeta),y,-\partial_y\Phi(x,y,\zeta))$$

*is a diffeomorphism onto the graph $\mathrm{Gr}(\varkappa)$ and $F\circ j_\Phi=\Phi|_{\mathcal{C}_\Phi}$. We say that Φ **locally generates** $\varkappa$ if it generates the restriction of $\varkappa$ to some open subset of U_2.*

Exercise 2.3.5. *Let Φ be a nondegenerate phase function.*

(i) Show that j_Φ has injective differential.

(ii) Assume that j_Φ is injective and assume that $j_\Phi(\mathcal{C}_\Phi)$ is the graph of some map $\varkappa:U_2\to U_1$, $U_j\subset T^*M_j$. Show that $\varkappa$ is a symplectomorphism.

(iii) Under the assumptions of part (ii), show that $d(\Phi\circ j_\Phi^{-1})=(\xi\,dx-\eta\,dy)|_{\mathrm{Gr}(\varkappa)}$, that is there is a choice of antiderivative F for which Φ generates $(\varkappa,F)$.

The key fact of (local) theory of Fourier integral operators is the following *change of parametrization*: if two phase functions Φ and Φ' generate the same exact symplectomorphism, and B is defined by (2.3.11) using Φ and some symbol b, then B is also defined by (2.3.11) using Φ' and some other symbol b', modulo an $\mathcal{O}(h^\infty)$ remainder. Because of this we associate Fourier integral operators to a symplectomorphism rather than to a particular phase function:

DEFINITION 2.3.6. *Let $\varkappa$ be an exact symplectomorphism. We say that $B : \mathcal{D}'(M_2) \to C_0^\infty(M_1)$ is a (compactly microlocalized)* ***Fourier integral operator associated to $\varkappa$****, denoted*

$$B \in I_h^{\mathrm{comp}}(\varkappa),$$

if B is the sum of finitely many operators of the form (2.3.11) for some phase functions Φ locally generating $\varkappa$, and a remainder with integral kernel in $\mathcal{O}(h^\infty)_{C_0^\infty(M_1\times M_2)}$.

Fourier integral operators enjoy many natural properties, in particular:

(1) Every $B \in I_h^{\mathrm{comp}}(\varkappa)$ is microlocalized on the graph of $\varkappa$ in the following sense:

$$\mathrm{Op}_h(a_1) B \,\mathrm{Op}_h(a_2) = \mathcal{O}(h^\infty) \quad \text{if } \varkappa(U_2 \cap \operatorname{supp} a_2) \cap \operatorname{supp} a_1 = \emptyset. \tag{2.3.12}$$

(2) If $\varkappa : T^*M \to T^*M$ is the identity map (with the 0 antiderivative) then $B \in I_h^{\mathrm{comp}}(\varkappa)$ if and only if B is a pseudodifferential operator, more precisely $B = \mathrm{Op}_h(b)$ for some symbol $b(x,\xi;h)$ compactly supported in (x,ξ). On $\mathbb{R}^n$ this is immediate from the standard quantization formula (2.2.1), as the function $\Phi(x,y,\zeta) = \langle x-y,\zeta\rangle$ generates $\varkappa$.

(3) The following version of Egorov's Theorem holds: if $a_j \in C_0^\infty(T^*M_j)$ satisfy $a_2 = a_1 \circ \varkappa$ on U_2, then

$$\mathrm{Op}_h(a_1) B = B\,\mathrm{Op}_h(a_2) + \mathcal{O}(h)_{I_h^{\mathrm{comp}}(\varkappa)} \quad \text{for all } B \in I_h^{\mathrm{comp}}(\varkappa). \tag{2.3.13}$$

Note that (2.3.13) implies (2.3.12) with an $\mathcal{O}(h)$ remainder.

(4) If $B_j \in I_h^{\mathrm{comp}}(\varkappa_j)$, $j = 1,2$, then the composition B_1B_2 lies in $I_h^{\mathrm{comp}}(\varkappa_1 \circ \varkappa_2)$.

(5) If $B \in I_h^{\mathrm{comp}}(\varkappa)$, then the adjoint B^* lies in $I_h^{\mathrm{comp}}(\varkappa^{-1})$.

(6) Every $B \in I_h^{\mathrm{comp}}(\varkappa)$ is bounded $L^2(M_2) \to L^2(M_1)$ uniformly in h. This follows immediately from the last two properties, as B^*B is a pseudodifferential operator.

To finish this subsection we give a few examples of Fourier integral operators (this list is by no means extensive – we merely include the examples which will be used later):

(1) Assume that (M,g) is a Riemannian manifold and $U(t) = \exp(-it\sqrt{-\Delta_g})$ is the wave propagator. Then

$$\mathrm{Op}_h(b)U(t),\ U(t)\,\mathrm{Op}_h(b) \in I_h^{\mathrm{comp}}(\varphi_t) \quad \text{for all } b \in C_0^\infty(T^*M \setminus 0) \tag{2.3.14}$$

where $\varphi_t = \exp(tH_p) : T^*M \setminus 0 \to T^*M \setminus 0$ is the homogeneous geodesic flow, see (1.1.9), and the antiderivative is identically 0. This together with (2.3.13) gives another way of proving Egorov's Theorem for the wave propagator, Proposition 1.1.3.

To prove (2.3.14) we need to provide an oscillatory integral representation of the form (2.3.11) for $\mathrm{Op}_h(b)U(t)$, at least for small t (the case of arbitrary t will then follow by the composition property). This is done by the *hyperbolic parametrix construction*, see for instance [**Zw12**, Theorem 10.4]. (A curious reader can try to prove this for $M = \mathbb{R}^n$ using the Fourier multiplier formula for $U(t)$.)

(2) Let $\mathcal{F}_h$ be the semiclassical Fourier transform. Then for each $b \in C_0^\infty(T^*\mathbb{R}^n)$ the operator $\mathrm{Op}_h(b)\mathcal{F}_h^*$ has the form (2.3.11):

$$\mathrm{Op}_h(b)\mathcal{F}_h^* f(x) = (2\pi h)^{-n/2} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x,y\rangle} b(x,y)\, f(y)\, dy.$$

Therefore $\mathrm{Op}_h(b)\mathcal{F}_h^* \in I_h^{\mathrm{comp}}(\varkappa)$ where $\varkappa(y,\eta) = (-\eta, y)$ is rotation by $\pi/2$. Same is true for $\mathcal{F}_h^* \mathrm{Op}_h(b)$.

(3) Assume that $\varkappa : U_2 \to U_1$ is a symplectomorphism where U_j are neighborhoods of some $(x_j, \xi_j) \in T^*M_j$ such that $\varkappa(x_2,\xi_2) = (x_1,\xi_1)$. Shrinking U_j we may assume that $\varkappa$ is exact; fix an arbitrary antiderivative. Then there exist

$$B \in I_h^{\mathrm{comp}}(\varkappa), \quad B' \in I_h^{\mathrm{comp}}(\varkappa^{-1})$$

which *quantize* $\varkappa$ *microlocally near* (x_j,ξ_j) in the following sense:

$$\begin{aligned} BB' &= I + \mathcal{O}(h^\infty) \quad \text{microlocally near } (x_1,\xi_1), \\ B'B &= I + \mathcal{O}(h^\infty) \quad \text{microlocally near } (x_2,\xi_2). \end{aligned} \tag{2.3.15}$$

More precisely, the first statement in (2.3.15) means that $BB' = \mathrm{Op}_h(b) + \mathcal{O}(h^\infty)$ for some $b \in C_0^\infty(T^*M_1)$ such that $b = 1$ in a neighborhood of (x_1,ξ_1).

If p is a symbol on T^*M_1 and $P = \mathrm{Op}_h(p)$ is the corresponding pseudodifferential operator, then Egorov's Theorem (2.3.13) implies that

$$B'PB = \mathrm{Op}_h(p \circ \varkappa) + \mathcal{O}(h) \quad \text{microlocally near } (x_2,\xi_2). \tag{2.3.16}$$

This gives rise to the following powerful technique in microlocal analysis: choose $\varkappa$ such that $p\circ\varkappa$ is in some normal form. Then the microlocal behavior of P near (x_1,ξ_1) is conjugated by B, B' to the behavior of the conjugated operator $\mathrm{Op}_h(p\circ\varkappa)$, which may be easier to analyze.

2.3.3. Semiclassical interpretation of generalized FUP. We now present a semiclassical interpretation of Proposition 2.3.2, similarly to how Proposition 2.2.1 is a semiclassical interpretation of Theorem 2.1.1. Let $\mathcal{B} = \mathcal{B}_h$ be given by (2.3.1) where

Φ, b satisfy conditions (1)–(3) in §2.3.1. We moreover assume that $b(x, y; h)$ is smooth in $h \in [0, h_0)$.

Assume that there exists a symplectomorphism $\varkappa : U_2 \to U_1$, where $U_j \subset T^*\mathbb{R}$,

$$(x, \xi) = \varkappa(y, \eta) \iff \xi = \partial_x \Phi(x, y),\ \eta = -\partial_y \Phi(x, y). \tag{2.3.17}$$

This can always be achieved if we shrink the domain U of Φ. Indeed, the right-hand side of (2.3.17) is a two-dimensional submanifold of $T^*\mathbb{R}^2$ which (as follows from the nondegeneracy condition (2.3.2)) locally projects diffeomorphically onto the (x, ξ) variables; the resulting $\varkappa$ is automatically a symplectomorphism.

Comparing the oscillatory integral expressions (2.3.1) and (2.3.11), we see that $\mathcal{B}$ is a Fourier integral operator associated to $\varkappa$. Take some $\mathcal{B}' \in I_h^{\mathrm{comp}}(\varkappa^{-1})$, then for each $X, Y \subset \mathbb{R}$ we have

$$\| \mathbb{1}_{X(h^\rho)} \mathcal{B}\, \mathbb{1}_{Y(h^\rho)} \mathcal{B}' \|_{L^2 \to L^2} \leq C \| \mathbb{1}_{X(h^\rho)} \mathcal{B}\, \mathbb{1}_{Y(h^\rho)} \|_{L^2 \to L^2}. \tag{2.3.18}$$

The right-hand side of (2.3.18) is estimated by the generalized FUP, Proposition 2.3.2, when X, Y are δ-regular. On the left-hand side, the operator $\mathbb{1}_{X(h^\rho)}$ is a quantization of the symbol $\mathbf{1}_{X(h^\rho)}(x)$ (ignoring that the latter is not smooth). Using Egorov's Theorem (2.3.13), we see that $\mathcal{B}\, \mathbb{1}_{Y(h^\rho)} \mathcal{B}'$ formally is a quantization of the symbol

$$\big(\mathbf{1}_{Y(h^\rho)} \circ \pi_x \circ \varkappa^{-1}\big) \sigma_h(\mathcal{B}\mathcal{B}'), \quad \pi_x(x, \xi) = x. \tag{2.3.19}$$

We thus introduce a quantization procedure which applies to (smoothened out) symbols of the form (2.3.19). This procedure (more precisely, its more general version in §2.3.4 below) is crucial for the applications of FUP, since it will let us quantize classical observables propagated for a long time under the geodesic flow.

Symbols of the type (2.3.19) lie in an anisotropic class similar to (2.2.3) and (2.2.4) except instead of horizontal/vertical direction, they are smooth in the direction given by the vector field $d\varkappa \cdot \partial_\xi$. This motivates the following

Definition 2.3.7. *Assume that $U \subset T^*\mathbb{R}$ is an open set and $L \subset TU$ is a* ***smooth one-dimensional foliation****, i.e. to each $(x, \xi) \in U$ corresponds a one-dimensional subspace*

$$L_{(x,\xi)} \subset T_{(x,\xi)}(T^*\mathbb{R}) \simeq \mathbb{R}^2.$$

Fix $\rho \in [0, 1)$. We say that a function $a(x, \xi; h)$ lies in the class $S^{\mathrm{comp}}_{L,\rho}(U)$ if the (x, ξ)-support of a is contained in an h-independent compact subset of U and the following derivative bounds hold:

$$|V_1 \dots V_j W_1 \dots W_k a| = \mathcal{O}(h^{-\rho j})$$

for all vector fields $V_1, \dots, V_j, W_1, \dots, W_k$ on U such that

$W_1, \dots, W_k$ are tangent to L.

In other words, we do not lose any power of h when differentiating along L and lose $h^{-\rho}$ when differentiating in other directions.

Note that the symbol classes $S_{L_0,\rho}$ and $S_{L_1,\rho}$ previously introduced in (2.2.3) and (2.2.4) correspond to the foliations

$$L_0 = \operatorname{span}(\partial_\xi), \quad L_1 = \operatorname{span}(\partial_x).$$

Let L be some foliation on $U \subset T^*\mathbb{R}$ and $a \in S^{\mathrm{comp}}_{L,\rho}(U)$. We define the quantization $\operatorname{Op}_h^L(a) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ following the steps below:

(1) If $L = L_0 = \operatorname{span}(\partial_\xi)$, that is a satisfies the derivative bounds (2.2.3), then we use the standard quantization $\operatorname{Op}_h^{L_0}(a) := \operatorname{Op}_h(a)$ as in §2.2.1.

(2) Assume that U is small. Then we can find an exact symplectomorphism onto its image $\varkappa : U \to T^*\mathbb{R}$ which straightens out the foliation L, so that $\varkappa_*(L) = L_0$. The function $a \circ \varkappa^{-1}$ lies in $S^{\mathrm{comp}}_{L_0}(T^*\mathbb{R})$, so it can be quantized using standard quantization, giving $\operatorname{Op}_h(a \circ \varkappa^{-1})$.

If $\operatorname{supp} a$ is small enough then we can find Fourier integral operators $B \in I_h^{\mathrm{comp}}(\varkappa)$, $B' \in I_h^{\mathrm{comp}}(\varkappa^{-1})$ which quantize $\varkappa$ near $\operatorname{supp} a$ in the sense of (2.3.15). We define

$$\operatorname{Op}_h^L(a) := B' \operatorname{Op}_h(a \circ \varkappa^{-1}) B. \tag{2.3.20}$$

Note that if a is a nice symbol, i.e. its derivatives are bounded uniformly in h, then by Egorov's Theorem (2.3.16) we have $\operatorname{Op}_h^L(a) = \operatorname{Op}_h(a) + \mathcal{O}(h)$.

(3) For the case of general a, we use a partition of unity to split it into pieces which can be quantized using step (2).

Of course the resulting quantization is non-canonical. However the corresponding class of operators (and the support of the corresponding full symbols) does not depend on the specific quantization procedure used; the principal symbol is canonically defined modulo $\mathcal{O}(h^{1-\rho})$. The constructed calculus satisfies the standard properties (1.1.3)–(1.1.5), with powers of h in the remainders replaced by powers of $h^{1-\rho}$. We refer the reader to [**DZ16**, §3] and [**DJ17b**, Appendix] for the proofs.

The following version of Egorov's Theorem holds: if $L \subset TU$, $L' \subset TU'$ are foliations and $\varkappa : U \to U'$ is an exact symplectomorphism such that $\varkappa_*(L) = L'$, then for each $B \in I_h^{\mathrm{comp}}(\varkappa)$, $B' \in I_h^{\mathrm{comp}}(\varkappa^{-1})$ and $a \in S^{\mathrm{comp}}_{L,\rho}(U)$, we have

$$\begin{gathered} B \operatorname{Op}_h^L(a) B' = \operatorname{Op}_h^{L'}(a_\varkappa) + \mathcal{O}(h^\infty)_{L^2 \to L^2} \\ \text{for some } a_\varkappa \in S^{\mathrm{comp}}_{L',\rho}(U'), \quad \operatorname{supp} a_\varkappa \subset \varkappa(\operatorname{supp} a). \end{gathered} \tag{2.3.21}$$

We are now ready to give the semiclassical formulation of Proposition 2.3.2:

Proposition 2.3.8. *Assume that $\delta, \rho \in (0,1)$ and*

- *$U \subset T^*\mathbb{R}$ is an open set, $\psi_X, \psi_Y : U \to \mathbb{R}$ are smooth functions, and*
$$\{\psi_X, \psi_Y\} \neq 0 \quad \text{on } U; \tag{2.3.22}$$
- *$L_X = \ker d\psi_X$, $L_Y = \ker d\psi_Y$ are the foliations corresponding to ψ_X, ψ_Y;*

- *$X, Y \subset \mathbb{R}$ are δ-regular with constant C_R on scales 0 to 1;*
- *$a_X \in S^{\mathrm{comp}}_{L_X,\rho}(U)$, $a_Y \in S^{\mathrm{comp}}_{L_Y,\rho}(U)$ and $\psi_X(\operatorname{supp} a_X) \subset X(h^\rho)$, $\psi_Y(\operatorname{supp} a_Y) \subset Y(h^\rho)$.*

Then there exists $\beta = \beta(\delta, C_R) > \max(0, \frac{1}{2} - \delta)$ such that

$$\| \operatorname{Op}_h^{L_X}(a_X) \operatorname{Op}_h^{L_Y}(a_Y)\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} = \mathcal{O}(h^{\beta-2(1-\rho)}) \quad \text{as } h \to 0. \tag{2.3.23}$$

Remark. Proposition 2.2.1 (for compactly supported a_0, a_1) is the special case when $U = T^*\mathbb{R}$, $\psi_X(x,\xi) = \xi$, $\psi_Y(x,\xi) = x$.

Proof. By a partition of unity we may reduce to the case when a_X, a_Y are supported in a small h-independent set. There exists an exact symplectomorphism which maps ψ_X to x and thus L_X to the vertical foliation $L_0 = \operatorname{span}(\partial_\xi)$. Conjugating by Fourier integral operators quantizing this symplectomorphism according to (2.3.15), and using Egorov's Theorem 2.3.21, we reduce to the case

$$\psi_X(x,\xi) = x.$$

Shrinking U if necessary, we fix an exact symplectomorphism $\varkappa : U \to T^*\mathbb{R}$ such that $\psi_Y \circ \varkappa^{-1} = x$ and thus $(\psi_Y)_* L_Y = L_0$. Then the conjugation procedure (2.3.20) gives

$$\operatorname{Op}_h^{L_Y}(a_Y) = B' \operatorname{Op}_h(\tilde a_Y) B \quad \text{where } B \in I_h^{\mathrm{comp}}(\varkappa),\ B' \in I_h^{\mathrm{comp}}(\varkappa^{-1}),\ \tilde a_Y = a_Y \circ \varkappa^{-1}.$$

The symbols $a_X, \tilde a_Y$ have the support property

$$\operatorname{supp} a_X \subset \{x \in X(h^\rho)\}, \quad \operatorname{supp} \tilde a_Y \subset \{x \in Y(h^\rho)\}. \tag{2.3.24}$$

Arguing as in the proof of Proposition 2.2.1, we estimate

$$\begin{aligned} \| \operatorname{Op}_h^{L_X}(a_X) \operatorname{Op}_h^{L_Y}(a_Y)\|_{L^2\to L^2} &= \| \operatorname{Op}_h(a_X) B' \operatorname{Op}_h(\tilde a_Y) B\|_{L^2\to L^2} \\ &\le C\| \operatorname{Op}_h(a_X) B' \operatorname{Op}_h(\tilde a_Y)\|_{L^2\to L^2} \\ &\le C\| \mathbb{1}_{X(2h^\rho)}\, B'\, \mathbb{1}_{Y(2h^\rho)} \|_{L^2\to L^2} + \mathcal{O}(h^\infty). \end{aligned} \tag{2.3.25}$$

The condition $\{\psi_X, \psi_Y\} \neq 0$ can be rewritten as $\partial_\xi(x \circ \varkappa) \neq 0$. Shrinking U if necessary, we see that the graph

$$\{(x,y,\xi,\eta) \in T^*\mathbb{R}^2 \mid (y,\eta) = \varkappa(x,\xi)\}$$

projects diffeomorphically onto the x, y variables. Therefore we can write this graph as

$$\{\xi = F_x(x,y),\ \eta = -F_y(x,y)\}. \tag{2.3.26}$$

Since $\varkappa$ is a symplectomorphism, the 1-form $F_x\, dx + F_y\, dy$ is exact, that is $F_x = \partial_x \Phi$ and $F_y = \partial_y \Phi$ for some function $\Phi(x,y)$. Then Φ and $\varkappa^{-1}$ are related by (2.3.17), meaning that $B' \in I_h^{\mathrm{comp}}(\varkappa^{-1})$ can be written in the form (2.3.1):

$$B' = \mathcal{B}_h \quad \text{for some choice of the symbol } b.$$

The function Φ satisfies the nondegeneracy condition $\partial^2_{xy}\Phi \neq 0$ since (2.3.26) projects diffeomorphically onto the (x,ξ) variables (being a graph).

Now the generalized FUP, Proposition 2.3.2, gives

$$\| \mathbb{1}_{X(2h^\rho)} B' \mathbb{1}_{Y(2h^\rho)} \|_{L^2\to L^2} = \mathcal{O}(h^{\beta-2(1-\rho)})$$

which together with (2.3.25) shows (2.3.23). □

2.3.4. General microlocal FUP. We finally extend the semiclassically interpreted generalized FUP (Proposition 2.3.8) to higher dimensional manifolds. The fractal directions are still one-dimensional (so Theorem 2.1.1 is still used), but we add extra directions in which essentially nothing happens. This needs to be done for the applications, since Proposition 2.3.8 concerns operators on $\mathbb{R}$ and the surfaces are two-dimensional. The main result of this section, Theorem 2.3.1, is the version of FUP which is used in later sections in applications.

We first discuss quantization of rough symbols in higher dimensions, generalizing the constructions of §§2.2.1,2.3.3. Fix $\rho \in [0,1)$. We start with the following generalization of (2.2.3): we say that

$$a(x,\xi;h) \in S_{L_0,\rho} \quad \text{if} \quad |\partial_x^\alpha \partial_\xi^\beta a| = \mathcal{O}(h^{-\rho|\alpha|}). \tag{2.3.27}$$

The standard quantization $\mathrm{Op}_h(a)$ of symbols in $S_{L_0,\rho}$ enjoys similar properties to the ones in §2.2.1 and will be used as a model for quantizing more general anisotropically bounded symbols.

For the general case, we consider pullbacks of compactly supported symbols in $S_{L_0,\rho}$ by some symplectomorphism $\varkappa$. Such symbols are regular along the foliation $\varkappa_*^{-1}L_0$ where L_0 is the vertical foliation on $T^*\mathbb{R}^n$:

$$L_0 = \ker(dx) = \mathrm{span}(\partial_{\xi_1},\dots,\partial_{\xi_n}).$$

This leads to the following

Definition 2.3.9. *Let $U \subset T^*M$ be an open set. A* ***Lagrangian foliation*** *L is a smooth map*

$$(x,\xi) \in U \ \mapsto\ L_{(x,\xi)} \subset T_{(x,\xi)}(T^*M)$$

such that:

- *L is integrable, i.e. the Lie bracket of any two vector fields tangent to L is also tangent to L;*
- *each subspace $L_{(x,\xi)}$ is Lagrangian.*

Remark. Any Lagrangian foliation can be locally mapped by a symplectomorphism to L_0. In particular, one can always locally write

$$L = \ker d(\psi_1,\dots,\psi_n) := \ker(d\psi_1) \cap \dots \cap \ker(d\psi_n)$$

where $\psi_1,\dots,\psi_n : U \to \mathbb{R}$ have linearly independent differentials and $\{\psi_j,\psi_k\} = 0$.

To each Lagrangian foliation L corresponds the symbol class $S^{\mathrm{comp}}_{L,\rho}(U)$ similarly to Definition 2.3.7. Following the procedure described in §2.3.3, we define a quantization procedure

$$a \in S^{\mathrm{comp}}_{L,\rho}(U) \ \mapsto\ \mathrm{Op}_h^L(a) : L^2(M) \to L^2(M).$$

The general microlocal FUP we use is

THEOREM 2.3.1. *Assume that $\rho,\delta \in (0,1)$, M is an n-dimensional manifold, and*

- *$U \subset T^*M$ is an open set, $\psi_X,\psi_Y,\psi_2,\dots,\psi_n : U \to \mathbb{R}$ are smooth functions with linearly independent differentials, and*
$$\{\psi_X,\psi_Y\} \neq 0; \quad \{\psi_X,\psi_j\} = \{\psi_Y,\psi_j\} = \{\psi_j,\psi_k\} = 0 \quad \text{on } U; \tag{2.3.28}$$
- *$L_X = \ker d(\psi_X,\psi_2,\dots,\psi_n)$, $L_Y = \ker d(\psi_Y,\psi_2,\dots,\psi_n)$ are the corresponding Lagrangian foliations;*
- *$X, Y \subset \mathbb{R}$ are δ-regular with constant C_R on scales 0 to 1;*
- *$a_X \in S^{\mathrm{comp}}_{L_X,\rho}(U)$, $a_Y \in S^{\mathrm{comp}}_{L_Y,\rho}(U)$ and $\psi_X(\operatorname{supp} a_X) \subset X(h^\rho)$, $\psi_Y(\operatorname{supp} a_Y) \subset Y(h^\rho)$.*

Then there exists $\beta = \beta(\delta, C_R) > \max(0,\frac{1}{2}-\delta)$ such that

$$\|\mathrm{Op}_h^{L_X}(a_X)\,\mathrm{Op}_h^{L_Y}(a_Y)\|_{L^2(M)\to L^2(M)} = \mathcal{O}(h^{\beta-2(1-\rho)}) \quad \text{as } h \to 0. \tag{2.3.29}$$

PROOF. We follow the proof of Proposition 2.3.8. First of all, using a partition of unity and conjugating by appropriately chosen Fourier integral operators, we reduce to the case when $M = \mathbb{R}^n$ and

$$\psi_X(x,\xi) = x_1,\ \ \psi_j(x,\xi) = x_j.$$

Fix an exact symplectomorphism $\varkappa : U \to T^*\mathbb{R}^n$ such that

$$\psi_Y = x_1 \circ \varkappa; \quad x_j = x_j \circ \varkappa, \quad j = 2,\dots,n.$$

We write vectors in $\mathbb{R}^n$ as (x_1,x') where $x' \in \mathbb{R}^{n-1}$. Since $\varkappa$ is a symplectomorphism, we have

$$\partial_{\xi'}(x_1\circ\varkappa) = \partial_{\xi'}(\xi_1\circ\varkappa) = 0, \quad \partial_{\xi'}(\xi'\circ\varkappa) = I.$$

Similarly to (2.3.25), we estimate for any $B' \in I_h^{\mathrm{comp}}(\varkappa^{-1})$ which is elliptic near $\operatorname{supp} a_Y$

$$\|\mathrm{Op}_h^{L_X}(a_X)\,\mathrm{Op}_h^{L_Y}(a_Y)\|_{L^2\to L^2} \leq C\|\mathbb{1}_{X(2h^\rho)}(x_1)B'\,\mathbb{1}_{Y(2h^\rho)}(x_1)\|_{L^2\to L^2} + \mathcal{O}(h^\infty). \tag{2.3.30}$$

Shrinking U if necessary and using the condition $\{\psi_X,\psi_Y\}\neq 0$, we see that the graph

$$\{(x,y,\xi,\eta) \in T^*\mathbb{R}^{2n} \mid (y,\eta) = \varkappa(x,\xi)\} \tag{2.3.31}$$

projects diffeomorphically onto the x, y_1, η' variables. Thus we write this graph as

$$\{\xi_1 = F_x(x,y_1),\ \xi' = \eta' + G(x,y_1),\ y' = x',\ \eta_1 = -F_y(x,y_1)\}. \tag{2.3.32}$$

Since $\varkappa$ is an exact symplectomorphism, the restriction to (2.3.32) of the 1-form $\xi_1\,dx_1 + \xi'\,dx' - \eta_1\,dy_1 + y'\,d\eta'$ is exact. That is, we have for some function $\Phi(x, y_1)$

$$F_x = \partial_{x_1}\Phi, \quad F_y = \partial_{y_1}\Phi, \quad G = \partial_{x'}\Phi.$$

Moreover, since $\varkappa$ is a symplectomorphism, we have $\partial^2_{x_1y_1}\Phi \neq 0$. The phase function

$$\Psi(x, y, \theta) = \Phi(x, y_1) + \langle x' - y', \theta\rangle, \quad \theta \in \mathbb{R}^{n-1}$$

parametrizes the symplectomorphism $\varkappa^{-1}$. Assuming without loss of generality that the symbol of B' depends only on x, y_1, we write

$$B'f(x) = (2\pi h)^{-(n-1/2)} \int_{\mathbb{R}^{2n-1}} e^{\frac{i}{h}\Psi(x,y,\theta)} b(x, y_1; h) f(y)\, dyd\theta,$$

that is, using Fourier inversion formula,

$$B'f(x_1, x') = (2\pi h)^{-1/2} \int_{\mathbb{R}} e^{\frac{i}{h}\Phi(x,y_1)} b(x, y_1; \theta) f(y_1, x')\, dy_1.$$

For each x', denote by $\mathcal{B}_{x'}$ the operator on $L^2(\mathbb{R})$ given by

$$\mathcal{B}_{x'} f(x_1) = (2\pi h)^{-1/2} \int_{\mathbb{R}} e^{\frac{i}{h}\Phi(x_1,x',y_1)} b(x_1, x', y_1; \theta) f(y_1)\, dy_1.$$

Then

$$(B'f)(\bullet, x') = \mathcal{B}_{x'}(f(\bullet, x')).$$

Now, the generalized uncertainty principle (Proposition 2.3.2) gives uniformly in x'

$$\| \mathbb{1}_{X(2h^\rho)}\, \mathcal{B}_{x'}\, \mathbb{1}_{Y(2h^\rho)} \|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} = \mathcal{O}(h^{\beta-2(1-\rho)}).$$

It follows that

$$\| \mathbb{1}_{X(2h^\rho)}(x_1) B'\, \mathbb{1}_{Y(2h^\rho)}(x_1) \|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} = \mathcal{O}(h^{\beta-2(1-\rho)})$$

and combining this with (2.3.30) we get (2.3.29). □

EXERCISE 2.3.10. *Let $\Phi(x_1, y_1; x')$ be the function constructed in the proof of Theorem 2.3.1. Show that for all $(x, \xi) \in U$ we have*

$$\partial^2_{x_1y_1}\Phi(\psi_X(x,\xi), \psi_Y(x,\xi); \psi_2(x,\xi), \dots, \psi_n(x,\xi)) = -\frac{1}{\{\psi_X, \psi_Y\}(x,\xi)}.$$

Deduce from here that for any $x_1, y_1, \tilde{x}_1, \tilde{y}_1, x'$, we have

$$\Phi(x_1, y_1; x') - \Phi(\tilde{x}_1, y_1; x') - \Phi(x_1, \tilde{y}_1; x') + \Phi(\tilde{x}_1, \tilde{y}_1; x') = \int_\gamma \omega$$

where ω is the symplectic form and γ is any simple embedded rectangle in T^*M whose 4 sides lie on the submanifolds

$$\{\psi_X = x_1\}, \quad \{\psi_Y = y_1\}, \quad \{\psi_X = \tilde{x}_1\}, \quad \{\psi_Y = \tilde{y}_1\}$$

intersected with $\{(\psi_2, \dots, \psi_n) = x'\}$.

CHAPTER 3

From FUP to eigenfunction control

3.1. Local dynamics of the geodesic flow

To prove Proposition 3.2.2, we first need to understand microlocal properties of the operators $A_1(t)$ when $|t| \leq \rho \log(1/h)$. Recall that by Egorov's Theorem $A_1(t) = \mathrm{Op}_h(a_1 \circ \varphi_t) + \mathcal{O}(h)$ as long as t is bounded independently of h. However we need to take t growing with h and this merits a separate discussion. The proof of Egorov's Theorem, sketched in (1.1.11), uses only basic properties of the quantization procedure, specifically the commutator formula. Thus the main difficulty with extending it to larger times is the ability to quantize the propagated symbol $a_1 \circ \varphi_t$. It turns out that this symbol lies in an anisotropic class similar to (2.2.3), (2.2.4).

We use the *stable/unstable decomposition*, which here is given by the frame

$$H_p, D, U_-, U_+ \in C^\infty(T^*M \setminus 0; T(T*^M \setminus 0))$$

where

- H_p is the generator of the homogeneous geodesic flow φ_t;
- $D = \xi \cdot \partial_\xi$ is the generator of dilations on the fibers of T^*M, which is preserved by the flow φ_t;
- U_- is the *unstable horocyclic vector field*, having the property
$$d\varphi_t(x,\xi) \cdot U_-(x,\xi) = e^t\, U_-(\varphi_t(x,\xi)), \tag{3.1.1}$$
that is U_- grows exponentially along the geodesic flow;
- U_+ is the *stable horocyclic vector field*, having the property
$$d\varphi_t(x,\xi) \cdot U_+(x,\xi) = e^{-t}\, U_+(\varphi_t(x,\xi)), \tag{3.1.2}$$
that is U_- decays exponentially along the geodesic flow.

See for instance [TODO] for the definition of the vector fields $U_\pm$.[1]

Using (3.1.1) and (3.1.2), we get the derivative bounds

$$|H_p^{k_0} D^{k_1} U_-^{k_-} U_+^{k_+}(a \circ \varphi_t)| = \mathcal{O}(e^{(k_- - k_+)t}). \tag{3.1.3}$$

[1]For hyperbolic surfaces the vector fields $U_\pm$ are given by explicit formulas and are smooth. For general negatively curved surfaces the analogues of $U_\pm$ are merely Hölder continuous. This lack of regularity is the main reason we carry out the arguments in constant curvature.

So if for instance $t \to \infty$, then the unstable derivatives of the propagated symbol grow exponentially but the rest of the derivatives stay bounded. This means that as long as $t \geq 0$ is bounded by the *Ehrenfest time*[2] $\rho \log(1/h)$, the symbol $a \circ \varphi_t$ satisfies derivative bounds

$$|H_p^{k_0} D^{k_1} U_-^{k_-} U_+^{k_+}(a \circ \varphi_t)| = \mathcal{O}(h^{-\rho(k_- + k_1)}). \tag{3.1.4}$$

We denote by $S_{L_s,\rho}$ the class of symbols (compactly supported on $T^*M \setminus 0$) which satisfy bounds (3.1.4). Here L_s and L_u denote the weak stable and unstable foliations:

$$L_s := \operatorname{span}(H_p, U_+), \quad L_u := \operatorname{span}(H_p, U_-).$$

Symbols in the class $S_{L_s,\rho}$ do not grow when differentiated along the leaves of L_s and are allowed to grow by $h^{-\rho}$ when differentiated in other directions. It may be useful to think of the support of such a symbol as a union of h^ρ-neighborhoods of local weak stable leaves. Similarly for negative times, if $-\rho \log(1/h) \leq t \leq 0$ then $a \circ \varphi_t$ lies in the class $S_{L_u,\rho}$ defined by replacing k_- by k_+ in (3.1.4).

There exist quantization procedures for symbols in the classes $S_{L_s,\rho}$ and $S_{L_u,\rho}$:

$$a \in S_{L,\rho} \mapsto \operatorname{Op}_h^L(a) : L^2(M) \to L^2(M), \quad L \in \{L_s, L_u\}.$$

This quantization procedure is constructed in [**DZ16**, §3]. Roughly speaking it works as follows. Take $L \in \{L_u, L_s\}$. Then the family of subspaces L gives rise to a foliation of T^*M with Lagrangian leaves, which means that near each point of $T^*M \setminus 0$ we can find a symplectomorphism $\varkappa_L : T^*M \to T^*\mathbb{R}^2$ which sends L to the vertical foliation:

$$(\varkappa_L)_*(L) = L_0 = \operatorname{span}(\partial_{\xi_1}, \partial_{\xi_2}). \tag{3.1.5}$$

The symplectomorphism $\varkappa_L$ can be quantized by a *Fourier integral operator* (see [**Zw12**, §11.2]), which we denote by

$$B_L : L^2(M) \to L^2(\mathbb{R}^2). \tag{3.1.6}$$

We then define for $a \in S_{L,\rho}$

$$\operatorname{Op}_h^L(a) := B_L^{-1} \operatorname{Op}_h(a \circ \varkappa_L^{-1}) B_L : L^2(M) \to L^2(M)$$

where $a \circ \varkappa_L^{-1}$ is a symbol on $T^*\mathbb{R}^2$ satisfying the anisotropic bound (2.2.3), and it is quantized using the standard procedure from (2.2.1).

The quantization procedure Op_h^L satisfies asymptotic expansions for products and adjoints if both symbols a, b lie in $S_{L,\rho}$ for the same choice of L. However, if one multiplies $\operatorname{Op}_h^{L_s}(a) \operatorname{Op}_h^{L_u}(b)$ for some $a \in S_{L_s,\rho}$, $b \in S_{L_u,\rho}$, the resulting symbol does not lie in a pseudodifferential calculus. In fact, the norm bound on such products is where the fractal uncertainty principle comes in the game – see [TODO] below.

[2]TODO regarding the Ehrenfest time

If $a \in S_{L,\rho}$ and p is as above [TODO], then $H_p a$ is bounded owing to (3.1.3). This makes it possible to show the commutation rule

$$[h\sqrt{-\Delta}, \operatorname{Op}_h^L(a)] = -ih \operatorname{Op}_h^L(\{p, a\}) + \mathcal{O}(h^{2-\rho}).$$

Arguing as in (1.1.11) we obtain the following version of Egorov's theorem:

$$A_1(t) = \operatorname{Op}_h^{L_s}(a_1 \circ \varphi_t) + \mathcal{O}(h^{1-\rho})_{L^2 \to L^2} \quad \text{for } 0 \leq t \leq \rho \log(1/h), \tag{3.1.7}$$

$$A_1(t) = \operatorname{Op}_h^{L_u}(a_1 \circ \varphi_t) + \mathcal{O}(h^{1-\rho})_{L^2 \to L^2} \quad \text{for } -\rho \log(1/h) \leq t \leq 0. \tag{3.1.8}$$

TODO remark about cheating with the remainder here.

3.2. The proof

We now give some ideas for how the fractal uncertainty principle (Theorem 2.1.1) can be used to obtain the results advertised in the preceding sections (Theorems 1.1.3 and 1.2.1).

We start with Theorem 1.1.3. Let M be a compact hyperbolic surface and fix $a \in C_0^\infty(T^*M)$ such that $a|_{S^*M} \not\equiv 0$. For simplicity we assume that $0 \leq a \leq 1$ and

$$a \equiv 1 \quad \text{on some open nonempty set } \mathcal{U} \subset S^*M.$$

The general case can be reduced to this one via the following consequence of the semiclassical elliptic estimate:

$$a, b \in C_0^\infty(T^*M),\ \operatorname{supp} a \subset \{b \neq 0\} \implies \|\operatorname{Op}_h(a)u\|_{L^2} \leq \|\operatorname{Op}_h(b)u\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2}.$$

We now introduce a pseudodifferential partition of unity induced by the partition $1 = a + (1-a)$. To simplify the notation below, denote

$$a_1 := a, \quad a_2 := 1 - a,$$

and take the operators

$$A_j := \operatorname{Op}_h(a_j), \quad A_1 + A_2 = I.$$

For simplicity we will assume that u is an eigenfunction. Then the statement of Theorem 1.1.3 takes the form

$$\|u\|_{L^2} \leq C\|A_1 u\|_{L^2} \quad \text{when } 0 < h \ll 1, \quad (-h^2\Delta - 1)u = 0. \tag{3.2.1}$$

Recall the wave group $U(t) = e^{-it\sqrt{-\Delta}}$. Since u is an eigenfunction, we have $U(t)u = e^{-it/h}u$. Denote

$$A_j(t) := U(-t)A_j U(t).$$

Then we control $A_1(t)u$ as follows:

$$\|A_1(t)u\|_{L^2} = \|A_1 U(t)u\|_{L^2} = \|A_1 u\|_{L^2}. \tag{3.2.2}$$

The proof of Theorem 1.1.3 uses the operators $A_1(t)$ with $|t| \le \log(1/h)$, that is time will grow as logarithm of the frequency. More specifically, define

$$N_1 := \lfloor \rho \log(1/h) \rfloor,$$

and consider the partition of unity $I = A_{\mathcal{X}} + A_{\mathcal{Y}}$ where

$$\begin{aligned} A_{\mathcal{X}} &:= A_2(N_1) \cdots A_2(1) A_2(0) A_2(-1) \cdots A_2(-N_1), \\ A_{\mathcal{Y}} &:= \sum_{\ell=-N_1}^{N_1} A_2(N_1) \cdots A_2(\ell+1) A_1(\ell). \end{aligned} \tag{3.2.3}$$

The function $A_{\mathcal{Y}} u$ is controlled as follows (here we use the norm bound $\|A_2\|_{L^2 \to L^2} = 1 + \mathcal{O}(h^{1/2})$, see [**Zw12**, Theorem 5.1]):

$$\|A_{\mathcal{Y}} u\|_{L^2} \le \sum_{\ell=-N_1}^{N_1} \|A_1(\ell) u\|_{L^2} \le C \log(1/h) \|A_1 u\|_{L^2}. \tag{3.2.4}$$

The right-hand side here is almost what we want except for the extra $\log(1/h)$ factor which would have to be put into the right-hand side of (1.1.15). This would have disastrous effects for some applications, for instance Theorem 1.1.5 would no longer follow. To remove the logarithmic factor, we need to revise the definition of the operators $A_{\mathcal{X}}, A_{\mathcal{Y}}$ in a way inspired by [**An08**], by fixing small $\alpha > 0$ and putting into $A_{\mathcal{X}}$ products of the form

$$A_{\mathbf{w}} = A_{w_{N_1}}(N_1) \cdots A_{w_{-N_1}}(-N_1)$$

for words $\mathbf{w} = w_{N_1} \dots w_{-N_1}$ such that at most αN_1 of the letters $w_{N_1}, \dots, w_{-N_1}$ are equal to 1. (This definition has to be further modified, see [**DJ17b**, TODO].) Then $A_{\mathcal{Y}} u$ will be the sum over words which have at least αN_1 letters equal to 1, and this can be estimated by just $\|A_1 u\|_{L^2}$, without the log factor. As for $A_{\mathcal{X}} u$, the number of words featured in $A_{\mathcal{X}}$ is at most $h^{-4\sqrt{\alpha}}$ by Stirling's formula, so for small enough $\alpha > 0$ depending on the fractal uncertainty exponent β, the estimate (3.2.6) below still gives decay with h. We do not provide a detailed argument for removing the log factor here, instead referring the reader to [**DJ17b**].

Exercise 3.2.1. *For general* u *(i.e. not an eigenfunction) revisit* (3.2.2) *and* (3.2.4) *to obtain*

$$\|A_{\mathcal{Y}} u\|_{L^2} \le C \log(1/h) \|A_1 u\|_{L^2} + C \frac{\log(1/h)^2}{h} \|(-h^2 \Delta - 1) u\|_{L^2}.$$

This explains (modulo the extra $\log(1/h)$ *factor discussed above) the second term on the right-hand side of* (1.1.15)*.*

It remains to estimate $A_{\mathcal{X}} u$. This is done by the following proposition, whose proof based on fractal uncertainty principle is sketched in §**??**. Note that this proposition never uses that u is an eigenfunction.

PROPOSITION 3.2.2. *There exists $\beta > 0$ depending only on M and the set $\mathcal{U}$ from [TODO] such that*

$$\|A_{\mathcal{X}}\|_{L^2\to L^2} \leq Ch^{\beta-2(1-\rho)} + Ch^{1-\rho}. \tag{3.2.5}$$

Choosing $\rho < 1$ sufficiently close to 1 (it is enough to take $1-\rho = \beta/3$), we see that

$$\|A_{\mathcal{X}}u\|_{L^2} \leq Ch^{1-\rho}\|u\|_{L^2}. \tag{3.2.6}$$

Together with the version of (3.2.4) which does not have the $\log(1/h)$ factor this gives

$$\|u\|_{L^2} \leq \|A_{\mathcal{X}}u\|_{L^2} + \|A_{\mathcal{Y}}u\|_{L^2} \leq C\|A_1u\|_{L^2} + Ch^{1-\rho}\|u\|_{L^2}.$$

The last term on the right-hand side can be removed for small h, giving Theorem 1.1.3.

3.3. Global dynamics and end of the proof

We now sketch the proof of Proposition 3.2.2 using the fractal uncertainty principle. By (3.1.7), (3.1.8), different directions of propagation lead to operators in different calculi. We thus write $A_{\mathcal{X}}$ as a product:

$$A_{\mathcal{X}} = A_-A_+,\quad A_- = A_1(N_1)\cdots A_1(1)A_1(0),\quad A_+ = A_1(1)\cdots A_1(-N_1).$$

Combining (3.1.7), (3.1.8), and the product rule for the $S_{L,\rho}$ calculus, we have

$$A_- = \operatorname{Op}_h^{L_s}(a_-) + \mathcal{O}(h^{1-\rho})_{L^2\to L^2},\quad a_- := \prod_{\ell=0}^{N_1}(a_1\circ\varphi_\ell);$$

$$A_+ = \operatorname{Op}_h^{L_u}(a_+) + \mathcal{O}(h^{1-\rho})_{L^2\to L^2},\quad a_+ := \prod_{\ell=1}^{N_1}(a_1\circ\varphi_{-\ell}).$$

Therefore to prove Proposition 3.2.2 it suffices to show that

$$\|\operatorname{Op}_h^{L_s}(a_-)\operatorname{Op}_h^{L_u}(a_+)\|_{L^2\to L^2} \leq Ch^{\beta-2(1-\rho)}. \tag{3.3.1}$$

We are finally ready to explain where the fractal structure appears, by establishing a porosity property for the supports of a_-, a_+. We use the following definition:

DEFINITION 3.3.1. *Let $\nu, h > 0$. A set $\Omega \subset \mathbb{R}$ is called ν-*porous *up to scale h if for each interval $I \subset R$ such that $h \leq |I| \leq 1$, there exists an interval $J \subset I$ such that $|J| = \nu|I|$ and $J \cap \Omega = \emptyset$.*

Porous sets need not be δ-regular, however they can be embedded into δ-regular sets for some $\delta > 1$:

LEMMA 3.3.2. *Let $\Omega \subset [0,1]$ be ν-porous up to scale h. Then we have Ω is contained in the h-neighborhood $X(h)$ of some set $X \subset [0,1]$ which is δ-regular with constant C_R on scales 0 to 1, where $\delta = \delta(\nu) < 1$ and $C_R = C_R(\nu)$.*

PROOF. TODO

□

Therefore, the fractal uncertainty principle holds with some $\beta = \beta(\nu) > 0$ for sets X, Y which are ν-porous up to scale h.

For small $\tau > 0$ and $(x, \xi) \in S^*M$, denote

$$W^\tau_\pm(x,\xi) := \{e^{s_1 H_p + s_2 U_\pm + s_3 D}(x,\xi) \colon |s_1| + |s_2| + |s_3| \le \tau\}.$$

We use the microlocal version of FUP from Proposition 2.2.1, which holds for X, Y which are ν-porous up to scale h^ρ. This crucial porosity property for $\operatorname{supp} a_\pm$ is given by

LEMMA 3.3.3. *There exist $\nu = \nu(\mathcal{U}) > 0$ and $\tau = \tau(\mathcal{U}) > 0$ such that for each $(x, \xi) \in S^*M$, the sets*

$$\Omega_\pm(x,\xi) := \{s \in \mathbb{R} \mid W^\tau_\mp(e^{sU_\pm}(x,\xi)) \cap \operatorname{supp} a_\pm \neq \emptyset\}$$

are ν-porous up to scale h^ρ.

PROOF. We just consider the case $\tau = 0$ where

$$\Omega_\pm(x,\xi) = \{s \in \mathbb{R} \mid e^{sU_\pm}(x,\xi) \in \operatorname{supp} a_\pm\}$$

TODO the rest □

Lemma 3.3.3 shows that by a partition of unity we may assume that for some $(x_0, \xi_0) \in S^*M$

$$\operatorname{supp} a_\pm \subset \bigcup_{s \in X_\pm} W^\tau_\mp(e^{sU_\pm}(x_0,\xi_0))$$

where $X_\pm \subset [0,1]$ are ν-porous up to scale h^ρ. In other words, $\operatorname{supp} a_\pm$ have the following structure:

- $\operatorname{supp} a_+$ is contained in a union of homogeneous weak unstable leaves indexed by a ν-porous set;
- $\operatorname{supp} a_-$ is contained in a union of homogeneous weak stable leaves indexed by a ν-porous set.

Now we can reduce (3.3.1), and thus Proposition 3.2.2, to Proposition 2.2.1, or rather its version with a different phase.

CHAPTER 4

Application to spectral gaps

4.1. Localization of resonant states

We now explain how the fractal uncertainty principle is used to prove the spectral gap in Theorem 1.2.1. Let M be a convex co-compact hyperbolic surface. To show that it has an essential spectral gap of some size β, one needs to exclude existence of resonances λ with $\operatorname{Im}\lambda \geq -\beta$ and large $\operatorname{Re}\lambda$. (One also needs the polynomial resolvent bound, which follows from the proof of resonance free region, but is not presented here.) To each resonance corresponds a *resonant state* $u \in C^\infty(M)$, which solves the equation

$$\Big(-\Delta - \frac{1}{4} - \lambda^2\Big)u = 0 \tag{4.1.1}$$

and satisfies the *outgoing condition* in each funnel end. [TODO say a bit more about this condition and about normalization of u, and why we can apply the wave group. Use a basic 1D case as an example.] Thus to show that λ cannot be a resonance it suffices to prove that the only outgoing solution to (4.1.1) is $u \equiv 0$.

Writing

$$h := (\operatorname{Re}\lambda)^{-1}, \quad \nu := -\operatorname{Im}\lambda,$$

and using semiclassical rescaling, we see that u solves the equation

$$(-h^2\Delta - \omega^2)u = 0, \quad \omega := \sqrt{(1-ih\nu)^2 + \frac{h^2}{4}} = 1 - ih\nu + \mathcal{O}(h^2). \tag{4.1.2}$$

We will assume from now on that $\omega = 1 - ih\nu$.

By elliptic estimate, we see that u is concentrated on S^*M in the following sense:

$$\operatorname{supp} a \cap S^*M = \emptyset \quad \Longrightarrow \quad \|\operatorname{Op}_h(a)u\| = \mathcal{O}(h^\infty)\|u\|.$$

As before, we use the wave group $U(t) = e^{-it\sqrt{-\Delta}} : L^2(M) \to L^2(M)$. Then

$$U(t)u = e^{-it\omega/h}u. \tag{4.1.3}$$

This gives the following version of propagation of singularities:

LEMMA 4.1.1. *Assume that $a, b \in C_0^\infty(T^*M \setminus 0)$ and for some $t \geq 0$*

$$\varphi_{-t}(\operatorname{supp} a) \subset \{b \neq 0\}.$$

Then

$$\|\operatorname{Op}_h(a)u\| \le Ce^{\nu t}\|\operatorname{Op}_h(b)u\| + \mathcal{O}(h^\infty)\|u\|.$$

Here $e^{\nu t}$ can be absorbed into the constant but we would like to keep it since we will later take t depending on h.

PROOF. We will only do the $\mathcal{O}(h)$ remainder. We write

$$\|U(-t)\operatorname{Op}_h(a)U(t)u\| = \|\operatorname{Op}_h(a)U(t)u\| = \|e^{-it\omega/h}\operatorname{Op}_h(a)u\| = e^{-\nu t}\|\operatorname{Op}_h(a)u\|.$$

Therefore

$$\|\operatorname{Op}_h(a)u\| = e^{\nu t}\|U(-t)\operatorname{Op}_h(a)U(t)u\|.$$

Now by Egorov's Theorem we have $U(-t)\operatorname{Op}_h(a)U(t) = \operatorname{Op}_h(a\circ\varphi_t) + \mathcal{O}(h)$ and $\operatorname{supp}(a\circ\varphi_t) \subset \{b \neq 0\}$. Thus by ellipticity

$$\|U(-t)\operatorname{Op}_h(a)U(t)u\| \le C\|\operatorname{Op}_h(b)u\| + \mathcal{O}(h)\|u\|.$$

Combining these formulas, we get the needed bound. □

We now introduce the *incoming/outgoing tails* $\Gamma_\pm \subset T^*M\setminus 0$ and the *trapped set* K:

$$\Gamma_\pm := \{(x,\xi)\in T^*M\setminus 0 \mid \varphi_t(x,\xi)\not\to\infty \quad \text{as } t\to\mp\infty\}, \quad K := \Gamma_+\cap\Gamma_-.$$

Note that $\Gamma_\pm$ are closed and $K\cap S^*M$ is compact, in fact it lies inside the convex core (the region outside of all the funnels). The outgoing condition on u is used in the following lemma which reduces analysis to a compact set:

PROPOSITION 4.1.2. *Let u be an outgoing solution to (4.1.2). Then:*

(1) *Assume that $a \in C_0^\infty(T^*M)$ is h-independent and* $\operatorname{supp} a \cap \Gamma_+ \cap S^*M = \emptyset$. *Then*
$$\|\operatorname{Op}_h(a)u\| = \mathcal{O}(h^\infty)\|u\|. \tag{4.1.4}$$
(2) *Assume that $b\in C_0^\infty(T^*M)$ is h-independent and $K\cap S^*M\subset\{b\neq 0\}$. Then for $h \ll 1$*
$$\|u\| \le C\|\operatorname{Op}_h(b)u\|. \tag{4.1.5}$$

PROOF. Complicated, using Vasy [**Va13a**, **Va13b**]. Define the *backwards directly escaping set* $\mathcal{E}_-\subset T^*M\setminus 0$ as follows: $(x,\xi)\in\mathcal{E}_-$ if

- x lies in some funnel end of M, and
- $H_p r(x,\xi) \le 0$ where r is a funnel coordinate.

We have the following dynamical statements:

(1) if $(x,\xi)\in\mathcal{E}_-$, then the geodesic ray $\varphi_t(x,\xi)$, $t\le 0$, escapes to infinity while staying inside $\mathcal{E}_-$;

(2) if $(x,\xi)\in T^*M\setminus 0$ and $(x,\xi)\notin\Gamma_+$, then $\varphi_t(x,\xi)\in\mathcal{E}_-$ for sufficiently large negative t;
(3) if $(x,\xi)\in\Gamma_+$, then $\varphi_t(x,\xi)\to K$ as $t\to-\infty$.

The basic input from the outgoing condition is the following statement:

$$\|\operatorname{Op}_h(a)u\|=\mathcal{O}(h^\infty)\|u\|\quad\text{if }\operatorname{supp}a\subset\mathcal{E}_-. \tag{4.1.6}$$

[TODO basic 1D case]

Now we prove (4.1.4) and (4.1.5) as follows:

- Assume first that $\operatorname{supp}a\cap S^*M\cap\Gamma_+=\emptyset$. By the elliptic estimate and a partition of unity we may assume that $\operatorname{supp}a\cap\Gamma_+=\emptyset$. Then, since $\operatorname{supp}a$ is compact, there exists t and $\tilde a$ such that

$$\varphi_{-t}(\operatorname{supp}a)\subset\{\tilde a\neq 0\},\quad \operatorname{supp}\tilde a\subset\mathcal{E}_-.$$

Then

$$\|\operatorname{Op}_h(a)u\|\leq C\|\operatorname{Op}_h(\tilde a)u\|+\mathcal{O}(h^\infty)\|u\|=\mathcal{O}(h^\infty)\|u\|.$$

- Assume now that $K\cap S^*M\subset\{b\neq 0\}$. First take arbitrary $a=a(x)\in C_0^\infty(M)$. Then we can break a into two pieces: $a=a_1+a_2$ where $\operatorname{supp}a\cap\Gamma_+\cap S^*M=\emptyset$ and $\varphi_{-t}(\operatorname{supp}a_2)\subset\{b\neq 0\}$ for some $t\geq 0$. Estimating $\operatorname{Op}_h(a_1)u$ as before and $\operatorname{Op}_h(a_2)u$ by propagation of singularities, we see that

$$\|\operatorname{Op}_h(a)u\|_{L^2}\leq Ce^{\nu t}\|\operatorname{Op}_h(b)u\|+\mathcal{O}(h^\infty)\|u\|.$$

Now, the entire u can be estimated by $\operatorname{Op}_h(a)u$ if the support of a is large enough (using the outgoing condition again). □

We now take h-dependent cutoffs. Fix $\chi\in C_0^\infty(T^*M\setminus 0)$ such that $\chi=1$ near $K\cap S^*M$ and put

$$\chi_\pm:=\chi(\chi\circ\varphi_{\mp T}),\quad T:=\rho\log(1/h).$$

Then $\chi_+\in S_{L_u,\rho}$ and $\chi_-\in S_{L_s,\rho}$. Moreover, $\operatorname{supp}\chi_\pm\subset\Gamma_\pm(h^\rho)$. Then Proposition 4.1.2 can be upgraded to the following statements:

$$\operatorname{Op}_h(\chi)u=\operatorname{Op}_h^{L_u}(\chi_+)u+\mathcal{O}(h^\infty)\|u\|, \tag{4.1.7}$$

$$\|u\|\leq Ce^{\nu T}\|\operatorname{Op}_h^{L_s}(\chi_-)\operatorname{Op}_h(\chi)u\|. \tag{4.1.8}$$

To see (4.1.7), (4.1.8), we argue as in the proofs (4.1.4), (4.1.5), using propagation of singularities for time T and the following two statements:

- for any $(x,\xi)\in\operatorname{supp}(\chi-\chi_+)$, $\varphi_{-T}(x,\xi)$ lies in some fixed compact set which does not intersect $\Gamma_+\cap S^*M$;
- for any (x,ξ) in a small enough h-independent neighborhood of K, $\varphi_{-T}(x,\xi)$ lies in $\{\chi_-\neq 0\}\cap\{\chi\neq 0\}$.

Combining (4.1.7) and (4.1.8), we get

$$\|u\| \leq Ch^{-\nu} \| \operatorname{Op}_h^{L_s}(\chi_-) \operatorname{Op}_h^{L_u}(\chi_+) u\|.$$

We use the fractal uncertainty principle to estimate $\operatorname{Op}_h^{L_s}(\chi_-) \operatorname{Op}_h^{L_u}(\chi_+)$. It shows that there exists $\beta > \max(0, \frac{1}{2} - \delta)$ such that

$$\| \operatorname{Op}_h^{L_s}(\chi_-) \operatorname{Op}_h^{L_u}(\chi_+)\|_{L^2 \to L^2} \leq Ch^{\beta - 2(1-\rho)}.$$

But then

$$\|u\| \leq Ch^{-\nu + \beta - 2(1-\rho)} \|u\|$$

and if $\nu < \beta$, then we can take ρ close enough to 1 to get a contradiction.

4.2. The limit set and end of proof

To get the fractal structure for $\operatorname{supp} \chi_\pm$, we lift M to the covering space $\mathbb{H}^2$ and use the maps

$$B_\pm : T^*(\mathbb{H}^2 \setminus 0) \to \mathbb{S}^1$$

which map (x, ξ) to the initial and terminal point of the geodesic $\varphi_t(x, \xi)$ on $\mathbb{H}^2$. Take the limit set $\Lambda_\Gamma \subset \mathbb{S}^1$. Then

$$B_\mp(\operatorname{supp} \chi_\pm) \subset \Lambda_\Gamma(h^\rho).$$

The set Λ_Γ is δ-regular as proved by Sullivan.

CHAPTER 5

A simple model: FUP for discrete Cantor sets

In this section we study discrete fractal uncertainty principle for Cantor sets, following [**DJ16**]. The proofs in this section are short and only use elementary tools which makes it a nice introduction to the subject. However, many phenomena seen in the case of Cantor sets are also present for more general fractal uncertainty principles, and the proofs in this special case can be thought of as a 'baby case' of the general proofs.

We next give an application of FUP for Cantor sets to spectral gaps for *open quantum baker's maps,* which are a simple model of quantum chaos that can be studied using elementary harmonic analysis and are also easy to model numerically.

5.1. Definition and basic properties

We use the following notation from discrete harmonic analysis:

- Take large $N \in \mathbb{N}$. This is the size of the matrices studied and it is also the frequency of oscillation. The relation to the semiclassical parameter h is $N = (2\pi h)^{-1}$. We are interested in the limit $N \to \infty$.
- Denote $\mathbb{Z}_N := \{0, \dots, N-1\} = \mathbb{Z}/(N\mathbb{Z})$, we use the ring structure of $\mathbb{Z}_N$.
- Denote $\ell^2_N := \ell^2(\mathbb{Z}_N) = \mathbb{C}^N$, we use the inner product

$$\langle u, v\rangle_{\ell^2_N} = \sum_{j\in\mathbb{Z}_N} u(j)\overline{v(j)}.$$

- Define the *discrete Fourier transform*

$$\mathcal{F}_N : \ell^2_N \to \ell^2_N, \quad \mathcal{F}_N u(j) = \frac{1}{\sqrt{N}} \sum_{\ell\in\mathbb{Z}_N} \exp\Big(-\frac{2\pi i j\ell}{N}\Big) u(\ell),$$

  it is a unitary operator.
- For a subset $X \subset \mathbb{Z}_N$, denote by $\mathbb{1}_X : \ell^2_N \to \ell^2_N$ the multiplication operator by the indicator function of X, that is

$$(\mathbb{1}_X\, u)(j) = \begin{cases} u(j), & j \in X; \\ 0, & j \notin X. \end{cases}$$

- For $u \in \ell^2_N$, define its support by

$$\operatorname{supp} u := \{j \in \mathbb{Z}_N \mid u(j) \neq 0\}.$$

- For $X \subset \mathbb{Z}_N$, denote by $|X|$ the number of elements in X.
- For $X \subset \mathbb{Z}_N$ and $\ell \in \mathbb{Z}_N$, denote

$$X + \ell := \{j + \ell \mid j \in X\}$$

where addition is in the group $\mathbb{Z}_N$.

We are now ready to define the uncertainty principle:

Definition 5.1.1. *Assume that $X = X(N), Y = Y(N)$ are families of subsets of $\mathbb{Z}_N$. We say that X, Y satisfy the* ***uncertainty principle with exponent*** *β if*

$$\| \mathbb{1}_X \mathcal{F}_N \mathbb{1}_Y \|_{\ell^2_N \to \ell^2_N} = \mathcal{O}(N^{-\beta}) \quad \text{as } N \to \infty. \tag{5.1.1}$$

Remarks. 1. More generally one can consider a sequence of values of N tending to infinity. This is what we will do for Cantor sets later, putting $N = M^k$ where M is fixed and $k \to \infty$.

2. In terms of linear algebra, $\| \mathbb{1}_X \mathcal{F}_N \mathbb{1}_Y \|$ is the norm of the matrix obtained from the matrix of $\mathcal{F}_N$ by only keeping rows in X and columns in Y.

3. Another interpretation of (5.1.1) is the following:

$$\| \mathbb{1}_X u\|_{\ell^2_N} = \mathcal{O}(N^{-\beta})\|u\|_{\ell^2_N} \quad \text{for all } u \in \ell^2_N,\ \operatorname{supp}(\mathcal{F}_N^* u) \subset Y. \tag{5.1.2}$$

Here are some basic properties of the norm in (5.1.1):

- *Trivial bound:*

$$\| \mathbb{1}_X \mathcal{F}_N \mathbb{1}_Y \|_{\ell^2_N \to \ell^2_N} \leq 1. \tag{5.1.3}$$

- *Volume bound:*

$$\| \mathbb{1}_X \mathcal{F}_N \mathbb{1}_Y \|_{\ell^2_N \to \ell^2_N} \leq \| \mathbb{1}_X \mathcal{F}_N \mathbb{1}_Y \|_{\mathrm{HS}} = \sqrt{\frac{|X| \cdot |Y|}{N}}. \tag{5.1.4}$$

  Here $\|A\|_{\mathrm{HS}}$ is the Hilbert–Schmidt norm of the matrix $A = (a_{j\ell}) : \ell^2_N \to \ell^2_N$, defined by

$$\|A\|^2_{\mathrm{HS}} = \sum_{j,\ell} |a_{j\ell}|^2.$$

  In particular, if $|X| = |Y| = 1$ then X, Y satisfy the uncertainty principle with $\beta = 1/2$.

- *Lower bound:* if $X, Y \neq \emptyset$ then

$$\| \mathbb{1}_X \mathcal{F}_N \mathbb{1}_Y \|_{\ell^2_N \to \ell^2_N} \geq \sqrt{\frac{\max(|X|, |Y|)}{N}}. \tag{5.1.5}$$

  To see this, it suffices to compute the ℓ^2 norms of rows and columns of the corresponding matrix.

- *Symmetry:* $\| \mathbb{1}_X \mathcal{F}_N \mathbb{1}_Y \|_{\ell^2_N \to \ell^2_N} = \| \mathbb{1}_Y \mathcal{F}_N \mathbb{1}_X \|_{\ell^2_N \to \ell^2_N}$.
- *Invariance under circular shifts*: for any $j, \ell \in \mathbb{Z}_N$

$$\| \mathbb{1}_{X+j} \mathcal{F}_N \mathbb{1}_{Y+\ell} \|_{\ell^2_N \to \ell^2_N} = \| \mathbb{1}_X \mathcal{F}_N \mathbb{1}_Y \|_{\ell^2_N \to \ell^2_N}. \tag{5.1.6}$$

  To see this, we use that the shift operator by ℓ is conjugated by the Fourier transform to the multiplication operator by the character $e_\ell(j) = e^{-2\pi i j \ell / N}$.
- *Monotonicity*: if $X \subset X'$, $Y \subset Y'$ then

$$\| \mathbb{1}_X \mathcal{F}_N \mathbb{1}_Y \|_{\ell^2_N \to \ell^2_N} \leq \| \mathbb{1}_{X'} \mathcal{F}_N \mathbb{1}_{Y'} \|_{\ell^2_N \to \ell^2_N}.$$

- *Triangle inequality*: if $X \subset X_1 \cup X_2$ then

$$\| \mathbb{1}_X \mathcal{F}_N \mathbb{1}_Y \|_{\ell^2_N \to \ell^2_N} \leq \| \mathbb{1}_{X_1} \mathcal{F}_N \mathbb{1}_Y \|_{\ell^2_N \to \ell^2_N} + \| \mathbb{1}_{X_2} \mathcal{F}_N \mathbb{1}_Y \|_{\ell^2_N \to \ell^2_N}$$

  and similarly if we take $Y \subset Y_1 \cup Y_2$.

We finish this section with two examples of sets whose uncertainty exponents β do not improve over the trivial bound and the volume bound:

Exercise 5.1.2 (Brick; see also Exercise 2.1.6). *Fix* $0 \leq \delta \leq 1$ *and consider the sets*

$$X = Y = \{0, 1, \dots, L-1\}, \quad L := \lfloor N^\delta \rfloor.$$

Show that as $N \to \infty$

$$\| \mathbb{1}_X \mathcal{F}_N \mathbb{1}_Y \|_{\ell^2_N \to \ell^2_N} \sim \begin{cases} N^{\delta - 1/2}, & 0 \leq \delta \leq 1/2; \\ 1, & 1/2 \leq \delta \leq 1. \end{cases}$$

Exercise 5.1.3 (Subgroup). *Assume that* $N = N_1 \cdot N_2$ *and let* X, Y *be additive subgroups of* $\mathbb{Z}_N$ *with* $|X| = N_1$, $|Y| = N_2$. *Show that* $\| \mathbb{1}_X \mathcal{F}_N \mathbb{1}_Y \|_{\ell^2_N \to \ell^2_N} = 1$.

5.2. Cantor sets and fractal uncertainty principle

We now state an uncertainty principle for the special class of *discrete Cantor sets*, which are basic examples of (discretized) fractal sets. To define these, fix

- an integer $M \geq 3$, called the *base*, and
- a nonempty subset $\mathcal{A} \subset \{0, \dots, M-1\}$, called the *alphabet*.

For an integer $k \geq 1$, called the *order*, put

$$N := M^k, \quad \mathcal{C}_k := \{a_0 + a_1 M + \dots + a_{k-1} M^{k-1} \mid a_0, \dots, a_{k-1} \in \mathcal{A}\} \subset \mathbb{Z}_N. \tag{5.2.1}$$

In other words, $\mathcal{C}_k$ is the set of numbers of length k base M with digits in $\mathcal{A}$. Note that $|\mathcal{C}_k| = |\mathcal{A}|^k = N^\delta$ where the dimension δ is defined by

$$\delta := \frac{\log |\mathcal{A}|}{\log M} \in [0, 1].$$

We have $0 < \delta < 1$ except in the trivial cases $|\mathcal{A}| = 1$ and $|\mathcal{A}| = M$. The number δ is the Hausdorff dimension of the limiting Cantor set

$$\mathcal{C}_\infty := \bigcap_{k\geq 1} \bigcup_{j\in\mathcal{C}_k} \Big[\frac{j}{M^k}, \frac{j+1}{M^k}\Big] \subset [0,1].$$

The standard (mid-third) Cantor set corresponds to $M = 3$, $\mathcal{A} = \{0, 2\}$.

EXERCISE 5.2.1. *Show that there exists some constant $C_R = C_R(M)$ such that:*

- *for each k, the set $\mathcal{C}_k$, considered as a subset of $\mathbb{Z}$, is δ-regular (in the sense of Definition 2.1.3) with constant C_R on scales 1 to M^k;*
- *the set $\mathcal{C}_\infty$ is δ-regular with constant C_R on scales 0 to 1.*

Find a sequence of pairs $(M, \mathcal{A})$ with $M \to \infty$, $\delta = 1/2$, such that the best regularity constant C_R goes to infinity.

We now study the uncertainty principle in the sense of Definition 5.1.1 with $X = Y = \mathcal{C}_k$. The upper bounds (5.1.3), (5.1.4) give

$$\| \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} \|_{\ell^2_N\to\ell^2_N} \leq N^{-\beta_{\min}} \quad \text{where } \beta_{\min} := \max\Big(0, \frac{1}{2} - \delta\Big)$$

while the lower bound (5.1.5) gives

$$\| \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} \|_{\ell^2_N\to\ell^2_N} \geq N^{-\beta_{\max}} \quad \text{where } \beta_{\max} := \frac{1-\delta}{2}.$$

Our uncertainty principle improves over $\beta_{\min}$ in the entire region $0 < \delta < 1$:

THEOREM 5.2.1. *Assume that $0 < \delta < 1$. Then there exists*

$$\beta = \beta(M, \mathcal{A}) > \max\Big(0, \frac{1}{2} - \delta\Big)$$

such that

$$\| \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} \|_{\ell^2_N\to\ell^2_N} = \mathcal{O}(N^{-\beta}) \quad \text{as } k \to \infty. \tag{5.2.2}$$

Remarks. 1. One could also consider X, Y to be Cantor sets with same base M and two different alphabets $\mathcal{A}, \mathcal{A}'$, assuming $|\mathcal{A}| = |\mathcal{A}'|$, and the proof of Theorem 5.2.1 still applies.

2. Exercises 5.1.2 and 5.1.3 show that the exponent $\beta_{\min}$ cannot be improved if we only use the size of $\mathcal{C}_k$. The proof of Theorem 5.2.1 uses the fractal structure of the Cantor set on every scale.

3. The best exponent β for which (5.2.2) holds (more precisely, the supremum of all β for which (5.2.2) holds) can be approximated numerically due to the submultiplicativity property discussed in the next section, see (5.3.2). The dependence of β on the alphabet can be quite complicated, see Figure 1. There exist various lower and upper bounds on β depending on M, δ, see [**DJ16**, §3].

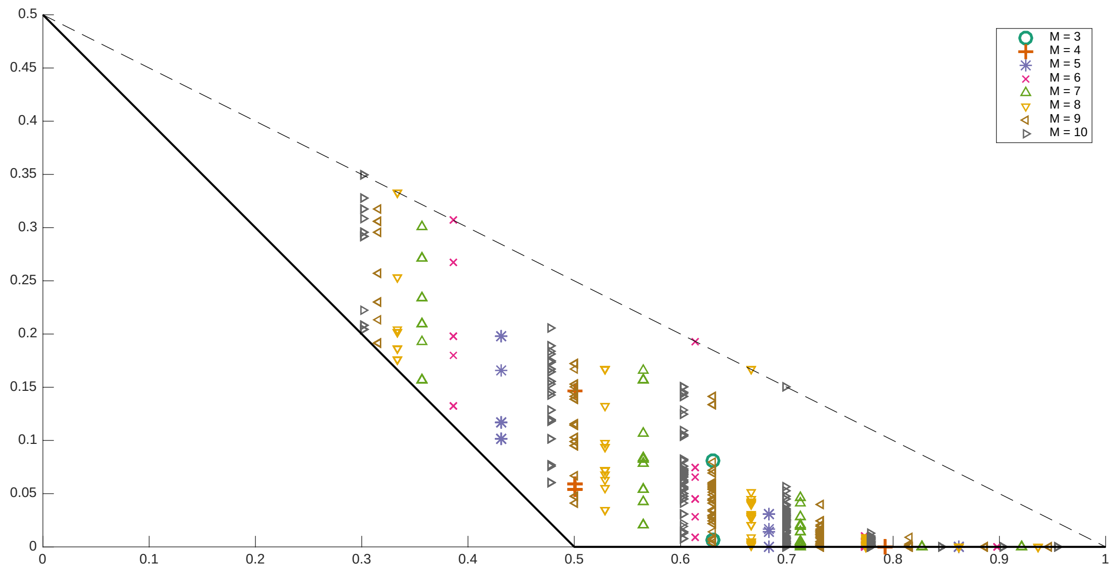


FIGURE 1. Numerically approximated fractal uncertainty exponents for all possible alphabets with $M \leq 10$ and $0 < \delta < 1$. Here the x axis represents δ and the y axis represents β. The solid black line is $\beta = \max(0, \frac{1}{2} - \delta)$ and the dashed line is $\beta = \frac{1-\delta}{2}$. See [**DJ16**, Figure 3] for details.

EXERCISE 5.2.2. *Show that* (5.2.2) *holds with* $\beta = \beta_{\max} = \frac{1-\delta}{2}$ *in the following cases:*

$$M = 6, \quad \mathcal{A} = \{1, 4\};$$
$$M = 8, \quad \mathcal{A} = \{2, 4\};$$
$$M = 8, \quad \mathcal{A} = \{1, 2, 5, 6\}.$$

The alphabets with this property are so-called ***spectral sets*** *and classifying them is closely related to Fuglede's conjecture – see* [**DJ16**, §3.5].

We finish this section with two open problems:

PROBLEM 5.2.3. *Fix* $\delta_\infty \in (1/2, 1)$*. Find a sequence of pairs* $(M_j, \mathcal{A}_j)$ *such that*

$$\delta(M_j, \mathcal{A}_j) \to \delta_\infty, \quad \beta(M_j, \mathcal{A}_j) \to 0.$$

(For $\delta_\infty \in (0, 1/2]$*, such a sequence with* $\beta(M_j, \mathcal{A}_j) \to \frac{1}{2} - \delta$ *is constructed in* [**DJ16**, Proposition 3.17].*)*

PROBLEM 5.2.4. *Fix* $M, \mathcal{A}$ *with* $0 < \delta < 1$*, fix* $\alpha \in [1, M)$*, and put*

$$N := \lfloor \alpha M^k \rfloor.$$

Show that there exists $\beta > \max(0, \frac{1}{2} - \delta)$ ***depending only on*** δ *such that for a* ***generic*** *choice of* α *we have* $\| \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} \|_{\ell^2_N \to \ell^2_N} = \mathcal{O}(N^{-\beta})$ *as* $k \to \infty$*. (Existence of*

β depending on $M, \mathcal{A}$ follows from general fractal uncertainty principles, see [**DJ17a**, §5]*. One expects the statement to be false for $\alpha = 1$, see Problem 5.2.3.)*

5.3. Proof of fractal uncertainty principle

We now give a proof of Theorem 5.2.1, following [**DJ16**, §3]. This proof is greatly simplified by the following property which uses the special structure of Cantor sets:

LEMMA 5.3.1 (Submultiplicativity). *Put*

$$r_k := \| \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} \|_{\ell^2_N \to \ell^2_N}.$$

Then for all k_1, k_2 we have

$$r_{k_1+k_2} \leq r_{k_1} \cdot r_{k_2}.$$

PROOF. Denote

$$k := k_1 + k_2, \quad N_j := M^{k_j}, \quad N := M^k.$$

We define the space

$$\ell^2(\mathcal{C}_k) = \{u \in \ell^2_N \mid \operatorname{supp} u \subset \mathcal{C}_k\}.$$

Then r_k is the norm of the operator

$$\mathcal{G}_k : \ell^2(\mathcal{C}_k) \to \ell^2(\mathcal{C}_k), \quad \mathcal{G}_k u = \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N u.$$

We now write $\mathcal{G}_k$ in terms of $\mathcal{G}_{k_1}, \mathcal{G}_{k_2}$ using a procedure similar to the one used in the Fast Fourier Transform (FFT) algorithm. Take

$$u \in \ell^2(\mathcal{C}_k), \quad v := \mathcal{G}_k u.$$

We associate to u, v the $|\mathcal{A}|^{k_1} \times |\mathcal{A}|^{k_2}$ matrices U, V defined as follows:

$$U_{ab} = u(N_1 \cdot b + a)$$
$$V_{ab} = v(N_2 \cdot a + b)$$

for all $a \in \mathcal{C}_{k_1}$ and $b \in \mathcal{C}_{k_2}$. Here we use the fact that

$$\mathcal{C}_k = N_2 \cdot \mathcal{C}_{k_1} + \mathcal{C}_{k_2} = N_1 \cdot \mathcal{C}_{k_2} + \mathcal{C}_{k_1}.$$

Note that the ℓ^2_N norms of u, v are equal to the Hilbert–Schmidt norms of U, V:

$$\|u\|^2_{\ell^2} = \sum_{a,b} |U_{ab}|^2, \quad \|v\|^2_{\ell^2} = \sum_{a,b} |V_{ab}|^2.$$

We now write the identity $v = \mathcal{G}_k u$ in terms of the matrices U, V:

$$V_{ab} = \frac{1}{\sqrt{N}} \sum_{\substack{p \in \mathcal{C}_{k_1} \\ q \in \mathcal{C}_{k_2}}} \exp\Big(- \frac{2\pi i (N_2 \cdot a + b)(N_1 \cdot q + p)}{N} \Big) U_{pq}.$$

Here is where a small miracle happens: the product of $N_2 \cdot a$ and $N_1 \cdot q$ is divisible by N, so it can be removed from the exponential. That is,

$$V_{ab} = \frac{1}{\sqrt{N}} \sum_{p,q} \exp\Big(-\frac{2\pi iap}{N_1}\Big) \exp\Big(-\frac{2\pi ibp}{N}\Big) \exp\Big(-\frac{2\pi ibq}{N_2}\Big) U_{pq}.$$

It follows that the matrix V can be obtained from U in the following three steps:

(1) Replace each row of U by its Fourier transform $\mathcal{G}_{k_2}$, obtaining the matrix

$$U'_{pb} = \frac{1}{\sqrt{N_2}} \sum_{q} \exp\Big(-\frac{2\pi ibq}{N_2}\Big) U_{pq}.$$

(2) Multiply the entries of U' by twist factors, obtaining the matrix

$$V'_{pb} = \exp\Big(-\frac{2\pi ibp}{N}\Big) U'_{pb}.$$

(3) Replace each column of V' by its Fourier transform $\mathcal{G}_{k_1}$, obtaining the matrix

$$V_{ab} = \frac{1}{\sqrt{N_1}} \sum_{p} \exp\Big(-\frac{2\pi iap}{N_1}\Big) V'_{pb}.$$

Now, we have

$$\|U'\|_{\mathrm{HS}} \leq r_{k_2} \|U\|_{\mathrm{HS}}, \quad \|V'\|_{\mathrm{HS}} = \|U'\|_{\mathrm{HS}}, \quad \|V\|_{\mathrm{HS}} \leq r_{k_1} \|V'\|_{\mathrm{HS}},$$

giving

$$\|v\|_{\ell^2_N} \leq r_{k_1} \cdot r_{k_2} \cdot \|u\|_{\ell^2_N}$$

which finishes the proof. □

Given Lemma 5.3.1, we see that it suffices to obtain the strict inequality

$$r_k := \| \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} \|_{\ell^2_N \to \ell^2_N} < \min(1, N^{\delta - 1/2}) \tag{5.3.1}$$

for just *one* value of k. Indeed, by Fekete's Lemma there exists a limit

$$\beta_0 = -\lim_{k\to\infty} \frac{\log r_k}{k \log M} = -\inf_{k\geq 1} \frac{\log r_k}{k \log M}. \tag{5.3.2}$$

Then the uncertainty bound (5.2.2) holds with any $\beta < \beta_0$, and (5.3.1) shows that $\beta_0 > \max(0, \frac{1}{2} - \delta)$, implying Theorem 5.2.1.

The inequality (5.3.1) is really two inequalities, proved below:

Lemma 5.3.2. *There exists k such that $r_k < 1$.*

Proof. By the trivial bound (5.1.3) we have $r_k \leq 1$. We argue by contradiction. Assume that $r_k = 1$. Then there exists

$$u \in \ell^2_N \setminus \{0\}, \quad \| \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} u\|_{\ell^2_N} = \|u\|_{\ell^2_N}.$$

Since $\mathcal{F}_N$ is unitary, this implies that

$$\operatorname{supp} u \subset \mathcal{C}_k, \tag{5.3.3}$$

$$\operatorname{supp}(\mathcal{F}_N u) \subset \mathcal{C}_k. \tag{5.3.4}$$

We now use the fact that discrete Fourier transform evaluates polynomials at roots of unity. Define the polynomial

$$p(z) := \sum_{\ell\in\mathbb{Z}_N} u(\ell) z^\ell.$$

Then

$$\mathcal{F}_N u(j) = \frac{1}{\sqrt{N}}\, p(e^{-2\pi i j/N}).$$

By (5.3.4) for each $j \in \mathbb{Z}_N \setminus \mathcal{C}_k$ we have $\mathcal{F}_N u(j) = 0$. It follows that the number of roots of p is bounded below by (here we use $\delta < 1$)

$$N - |\mathcal{C}_k| \geq M^k - (M-1)^k.$$

On the other hand, the set $\mathbb{Z}_N \setminus \mathcal{C}_k$ contains M^{k-1} consecutive numbers (specifically $aM^{k-1}, \dots, (a+1)M^{k-1} - 1$ where $a \in \mathbb{Z}_M \setminus \mathcal{A}$). We shift $\mathcal{C}_k$ circularly (which does not change the norm r_k) to map these numbers to $(M-1)M^{k-1}, \dots, M^k - 1$. Then the degree of p is smaller than $(M-1)M^{k-1}$.

Now, for k large enough we have

$$M^k - (M-1)^k \geq (M-1)M^{k-1}.$$

Then the number of roots of p is higher than its degree, giving a contradiction. □

Lemma 5.3.3. *For $k \geq 2$ we have $r_k < N^{\delta - 1/2}$.*

Proof. Recall from (5.1.4) that $N^{\delta-1/2}$ is the Hilbert–Schmidt norm of $\mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k}$, while r_k is its operator norm. We again arguing by contradiction, assuming that $r_k = N^{\delta-1/2}$. Then $\mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k}$ is a rank 1 operator; indeed, the sum of the squares of its singular values is equal to the square of the maximal singular value. It follows that each rank 2 minor of $\mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k}$ is equal to zero, namely

$$\det \begin{pmatrix} e^{-2\pi i j\ell/N} & e^{-2\pi i j\ell'/N} \\ e^{-2\pi i j'\ell/N} & e^{-2\pi i j'\ell'/N} \end{pmatrix} = 0 \quad \text{for all } j, j', \ell, \ell' \in \mathcal{C}_k.$$

Computing the determinant we see that

$$(j - j')(\ell - \ell') \in N\mathbb{Z} \quad \text{for all } j, j', \ell, \ell' \in \mathcal{C}_k.$$

However, if $k \geq 2$ we may take $j = \ell, j' = \ell' \in \mathcal{C}_k$ such that (here we use that $\delta > 0$)

$$0 < |j - j'| < M \leq \sqrt{N},$$

giving a contradiction. □

5.4. Application to open quantum maps

TODO

Bibliography

[An08] Nalini Anantharaman, *Entropy and the localization of eigenfunctions,* Ann. of Math. **168**(2008), 435–475.

[AN07] Nalini Anantharaman and Stéphane Nonnenmacher, *Half-delocalization of eigenfunctions of the Laplacian on an Anosov manifold,* Ann. Inst. Fourier **57**(2007), 2465–2523.

[AS13] Nalini Anantharaman and Lior Silberman, *A Haar component for quantum limits on locally symmetric spaces,* Israel J. Math. **195**(2013), 393–447.

[BV89] Nándor L. Balázs and André Voros, *The quantized baker's transformation,* Ann. Phys. **190** (1989), 1–31.

[B$^+$13] Sonja Barkhofen, Tobias Weich, Alexander Potzuweit, Hans-Jürgen Stöckmann, Ulrich Kuhl, and Maciej Zworski, *Experimental observation of the spectral gap in microwave n-disk systems,* Phys. Rev. Lett. **110**(2013), 164102.

[Bo16] David Borthwick, *Spectral theory of infinite-area hyperbolic surfaces,* second edition, Birkhäuser, 2016.

[Bo14] David Borthwick, *Distribution of resonances for hyperbolic surfaces,* Experimental Math. **23**(2014), 25–45.

[BW14] David Borthwick and Tobias Weich, *Symmetry reduction of holomorphic iterated function schemes and factorization of Selberg zeta functions,* J. Spectr. Th. **6**(2016), 267–329.

[BD16] Jean Bourgain and Semyon Dyatlov, *Spectral gaps without the pressure condition,* preprint, arXiv:1612.09040.

[BD17] Jean Bourgain and Semyon Dyatlov, *Fourier dimension and spectral gaps for hyperbolic surfaces,* to appear in Geom. Funct. Anal., arXiv:1704.02909.

[BGS11] Jean Bourgain, Alex Gamburd, and Peter Sarnak, *Generalization of Selberg's 3/16 theorem and affine sieve,* Acta Math. **207**(2011), 255–290.

[Br15] Shimon Brooks, *Logarithmic-scale quasimodes that do not equidistribute,* Int. Math. Res. Not. (2015), 11934–11960.

[BGH10] Nicolas Burq, Colin Guillarmou, and Andrew Hassell, *Strichartz estimates without loss on manifolds with hyperbolic trapped geodesics,* Geom. Funct. Anal. **20**(2010), 627–656.

[BZ04] Nicolas Burq and Maciej Zworski, *Geometric control in the presence of a black box,* J. Amer. Math. Soc. **17**(2004), 443–471.

[CW15] H. Cao and J. Wiersig, *Dielectric microcavities: model systems for wave chaos and non-Hermitian physics,* Rev. Mod. Phys. **87**(2015), 61–111.

[CdV85] Yves Colin de Verdière, *Ergodicité et fonctions propres du Laplacien,* Comm. Math. Phys. **102**(1985), 497–502.

[Da09] Kiril Datchev, *Local smoothing for scattering manifolds with hyperbolic trapped sets,* Comm. Math. Phys. **286** (2009) 837–850.

[DD13] Kiril Datchev and Semyon Dyatlov, *Fractal Weyl laws for asymptotically hyperbolic manifolds,* Geom. Funct. Anal. **23**(2013), 1145–1206.

[DNW06] Mirko Degli Esposti, Stéphane Nonnenmacher, and Brian Winn, *Quantum variance and ergodicity for the baker's map,* Comm. Math. Phys. **263**(2006), 325–352.

[Do98] Dmitry Dolgopyat, *On decay of correlations in Anosov flows,* Ann. Math. (2) **147**(1998), 357–390.

[Do03] Harold Donnelly, *Quantum unique ergodicity,* Proc. Amer. Math. Soc. **131**(2003), 2945–2951.

[DG17] Semyon Dyatlov and Jeffrey Galkowski, *Fractal Weyl laws and wave decay for general trapping,* to appear in Nonlinearity, arXiv:1703.06515.

[DJ16] Semyon Dyatlov and Long Jin, *Resonances for open quantum maps and a fractal uncertainty principle,* Comm. Math. Phys., published online, arXiv:1608.02238.

[DJ17a] Semyon Dyatlov and Long Jin, *Dolgopyat's method and the fractal uncertainty principle,* preprint, arXiv:1702.03619.

[DJ17b] Semyon Dyatlov and Long Jin, *Semiclassical measures on hyperbolic surfaces have full support,* preprint, arXiv:1705.05019.

[DZ16] Semyon Dyatlov and Joshua Zahl, *Spectral gaps, additive energy, and a fractal uncertainty principle,* Geom. Funct. Anal. **26**(2016), 1011–1094.

[DZ$^{\mathrm{B}}$] Semyon Dyatlov and Maciej Zworski, *Mathematical theory of scattering resonances,* book in progress, `http://math.mit.edu/~dyatlov/res/`

[EFS15] Leonardo Ermann, Klaus M. Frahm, and Dima L. Shepelyansky, *Google matrix analysis of directed networks,* Rev. Mod. Phys. **87**(2015), 1261.

[EN15] Suresh Eswarathasan and Stéphane Nonnenmacher, *Strong scarring of logarithmic quasimodes,* to appear in Ann. Inst. Fourier, arXiv:1507.08371.

[ES16] Suresh Eswarathasan and Lior Silberman, *Scarring of quasimodes on hyperbolic manifolds,* preprint, arXiv:1609.04912.

[FW15] Frédéric Faure and Tobias Weich, *Asymptotic spectral gap for open partially expanding maps,* preprint, arXiv:1504.06728.

[GR89] Pierre Gaspard and Stuart Rice, *Scattering from a classically chaotic repeller,* J. Chem. Phys. **90**(1989), 2225–2241.

[Gu05] Colin Guillarmou, *Meromorphic properties of the resolvent on asymptotically hyperbolic manifolds,* Duke Math. J. **129**:1(2005), 1–37.

[GLZ04] Laurent Guillopé, Kevin K. Lin, and Maciej Zworski, *The Selberg zeta function for convex co-compact Schottky groups,* Comm. Math. Phys. **245**:1(2004), 149–176.

[GuZw95a] Laurent Guillopé and Maciej Zworski, *Polynomial bounds on the number of resonances for some complete spaces of constant negative curvature near infinity,* Asymptotic Anal. **11**:1(1995), 1–22.

[Ha10] Andrew Hassell, *Ergodic billiards that are not quantum unique ergodic,* Ann. of Math. **171**(2010), 605–618.

[HMR87] Bernard Helffer, André Martinez, and Didier Robert, *Ergodicité en limite semi-classique,* Comm. Math. Phys. **109**(1987), 313–326.

[HV16] Peter Hintz and András Vasy, *The global non-linear stability of the Kerr–de Sitter family of black holes,* preprint, arXiv:1606.04014.

[HöIII] Lars Hörmander, *The Analysis of Linear Partial Differential Operators III. Pseudo-Differential Operators,* Springer, 1994.

[Ik88] Mitsuru Ikawa, *Decay of solutions of the wave equation in the exterior of several convex bodies,* Ann. Inst. Fourier **38**(1988), 113–146.

[Ji17] Long Jin, *Control for Schrödinger equation on hyperbolic surfaces,* preprint, arXiv:1707.04990.

[JZ17] Long Jin and Ruixiang Zhang, *Fractal uncertainty principle with explicit exponent,* preprint, arXiv:1710.00250.

[K+08] Jonathan P. Keating, Stéphane Nonnenmacher, Marcel Novaes, and Martin Sieber, *On the resonance eigenstates of an open quantum baker map,* Nonlinearity **21** (2008), 2591–2624.

[K+06] Jonathan P. Keating, Marcel Novaes, Sandra D. Prado, and Martin Sieber, *Semiclassical Structure of Chaotic Resonance Eigenfunctions,* Phys. Rev. Lett. **97** (2006), 150406.

[Li06] Elon Lindenstrauss, *Invariant measures and arithmetic quantum unique ergodicity,* Ann. of Math. **163**(2006), 165–219.

[LSZ03] Wentao Lu, Srinivas Sridhar, and Maciej Zworski, *Fractal Weyl laws for chaotic open systems,* Phys. Rev. Lett. **91**(2003), 154101.

[MOW14] Michael Magee, Hee Oh, and Dale Winter, *Uniform congruence counting for Schottky semigroups in* $\mathrm{SL}_2(\mathbb{Z})$*,* preprint, arXiv:1412.4284.

[Ma06] Jens Marklof, *Arithmetic quantum chaos,* Encyclopedia of Mathematical Physics, Oxford: Elsevier **1**(2006), 212–220.

[MK05] Jens Marklof and Stephen O'Keefe, *Weyl's law and quantum ergodicity for maps with divided phase space,* with an appendix by Steve Zelditch, Nonlinearity **18**(2005), 277–304.

[MM87] Rafe Mazzeo and Richard Melrose, *Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature,* J. Funct. Anal. **75**(1987), 260–310.

[Na05] Frédéric Naud, *Expanding maps on Cantor sets and analytic continuation of zeta functions,* Ann. de l'ENS (4) **38**(2005), 116–153.

[Na14] Frédéric Naud, *Density and location of resonances for convex co-compact hyperbolic surfaces,* Invent. Math. **195**(2014), 723–750.

[No11a] Stéphane Nonnenmacher, *Spectral problems in open quantum chaos,* Nonlinearity **24**(2011), R123.

[No11b] Stéphane Nonnenmacher, *Spectral theory of damped quantum chaotic systems,* preprint, arXiv:1109.0930.

[NR07] Stéphane Nonnenmacher and Mathieu Rubin, *Resonant eigenstates in quantum chaotic scattering,* Nonlinearity **20**(2007), 1387–1420.

[NSZ11] Stéphane Nonnenmacher, Johannes Sjöstrand, and Maciej Zworski, *From open quantum systems to open quantum maps,* Comm. Math. Phys. **304**:1(2011), 1–48.

[NSZ14] Stéphane Nonnenmacher, Johannes Sjöstrand, and Maciej Zworski, *Fractal Weyl law for open quantum chaotic maps,* Ann. of Math.(2) **179**(2014), 179–251.

[NZ07] Stéphane Nonnenmacher and Maciej Zworski, *Distribution of resonances for open quantum maps,* Comm. Math. Phys. **269**(2007), 311–365.

[NZ09] Stéphane Nonnenmacher and Maciej Zworski, *Quantum decay rates in chaotic scattering,* Acta Math. **203**(2009), 149–233.

[OW14] Hee Oh and Dale Winter, *Uniform exponential mixing and resonance free regions for convex cocompact congruence subgroups of* $\mathrm{SL}_2(\mathbb{Z})$*,* J. Amer. Math. Soc. **29**(2016), 1069–1115.

[Pa76] Samuel James Patterson, *The limit set of a Fuchsian group,* Acta Math. **136**(1976), 241–273.

[PS10] Vesselin Petkov and Luchezar Stoyanov, *Analytic continuation of the resolvent of the Laplacian and the dynamical zeta function,* Anal. PDE **3**(2010), 427–489.

[P+12] Alexander Potzuweit, Tobias Weich, Sonja Barkhofen, Ulrich Kuhl, Hans-Jürgen Stöckmann, and Maciej Zworski, *Weyl asymptotics: from closed to open systems,* Phys. Rev. E. **86**(2012), 066205.

[Ri10a] Gabriel Rivière, *Entropy of semiclassical measures in dimension 2,* Duke Math. J. **155**(2010), 271–335.

[Ri10b] Gabriel Rivière, *Entropy of semiclassical measures for nonpositively curved surfaces,* Ann. Henri Poincaré **11**(2010), 1085–1116.

[RS94] Zeév Rudnick and Peter Sarnak, *The behaviour of eigenstates of arithmetic hyperbolic manifolds,* Comm. Math. Phys. **161**(1994), 195–213.

[Sa90] Marcos Saraceno, *Classical structures in the quantized baker transformation,* Ann. Phys. **199**(1990), 37–60.

[SV94] Marcos Saraceno and André Voros, *Towards a semiclassical theory of the quantum baker's map,* Physica D: Nonlinear Phenomena **79**(1994), 206–268.

[Sa11] Peter Sarnak, *Recent progress on the quantum unique ergodicity conjecture,* Bull. Amer. Math. Soc. **48**(2011), 211–228.

[Sa13] Peter Sarnak, *Notes on thin matrix groups,* MSRI Publications **61**(2013).

[Sh74] Alexander Shnirelman, *Ergodic properties of eigenfunctions,* Usp. Mat. Nauk. **29**(1974), 181–182.

[Sj90] Johannes Sjöstrand, *Geometric bounds on the density of resonances for semiclassical problems,* Duke Math. J. **60**:1(1990), 1–57.

[SZ07] Johannes Sjöstrand and Maciej Zworski, *Fractal upper bounds on the density of semiclassical resonances,* Duke Math. J. **137**(2007), 381–459.

[So14] Christopher Sogge, *Hangzhou lectures on eigenfunctions of the Laplacian,* Ann. Math. St. **188**, Princeton Univ. Press, 2014.

[So10] Kannan Soundararajan, *Quantum unique ergodicity for* $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$*,* Ann. of Math. **172**(2010), 1529–1538.

[St11] Luchezar Stoyanov, *Spectra of Ruelle transfer operators for axiom A flows,* Nonlinearity **24**(2011), 1089–1120.

[St12] Luchezar Stoyanov, *Non-integrability of open billiard flows and Dolgopyat-type estimates,* Erg. Theory Dyn. Syst. **32**(2012), 295–313.

[Su79] Dennis Sullivan, *The density at infinity of a discrete group of hyperbolic motions,* Publ. Math. de l'IHES **50**(1979), 171–202.

[Va13a] András Vasy, *Microlocal analysis of asymptotically hyperbolic and Kerr–de Sitter spaces,* with an appendix by Semyon Dyatlov, Invent. Math. **194**(2013), 381–513.

[Va13b] András Vasy, *Microlocal analysis of asymptotically hyperbolic spaces and high energy resolvent estimates,* Inverse Problems and Applications. Inside Out II, Gunther Uhlmann (ed.), MSRI publications **60**, Cambridge Univ. Press, 2013.

[Wa17] Jian Wang, *Strichartz estimates for convex co-compact hyperbolic surfaces,* preprint, arXiv:1707.06310.

[Ze87] Steve Zelditch, *Uniform distribution of eigenfunctions on compact hyperbolic surfaces,* Duke Math. J. **55**(1987), 919–941.

[Ze09] Steve Zelditch, *Recent developments in mathematical quantum chaos,* Curr. Dev. Math. 2009, 115–204.

[ZZ96] Steve Zelditch and Maciej Zworski, *Ergodicity of eigenfunctions for ergodic billiards,* Comm. Math. Phys. **175**(1996), 673–682.

[Zw99] Maciej Zworski, *Dimension of the limit set and the density of resonances for convex co-compact hyperbolic surfaces,* Invent. Math. **136**(1999), 353–409.

[Zw12] Maciej Zworski, *Semiclassical analysis,* Graduate Studies in Mathematics **138**, AMS, 2012.

[Zw16a] Maciej Zworski, *Resonances for asymptotically hyperbolic manifolds: Vasy's method revisited,* J. Spect. Th. **6**(2016), 1087–1114.

[Zw16b] Maciej Zworski, *Mathematical study of scattering resonances,* Bull. Math. Sci. (2017) 7:1–85.