Spectral gaps via additive combinatorics

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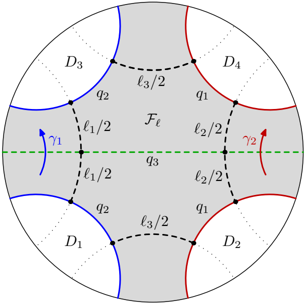
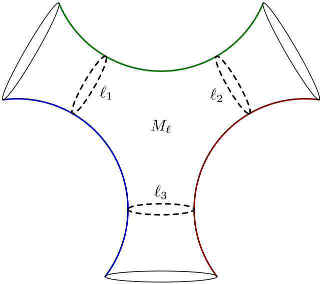
joint work with Joshua Zahl (MIT)

August 24, 2015

Setting: hyperbolic surfaces

$(M, g) = \Gamma\backslash\mathbb{H}^2$ convex co-compact hyperbolic surface





Resonances: poles of the scattering resolvent

$$R(\lambda) = \Big(-\Delta_g - \frac{1}{4} - \lambda^2 \Big)^{-1} : \begin{cases} L^2(M) \to L^2(M), & \operatorname{Im}\lambda > 0 \\ L^2_{\text{comp}}(M) \to L^2_{\text{loc}}(M), & \operatorname{Im}\lambda \le 0 \end{cases}$$

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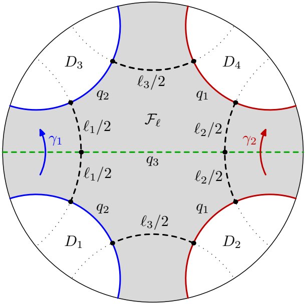

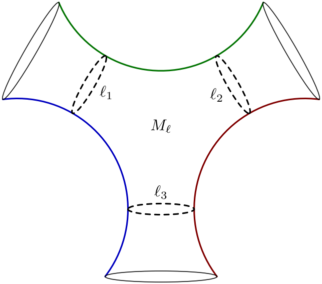

Resonances: poles of the scattering resolvent

Also correspond to poles of the Selberg zeta function

Existence of meromorphic continuation: Patterson '75,'76, Perry '87,'89, Mazzeo–Melrose '87, Guillopé–Zworski '95, Guillarmou '05, Vasy '13

Resonances: poles of the scattering resolvent $R(\lambda)$

Featured in resonance expansions of waves:

$$\operatorname{Re}\lambda = \text{rate of oscillation}, \quad -\operatorname{Im}\lambda = \text{rate of decay}$$

Borthwick '13, Borthwick–Weich '14: numerics for resonances

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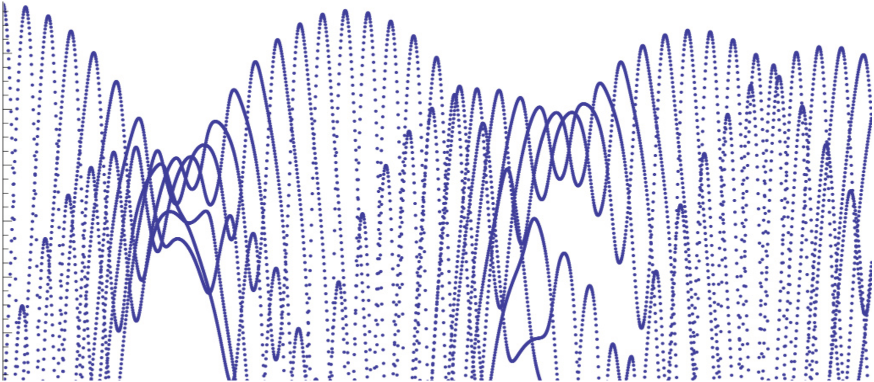

Pictures courtesy of David Borthwick

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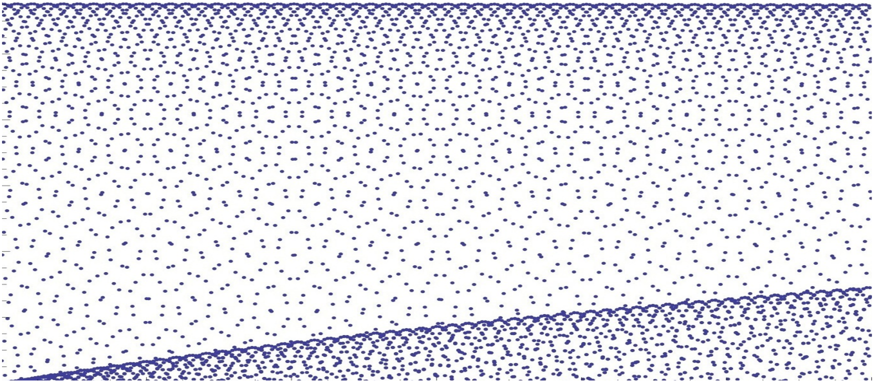

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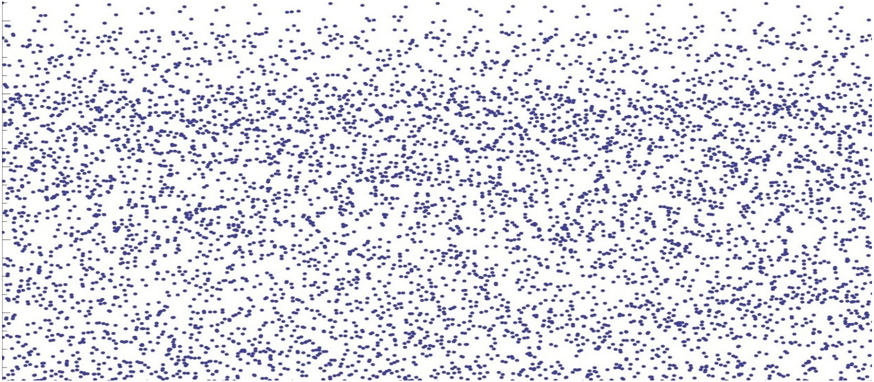

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Spectral gaps

Essential spectral gap of size $\beta > 0$:

only finitely many resonances with $\operatorname{Im}\lambda > -\beta$

One application: exponential decay of linear waves

Patterson–Sullivan theory: the topmost resonance is at $\lambda = i(\delta - \frac{1}{2})$, where $\delta \in (0,1)$ is defined below $\Rightarrow$ gap of size $\beta = \max\left(0, \frac{1}{2} - \delta\right)$.

$\delta > \frac{1}{2}$ $\delta < \frac{1}{2}$

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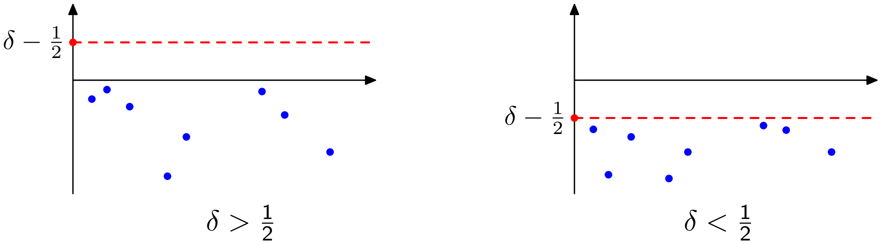
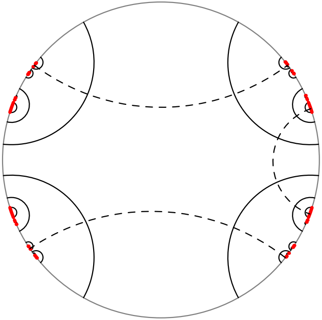


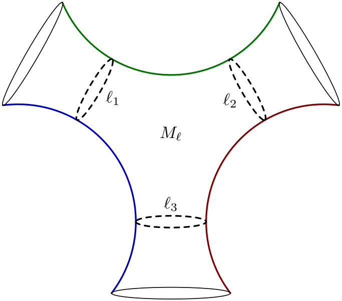
The limit set and δ

$M = \Gamma\backslash\mathbb{H}^2$ hyperbolic surface

$\Lambda_\Gamma \subset \mathbb{S}^1$ the limit set

$\delta := \dim_H(\Lambda_\Gamma) \in (0,1)$



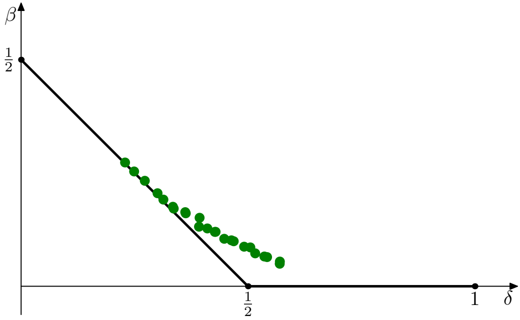


Trapped geodesics: those with endpoints in Λ_Γ

Beyond the Patterson–Sullivan gap

Patterson–Sullivan gap: $\beta = \max(0, \frac{1}{2} - \delta)$

Borthwick–Weich '14: numerics for symmetric 3- and 4-funneled surfaces



Dolgopyat '98, Naud '04, Stoyanov '11,'13, Petkov–Stoyanov '10:
for $0 < \delta \leq \frac{1}{2}$, gap of size $\frac{1}{2} - \delta + \varepsilon$, where $\varepsilon > 0$ depends on the surface

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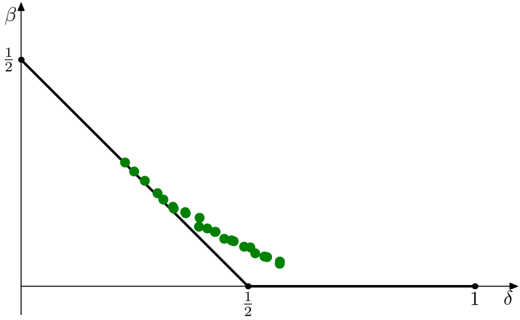

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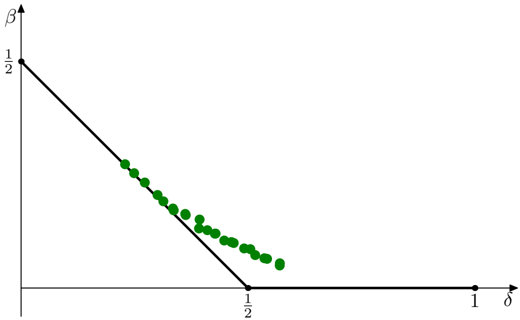

Bourgain–Gamburd–Sarnak '11, Oh–Winter '14: gaps for the case of congruence quotients

Patterson–Sullivan gap: $\beta = \max(0, \frac{1}{2} - \delta)$

Theorem [D–Zahl '15]

There is an essential spectral gap with polynomial resolvent bound of size

$$\beta = \frac{3}{8}\Big(\frac{1}{2} - \delta\Big) + \frac{\beta_E}{16}$$

where $\beta_E \in [0, \delta]$ is the improvement in the asymptotic of additive energy of the limit set (explained below).

Theorem [D–Zahl '15]

$$\beta_E > \delta \exp\big[-K(1-\delta)^{-28} \log^{14}(1+C)\big]$$

where C is the constant in the δ-regularity of the limit set and K is a global constant.

C depends continuously on the surface $\Longrightarrow$ examples of cases when $\delta > \frac{1}{2}$, but there is a gap

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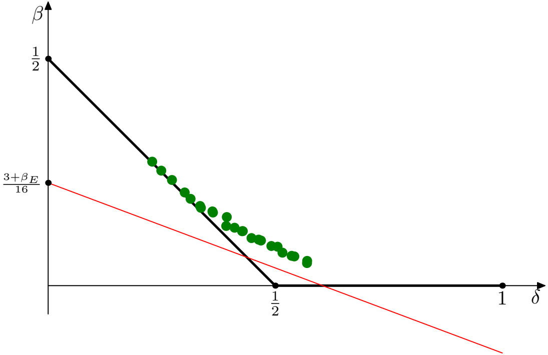
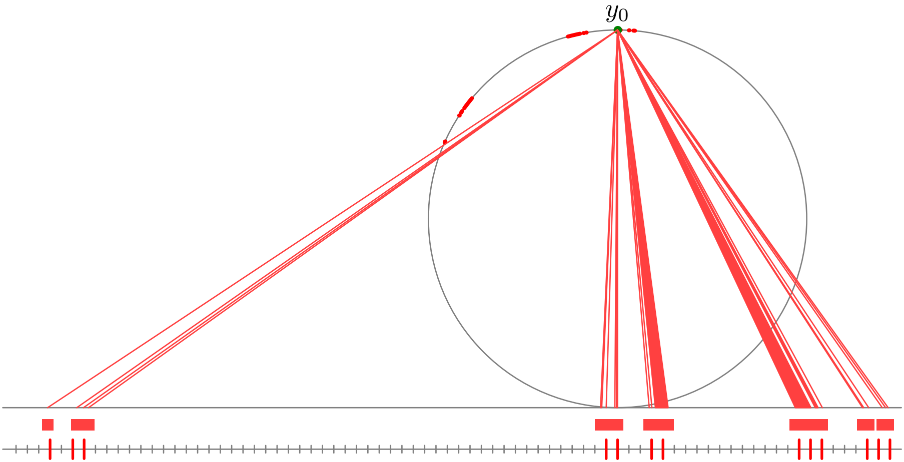


Numerics by Borthwick–Weich '14 + our gap for $\beta_E := \delta$

Additive energy

$X(y_0, \alpha) \subset \alpha\mathbb{Z} \cap [-1, 1]$ the discretization of Λ_Γ projected from $y_0 \in \Lambda_\Gamma$



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Additive energy:

$$E_A(y_0, \alpha) = \#\{(a, b, c, d) \in X(y_0, \alpha)^4 \mid a + b = c + d\}$$
$$|X(y_0, \alpha)| \sim \alpha^{-\delta}, \quad \alpha^{-2\delta} \lesssim E_A(y_0, \alpha) \lesssim \alpha^{-3\delta}$$

Definition

Λ_Γ has improved additive energy with exponent $\beta_E \in [0, \delta]$, if

$$E_A(y_0, \alpha) \leq C\alpha^{-3\delta+\beta_E}, \quad 0 < \alpha < 1,$$

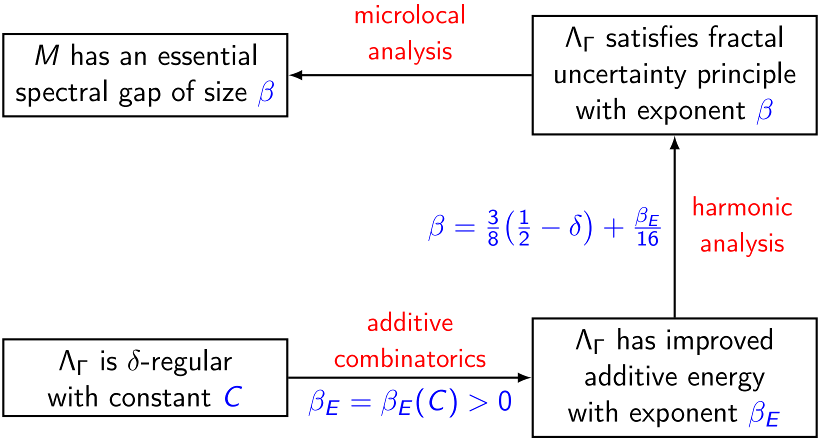
where C does not depend on y_0.

Random sets have improved additive energy with $\beta_E = \min(\delta, 1 - \delta)$

Scheme of the proof

$M = \Gamma \backslash \mathbb{H}^2$ convex co-compact hyperbolic surface

$\Lambda_\Gamma \subset \mathbb{S}^1$ limit set, $\delta = \dim_H(\Lambda_\Gamma)$



Spectral gap via fractal uncertainty principle

To prove that M has an essential spectral gap of size β, enough to show

$$\left.\begin{array}{c}\Big(-\Delta_g-\dfrac{1}{4}-\lambda^2\Big)u=0,\quad u\ \text{outgoing}\\ \operatorname{Im}\lambda>-\beta,\quad \operatorname{Re}\lambda\gg 1\end{array}\right\}\quad\Longrightarrow\quad u\equiv 0$$

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$$\left.\begin{array}{c}\Big(-h^2\Delta_g-\dfrac{h^2}{4}-\omega^2\Big)u=0,\quad u\text{ outgoing}\\ \operatorname{Im}\omega>-\beta h,\quad \operatorname{Re}\omega=1,\quad 0<h\ll 1\end{array}\right\}\quad\Longrightarrow\quad u\equiv 0$$

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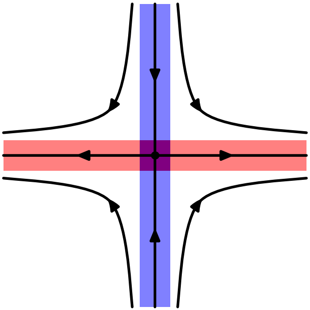
Using Vasy '13 and propagation of singularities up to time t, get

$$\|(1 - \operatorname{Op}_h(\chi_+))u\|_{L^2} = \mathcal{O}(h^\infty)\|u\|_{L^2}$$
$$\|\operatorname{Op}_h(\chi_-)u\|_{L^2} \gtrsim e^{\frac{\operatorname{Im}\omega}{h}t}\|u\|_{L^2}$$

where χ_+, χ_- live e^{-t}-close to the outgoing/incoming tails $\Gamma_+, \Gamma_- \subset T^*M$



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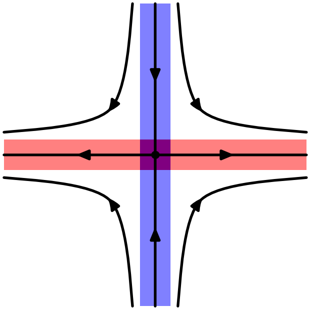
Using Vasy '13 and propagation of singularities up to time $t=\log(1/h)$, get

$$\|(1-\operatorname{Op}_h^{L_u}(\chi_+))u\|_{L^2}=\mathcal{O}(h^\infty)\|u\|_{L^2}$$
$$\|\operatorname{Op}_h^{L_s}(\chi_-)u\|_{L^2}\gg h^\beta\|u\|_{L^2}$$

where χ_+,χ_- live h-close to the outgoing/incoming tails $\Gamma_+,\Gamma_-\subset T^*M$ and L_u, L_s are weak unstable/stable Lagrangian foliations, tangent to Γ_+, Γ_-



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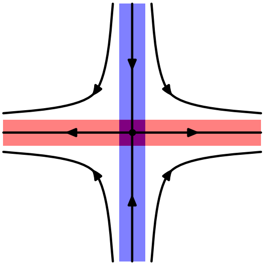
$$\|\mathrm{Op}_h^{L_s}(\chi_-)\mathrm{Op}_h^{L_u}(\chi_+)\|_{L^2 \to L^2} \leq Ch^\beta$$



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for all $\chi \in C_0^\infty(\mathbb{S}^1 \times \mathbb{S}^1 \setminus \{y = y'\})$, where $\Lambda_\Gamma(h) \subset \mathbb{S}^1$ is the h-neighborhood of the limit set Λ_Γ

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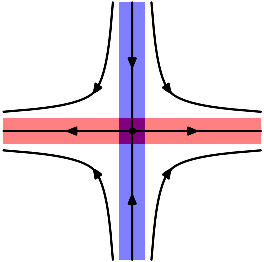

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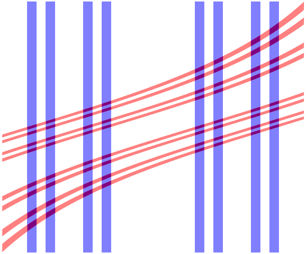
Theorem [D–Zahl '15]

If Λ_Γ satisfies the fractal uncertainty principle with exponent β, then $M = \Gamma\backslash\mathbb{H}^2$ has an essential spectral gap of size β

How to prove a F.U.P.?

- $\|\mathcal{B}_\chi\|_{L^2\to L^2}$: $\beta = 0$
- $\|\mathcal{B}_\chi\|_{L^1\to L^\infty}$: $\beta = \frac{1}{2}-\delta$
- L^4: $\beta = \frac{3}{8}\left(\frac{1}{2}-\delta\right)+\frac{\beta_E}{16}$, where β_E comes from the improvement on additive energy



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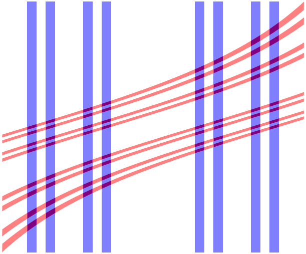

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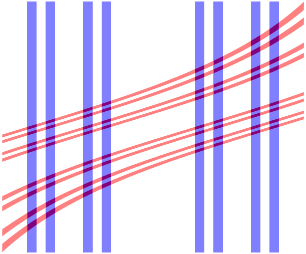

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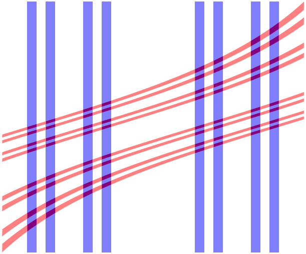

From regularity to an additive energy bound

Definition

$\Lambda_\Gamma \subset \mathbb{S}^1$ is δ-regular with constant C, if (μ_δ is the Hausdorff measure)

$$C^{-1}\alpha^\delta \le \mu_\delta(\Lambda_\Gamma \cap B(y_0,\alpha)) \le C\alpha^\delta, \quad y_0 \in \Lambda_\Gamma, \quad \alpha \in (0,1)$$

Theorem [D–Zahl '15]

If Λ_Γ is δ-regular with constant C, then it has improved additive energy with exponent (here K is a global constant)

$$\beta_E = \delta \exp\big[-K(1-\delta)^{-28}\log^{14}(1+C)\big]$$

- δ-regularity $+$ $\delta < 1 \Rightarrow \Lambda_\Gamma$ does not have long arithmetic progressions
- A version of Freĭman's theorem $\Rightarrow \Lambda_\Gamma$ cannot have maximal additive energy at a small enough scale
- Induction on scale $\Rightarrow$ improvement in the additive energy at all scales

Thank you for your attention!