A microlocal toolbox for hyperbolic dynamics

Semyon Dyatlov (MIT)

December 4, 2014

Motivation: statistics for billiards

One billiard ball in a Sinai billiard with finite horizon

10000 billiard balls in a Sinai billiard with finite horizon

#(balls in the box) $\to$ volume of the box

velocity angles distribution $\to$ uniform measure

10000 billiard balls in a three-disk system

$$\#(\text{balls in the box}) \quad \to \quad 0 \text{ exponentially}$$

$$\text{velocity angles distribution} \quad \to \quad \text{some fractal measure}$$

10000 billiard balls in a rectangle

#(balls in the box), velocity angles distribution have no limit

Correlations (closed billiards)

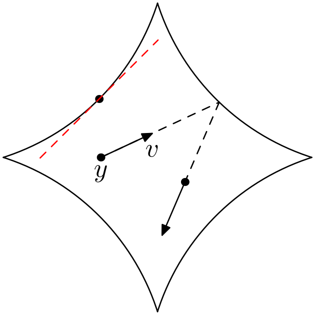
$M \subset \mathbb{R}^2$ a closed billiard

$\mathcal{U} = \{(y,v) \mid y \in M,\ v \in \mathbb{S}^1\}$

$\varphi^t : \mathcal{U} \to \mathcal{U}$ billiard flow

$f, g \in C^\infty(\mathcal{U})$

$$\rho_{f,g}(t) = \int_{\mathcal{U}} (f \circ \varphi^{-t}) g \, dxdv$$



Mixing: $\rho_{f,g}(t) = c\Big(\int_{\mathcal{U}} f\,dxdv\Big)\Big(\int_{\mathcal{U}} g\,dxdv\Big) + o(1)$ as $t \to +\infty$

Sinai '70, Bunimovich '74, Young '98, Chernov '99 . . .

Climate models: Chekroun–Neelin–Kondrashov–McWilliams–Ghil '14

Hard because of glancing rays – consider a model without boundary instead

Correlations (closed billiards)

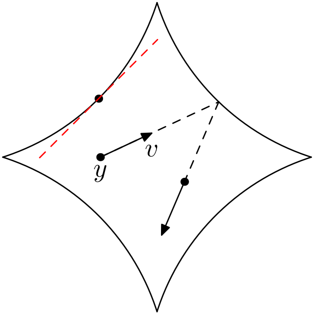
$M \subset \mathbb{R}^2$ a closed billiard

$\mathcal{U} = \{(y, v) \mid y \in M,\ v \in \mathbb{S}^1\}$

$\varphi^t : \mathcal{U} \to \mathcal{U}$ billiard flow

$f, g \in C^\infty(\mathcal{U})$

$$\rho_{f,g}(t) = \int_{\mathcal{U}} (f \circ \varphi^{-t}) g \, dxdv$$



Mixing: $\rho_{f,g}(t) = c\Big(\int_{\mathcal{U}} f \, dxdv\Big)\Big(\int_{\mathcal{U}} g \, dxdv\Big) + o(1)$ as $t \to +\infty$

Sinai '70, Bunimovich '74, Young '98, Chernov '99 . . .

Climate models: Chekroun–Neelin–Kondrashov–McWilliams–Ghil '14

Hard because of glancing rays – consider a model without boundary instead

Correlations (closed billiards)

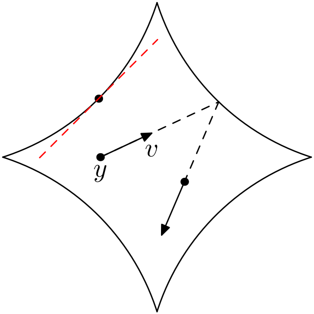
$M \subset \mathbb{R}^2$ a closed billiard

$\mathcal{U} = \{(y, v) \mid y \in M,\ v \in \mathbb{S}^1\}$

$\varphi^t : \mathcal{U} \to \mathcal{U}$ billiard flow

$f, g \in C^\infty(\mathcal{U})$

$$\rho_{f,g}(t) = \int_{\mathcal{U}} (f \circ \varphi^{-t}) g \, dxdv$$



Mixing: $\rho_{f,g}(t) = c\Big(\int_{\mathcal{U}} f \, dxdv\Big)\Big(\int_{\mathcal{U}} g \, dxdv\Big) + o(1)$ as $t \to +\infty$

Sinai '70, Bunimovich '74, Young '98, Chernov '99 . . .

Climate models: Chekroun–Neelin–Kondrashov–McWilliams–Ghil '14

Hard because of glancing rays – consider a model without boundary instead

Correlations (compact manifolds)

M compact manifold

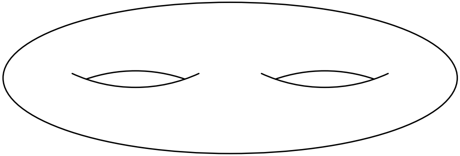
g negatively curved metric

$\mathcal{U} = \{(y, v) \in TM : |v|_g = 1\}$

$\varphi^t : \mathcal{U} \to \mathcal{U}$ geodesic flow

$f, g \in C^\infty(\mathcal{U})$

μ Liouville measure



$$\rho_{f,g}(t) = \int_{\mathcal{U}} (f \circ \varphi^{-t}) g \, d\mu$$

Exponential mixing: $\rho_{f,g}(t) = c\Big(\int_{\mathcal{U}} f\, d\mu\Big)\Big(\int_{\mathcal{U}} g\, d\mu\Big) + \mathcal{O}(e^{-\nu t})$

Moore '87, Ratner '87, Chernov '98, Dolgopyat '98, Liverani '04

Correlations (compact manifolds)

M compact manifold

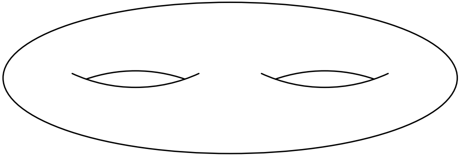
g negatively curved metric

$\mathcal{U} = \{(y, v) \in TM : |v|_g = 1\}$

$\varphi^t : \mathcal{U} \to \mathcal{U}$ geodesic flow

$f, g \in C^\infty(\mathcal{U})$

μ Liouville measure



$$\rho_{f,g}(t) = \int_{\mathcal{U}} (f \circ \varphi^{-t}) g \, d\mu$$

Exponential mixing: $\rho_{f,g}(t) = c\Big(\int_{\mathcal{U}} f \, d\mu\Big)\Big(\int_{\mathcal{U}} g \, d\mu\Big) + \mathcal{O}(e^{-\nu t})$

Moore '87, Ratner '87, Chernov '98, Dolgopyat '98, Liverani '04

From correlations to resonances

Correlation:

$$\rho_{f,g}(t) = \int_{\mathcal{U}} (f \circ \varphi^{-t}) g \, d\mu, \quad f, g \in C_0^\infty(\mathcal{U})$$

Fourier–Laplace transform:

$$\hat{\rho}_{f,g}(\lambda) = \int_0^\infty e^{-\lambda t} \rho_{f,g}(t) \, dt, \quad \operatorname{Re} \lambda > C_0 \gg 1,$$

$$\rho_{f,g}(t) = \frac{i}{2\pi} \int_{\operatorname{Re} \lambda = C_0} e^{\lambda t} \hat{\rho}_{f,g}(\lambda) \, d\lambda$$

Goal: continue $\hat{\rho}_{f,g}(\lambda)$ meromorphically to $\lambda \in \mathbb{C}$

Poles of $\hat{\rho}_{f,g}(\lambda)$: Pollicott–Ruelle resonances

From correlations to resonances

Correlation:

$$\rho_{f,g}(t) = \int_{\mathcal{U}} (f \circ \varphi^{-t}) g \, d\mu, \quad f, g \in C_0^\infty(\mathcal{U})$$

Fourier–Laplace transform:

$$\hat{\rho}_{f,g}(\lambda) = \int_0^\infty e^{-\lambda t} \rho_{f,g}(t) \, dt, \quad \operatorname{Re} \lambda > C_0 \gg 1,$$

$$\rho_{f,g}(t) = \frac{i}{2\pi} \int_{\operatorname{Re} \lambda = C_0} e^{\lambda t} \hat{\rho}_{f,g}(\lambda) \, d\lambda$$

Goal: continue $\hat{\rho}_{f,g}(\lambda)$ meromorphically to $\lambda \in \mathbb{C}$

Poles of $\hat{\rho}_{f,g}(\lambda)$: Pollicott–Ruelle resonances

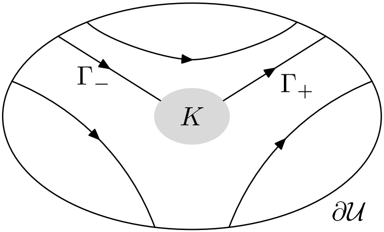
Setup: hyperbolic flows

$\overline{\mathcal{U}}$ compact manifold with boundary

$\varphi^t = e^{tX} : \mathcal{U} \to \mathcal{U}$ a C^∞ flow

$\partial\mathcal{U}$ strictly convex

$$K = \bigcap_{\pm t \in \mathbb{R}} \varphi^t(\overline{\mathcal{U}})$$



K hyperbolic ($\theta > 0$):

$$x \in K \implies T_x\mathcal{U} = \mathbb{R}X(x) \oplus E_u(x) \oplus E_s(x),$$

$$|d\varphi^t(x) \cdot w| \le Ce^{-\theta|t|}|w|, \quad \begin{cases} w \in E_s(x), & t > 0; \\ w \in E_u(x), & t < 0. \end{cases}$$

Anosov flow = closed hyperbolic system: $\partial\mathcal{U} = \emptyset$, $K = \mathcal{U}$

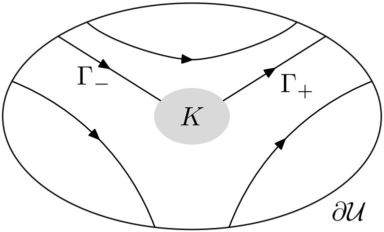
General case closely related to Axiom A basic sets [Smale '67]

Setup: hyperbolic flows

$\overline{\mathcal{U}}$ compact manifold with boundary

$\varphi^t = e^{tX} : \mathcal{U} \to \mathcal{U}$ a C^∞ flow

$\partial\mathcal{U}$ strictly convex



$$K = \bigcap_{\pm t \in \mathbb{R}} \varphi^t(\overline{\mathcal{U}})$$

K hyperbolic ($\theta > 0$):

$$x \in K \implies T_x\mathcal{U} = \mathbb{R}X(x) \oplus E_u(x) \oplus E_s(x),$$

$$|d\varphi^t(x) \cdot w| \leq Ce^{-\theta|t|}|w|, \quad \begin{cases} w \in E_s(x), & t > 0; \\ w \in E_u(x), & t < 0. \end{cases}$$

Anosov flow = closed hyperbolic system: $\partial\mathcal{U} = \emptyset, \quad K = \mathcal{U}$

General case closely related to Axiom A basic sets [Smale '67]

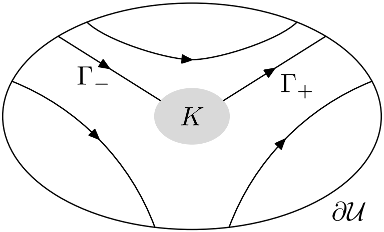
Setup: hyperbolic flows

$\overline{\mathcal{U}}$ compact manifold with boundary

$\varphi^t = e^{tX} : \mathcal{U} \to \mathcal{U}$ a C^∞ flow

$\partial\mathcal{U}$ strictly convex

$$K = \bigcap_{\pm t \in \mathbb{R}} \varphi^t(\overline{\mathcal{U}})$$



K hyperbolic ($\theta > 0$):

$$x \in K \implies T_x\mathcal{U} = \mathbb{R}X(x) \oplus E_u(x) \oplus E_s(x),$$

$$|d\varphi^t(x) \cdot w| \le Ce^{-\theta|t|}|w|, \quad \begin{cases} w \in E_s(x), & t > 0; \\ w \in E_u(x), & t < 0. \end{cases}$$

Anosov flow = closed hyperbolic system: $\partial\mathcal{U} = \emptyset$, $K = \mathcal{U}$

General case closely related to Axiom A basic sets [Smale '67]

Resonances and correlations

$\varphi^t = e^{tX} : \mathcal{U} \to \mathcal{U}$, $\quad f, g \in C_0^\infty(\mathcal{U})$. For $\operatorname{Re}\lambda \gg 1$,

$$\hat{\rho}_{f,g}(\lambda) = \int_0^\infty e^{-\lambda t} \int_{\mathcal{U}} (f \circ \varphi^{-t}) g \, d\mu dt$$

Resonances and correlations

$\varphi^t = e^{tX} : \mathcal{U} \to \mathcal{U}$, $\quad f, g \in C_0^\infty(\mathcal{U})$. For $\operatorname{Re}\lambda \gg 1$,

$$\hat{\rho}_{f,g}(\lambda) = \int_0^\infty e^{-\lambda t} \int_{\mathcal{U}} (e^{-tX} f) g \, d\mu dt$$

Resonances and correlations

$\varphi^t = e^{tX} : \mathcal{U} \to \mathcal{U}, \quad f, g \in C_0^\infty(\mathcal{U}).$ For $\operatorname{Re}\lambda \gg 1$,

$$\hat{\rho}_{f,g}(\lambda) = \int_0^\infty e^{-\lambda t} \int_{\mathcal{U}} (e^{-tX} f) g \, d\mu dt = \int_{\mathcal{U}} (R(\lambda) f) g \, d\mu,$$

$$R(\lambda) = \int_0^\infty e^{-t(X+\lambda)} \, dt = (X+\lambda)^{-1} : L^2(\mathcal{U}) \to L^2(\mathcal{U}).$$

Resonances and correlations

$\varphi^t = e^{tX} : \mathcal{U} \to \mathcal{U}, \quad f, g \in C_0^\infty(\mathcal{U}).$ For $\operatorname{Re}\lambda \gg 1$,

$$\hat\rho_{f,g}(\lambda) = \int_0^\infty e^{-\lambda t} \int_{\mathcal{U}} (e^{-tX} f) g \, d\mu dt = \int_{\mathcal{U}} (R(\lambda) f) g \, d\mu,$$

$$R(\lambda) = \int_0^\infty e^{-t(X+\lambda)} \, dt = (X+\lambda)^{-1} : L^2(\mathcal{U}) \to L^2(\mathcal{U}).$$

Theorem 1 [D–Guillarmou '14]

The operator $R(\lambda) : C_0^\infty(\mathcal{U}) \to \mathcal{D}'(\mathcal{U})$ continues meromorphically to $\lambda \in \mathbb{C}$. Its poles are called Pollicott–Ruelle resonances.

Resonances and correlations

$\varphi^t = e^{tX} : \mathcal{U} \to \mathcal{U}$, $f, g \in C_0^\infty(\mathcal{U})$. For $\operatorname{Re}\lambda \gg 1$,

$$\hat{\rho}_{f,g}(\lambda) = \int_0^\infty e^{-\lambda t} \int_{\mathcal{U}} (e^{-tX} f) g \, d\mu dt = \int_{\mathcal{U}} (R(\lambda) f) g \, d\mu,$$

$$R(\lambda) = \int_0^\infty e^{-t(X+\lambda)} \, dt = (X+\lambda)^{-1} : L^2(\mathcal{U}) \to L^2(\mathcal{U}).$$

Theorem 1 [D–Guillarmou '14]

The operator $R(\lambda) : C_0^\infty(\mathcal{U}) \to \mathcal{D}'(\mathcal{U})$ continues meromorphically to $\lambda \in \mathbb{C}$. Its poles are called Pollicott–Ruelle resonances.

Ruelle '76, Rugh '92, Fried '95 (analytic),
Pollicott '85,'86, Ruelle '86,'87, Parry–Pollicott '90 (C^∞ small strip),
Liverani '04, Butterley–Liverani '07, Faure–Sjöstrand '11 (Anosov)

Resonances and zeta functions

Ruelle zeta function: $T_{\gamma^\sharp}$ periods of primitive closed trajectories

$$\zeta_R(\lambda) = \prod_{\gamma^\sharp} (1 - e^{-\lambda T_{\gamma^\sharp}}), \quad \operatorname{Re} \lambda \gg 1$$

Theorem 2 [D–Guillarmou '14]

$\zeta_R(\lambda)$ extends meromorphically to $\lambda \in \mathbb{C}$.

- Poles are Pollicott–Ruelle resonances on certain vector bundles
- Classical application: counting primitive closed trajectories

$$\#\{T_{\gamma_\sharp} \leq T\} = \frac{e^{h_{\mathrm{top}} T}}{h_{\mathrm{top}} T}(1 + o(1))$$

where h_{top} is the topological entropy (first pole of ζ_R).
First proved by Margulis

Smale '67 (conjectured),
Giulietti–Liverani–Pollicott '12, D–Zworski '13 (Anosov)

Resonances and zeta functions

Ruelle zeta function: $T_{\gamma^\sharp}$ periods of primitive closed trajectories

$$\zeta_R(\lambda) = \prod_{\gamma^\sharp} (1 - e^{-\lambda T_{\gamma^\sharp}}), \quad \operatorname{Re}\lambda \gg 1$$

Theorem 2 [D–Guillarmou '14]

$\zeta_R(\lambda)$ extends meromorphically to $\lambda \in \mathbb{C}$.

- Poles are Pollicott–Ruelle resonances on certain vector bundles
- Classical application: counting primitive closed trajectories

$$\#\{T_{\gamma_\sharp} \leq T\} = \frac{e^{h_{\mathrm{top}} T}}{h_{\mathrm{top}} T}(1 + o(1))$$

where h_{top} is the topological entropy (first pole of ζ_R).
First proved by Margulis

Smale '67 (conjectured),
Giulietti–Liverani–Pollicott '12, D–Zworski '13 (Anosov)

Resonances and zeta functions

Ruelle zeta function: $T_{\gamma^\sharp}$ periods of primitive closed trajectories

$$\zeta_R(\lambda) = \prod_{\gamma^\sharp} (1 - e^{-\lambda T_{\gamma^\sharp}}), \quad \operatorname{Re}\lambda \gg 1$$

Theorem 2 [D–Guillarmou '14]

$\zeta_R(\lambda)$ extends meromorphically to $\lambda \in \mathbb{C}$.

- Poles are Pollicott–Ruelle resonances on certain vector bundles
- Classical application: counting primitive closed trajectories

$$\#\{T_{\gamma_\sharp} \le T\} = \frac{e^{h_{\mathrm{top}} T}}{h_{\mathrm{top}} T}(1 + o(1))$$

where h_{top} is the topological entropy (first pole of ζ_R).
First proved by Margulis

Smale '67 (conjectured),
Giulietti–Liverani–Pollicott '12, D–Zworski '13 (Anosov)

Three stages of Pollicott–Ruelle resonances

- Ruelle '76,'86,'87, Pollicott '85,'86, Parry–Pollicott '90, Fried '95, Dolgopyat '98, Naud '05, Stoyanov '11,'13 (spectral gaps)...
  symbolic dynamics and transfer operators
- Rugh '92, Kitaev '99, Blank–Keller–Liverani '02, Liverani '04,'05, Gouëzel–Liverani '06, Baladi–Tsujii '07 Butterley–Liverani '07, Giulietti–Liverani–Pollicott '12...
  resonances = spectrum of $-X$, $\varphi^t = e^{tX}$, on an anisotropic space:
  $$X + \lambda : \mathcal{H}^r \to \mathcal{H}^r \quad \text{Fredholm}, \quad r \gg \max(1, -\operatorname{Re}\lambda)$$
- Faure–Roy–Sjöstrand '08, Faure–Sjöstrand '11, Tsujii '12, Faure–Tsujii '11,'13, Datchev–D–Zworski '12, D–Zworski '13,'14, D–Faure–Guillarmou '14, D–Guillarmou '14, Guillarmou '14, Jin–Zworski '14 (various applications)...
  $(X + \lambda)u = f$ is a scattering problem in the phase space and $u \in \mathcal{H}^r$ is the outgoing/radiation condition. Relies on microlocal analysis.

Microlocal analysis (semiclassical version)

- Phase space: $T^*\mathcal{U} \ni (x, \xi)$
- Semiclassical parameter: $h \to 0$, the effective wavelength
- Classical observables: $a(x, \xi) \in C^\infty(T^*\mathcal{U})$
- Quantization: $\mathrm{Op}_h(a) = a\big(x, \frac{h}{i}\partial_x\big) : C^\infty(\mathcal{U}) \to C^\infty(\mathcal{U})$, semiclassical pseudodifferential operator

Basic examples

- $a(x, \xi) = x_j \quad \Longrightarrow \quad \mathrm{Op}_h(a) = x_j$ multiplication operator
- $a(x, \xi) = \xi_j \quad \Longrightarrow \quad \mathrm{Op}_h(a) = \frac{h}{i}\partial_{x_j}$

Classical-quantum correspondence

- $[\mathrm{Op}_h(a), \mathrm{Op}_h(b)] = \frac{h}{i}\mathrm{Op}_h(\{a, b\}) + \mathcal{O}(h^2)$
- $\{a, b\} = \partial_\xi a \cdot \partial_x b - \partial_x a \cdot \partial_\xi b = H_a b, \quad e^{tH_a}$ Hamiltonian flow
- Example: $[\mathrm{Op}_h(\xi_k), \mathrm{Op}_h(x_j)] = \frac{h}{i}\delta_{jk}$

Microlocal analysis (semiclassical version)

- Phase space: $T^*\mathcal{U} \ni (x, \xi)$
- Semiclassical parameter: $h \to 0$, the effective wavelength
- Classical observables: $a(x, \xi) \in C^\infty(T^*\mathcal{U})$
- Quantization: $\mathrm{Op}_h(a) = a\big(x, \frac{h}{i}\partial_x\big) : C^\infty(\mathcal{U}) \to C^\infty(\mathcal{U})$, semiclassical pseudodifferential operator

Basic examples

- $a(x, \xi) = x_j \quad \Longrightarrow \quad \mathrm{Op}_h(a) = x_j$ multiplication operator
- $a(x, \xi) = \xi_j \quad \Longrightarrow \quad \mathrm{Op}_h(a) = \frac{h}{i}\partial_{x_j}$

Classical-quantum correspondence

- $[\mathrm{Op}_h(a), \mathrm{Op}_h(b)] = \frac{h}{i}\mathrm{Op}_h(\{a, b\}) + \mathcal{O}(h^2)$
- $\{a, b\} = \partial_\xi a \cdot \partial_x b - \partial_x a \cdot \partial_\xi b = H_a b, \quad e^{tH_a}$ Hamiltonian flow
- Example: $[\mathrm{Op}_h(\xi_k), \mathrm{Op}_h(x_j)] = \frac{h}{i}\delta_{jk}$

Microlocal analysis (semiclassical version)

- Phase space: $T^*\mathcal{U} \ni (x, \xi)$
- Semiclassical parameter: $h \to 0$, the effective wavelength
- Classical observables: $a(x, \xi) \in C^\infty(T^*\mathcal{U})$
- Quantization: $\mathrm{Op}_h(a) = a\big(x, \frac{h}{i}\partial_x\big) : C^\infty(\mathcal{U}) \to C^\infty(\mathcal{U})$, semiclassical pseudodifferential operator

Basic examples

- $a(x, \xi) = x_j \quad \Longrightarrow \quad \mathrm{Op}_h(a) = x_j$ multiplication operator
- $a(x, \xi) = \xi_j \quad \Longrightarrow \quad \mathrm{Op}_h(a) = \frac{h}{i}\partial_{x_j}$

Classical-quantum correspondence

- $[\mathrm{Op}_h(a), \mathrm{Op}_h(b)] = \frac{h}{i}\mathrm{Op}_h(\{a, b\}) + \mathcal{O}(h^2)$
- $\{a, b\} = \partial_\xi a \cdot \partial_x b - \partial_x a \cdot \partial_\xi b = H_a b, \quad e^{tH_a}$ Hamiltonian flow
- Example: $[\mathrm{Op}_h(\xi_k), \mathrm{Op}_h(x_j)] = \frac{h}{i}\delta_{jk}$

Standard semiclassical estimates

General question

$$P = \mathrm{Op}_h(p), \quad Pu = f \quad \Longrightarrow \quad \|u\| \lesssim \|f\| \ ?$$

Control u microlocally: $\|\mathrm{Op}_h(a)u\| \lesssim \qquad \|f\| + \mathcal{O}(h^\infty)\|u\|$

Elliptic estimate

Propagation of singularities

Standard semiclassical estimates

General question

$P = \mathrm{Op}_h(p), \quad Pu = f \quad \Longrightarrow \quad \|u\| \lesssim \|f\|$?

Control u microlocally: $\|\mathrm{Op}_h(a)u\| \lesssim \qquad \|f\| + \mathcal{O}(h^\infty)\|u\|$

Elliptic estimate

Propagation of singularities

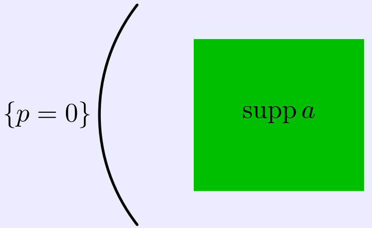
Standard semiclassical estimates

General question

$P = \mathrm{Op}_h(p), \quad Pu = f \quad \Longrightarrow \quad \|u\| \lesssim \|f\| \ ?$

Control u microlocally: $\|\mathrm{Op}_h(a)u\| \lesssim \quad \|f\| + \mathcal{O}(h^\infty)\|u\|$

Elliptic estimate



Propagation of singularities

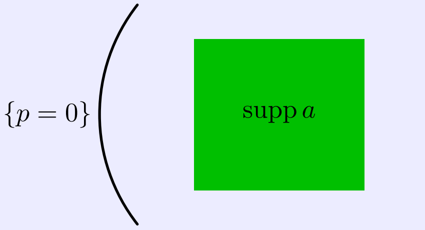
Standard semiclassical estimates

General question

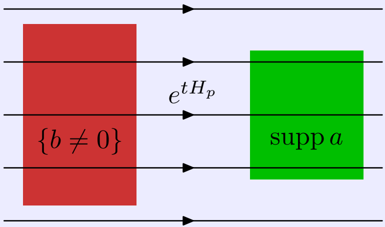
$P = \mathrm{Op}_h(p), \quad Pu = f \quad \Longrightarrow \quad \|u\| \lesssim \|f\|$?

Control u microlocally: $\|\mathrm{Op}_h(a)u\| \lesssim \|\mathrm{Op}_h(b)u\| + h^{-1}\|f\| + \mathcal{O}(h^\infty)\|u\|$

Elliptic estimate



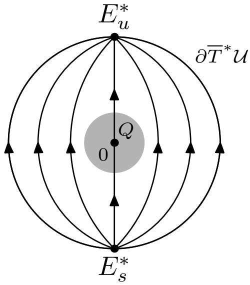
Propagation of singularities



The microlocal picture, Anosov case

$P(\lambda) := \frac{h}{i}(X + \lambda) = \mathrm{Op}_h(p)$,
$p(x,\xi) = \langle \xi, X(x)\rangle$
$e^{tH_p} : (x,\xi) \mapsto (\varphi^t(x), (d\varphi^t(x))^{-T}\xi)$

$$T^*\mathcal{U} \;=\; E_0^* \oplus E_s^* \oplus E_u^*,$$
$$\{p = 0\} \;=\; E_s^* \oplus E_u^*$$



Theorem 1

$P(\lambda)^{-1} : \mathcal{H}^r \to \mathcal{H}^r$ continues meromorphically to $\lambda \in \mathbb{C}$.

By Fredholm theory, enough to prove for $Q = \mathrm{Op}_h(q)$, $q \in C_0^\infty(T^*\mathcal{U})$

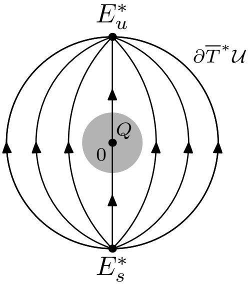
$$\|u\|_{\mathcal{H}^r} \le Ch^{-1}\|Pu\|_{\mathcal{H}^r} + C\|Qu\|_{\mathcal{H}^r}$$

where $\mathcal{H}^r$ is an anisotropic Sobolev space

The microlocal picture, Anosov case

$P(\lambda) := \frac{h}{i}(X + \lambda) = \mathrm{Op}_h(p)$,
$p(x,\xi) = \langle \xi, X(x) \rangle$
$e^{tH_p} : (x,\xi) \mapsto (\varphi^t(x), (d\varphi^t(x))^{-T}\xi)$

$$\begin{aligned} T^*\mathcal{U} &= E_0^* \oplus E_s^* \oplus E_u^*, \\ \{p = 0\} &= E_s^* \oplus E_u^* \end{aligned}$$



Theorem 1

$P(\lambda)^{-1} : \mathcal{H}^r \to \mathcal{H}^r$ continues meromorphically to $\lambda \in \mathbb{C}$.

By Fredholm theory, enough to prove for $Q = \mathrm{Op}_h(q)$, $q \in C_0^\infty(T^*\mathcal{U})$

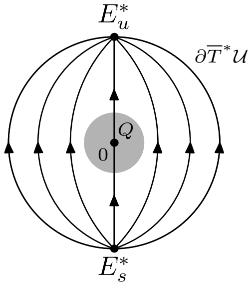
$$\|u\|_{\mathcal{H}^r} \leq Ch^{-1}\|Pu\|_{\mathcal{H}^r} + C\|Qu\|_{\mathcal{H}^r}$$

where $\mathcal{H}^r$ is an anisotropic Sobolev space

The microlocal picture, Anosov case

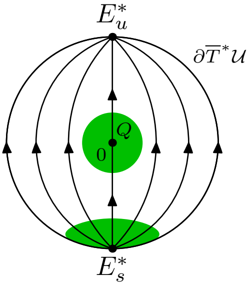
$P(\lambda) := \frac{h}{i}(X + \lambda) = \mathrm{Op}_h(p)$,
$p(x, \xi) = \langle \xi, X(x) \rangle$
$e^{tH_p} : (x, \xi) \mapsto (\varphi^t(x), (d\varphi^t(x))^{-T}\xi)$

$$\|u\|_{\mathcal{H}^r} \leq Ch^{-1}\|Pu\|_{\mathcal{H}^r} + C\|Qu\|_{\mathcal{H}^r}$$



The microlocal picture, Anosov case

$P(\lambda) := \frac{h}{i}(X + \lambda) = \mathrm{Op}_h(p)$,
$p(x, \xi) = \langle \xi, X(x) \rangle$
$e^{tH_p} : (x, \xi) \mapsto (\varphi^t(x), (d\varphi^t(x))^{-T}\xi)$

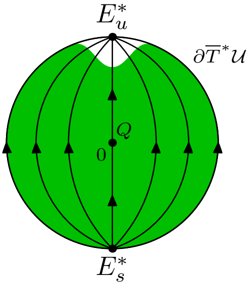


$$\|\mathrm{Op}_h(a)u\|_{\mathcal{H}^r} \leq Ch^{-1}\|Pu\|_{\mathcal{H}^r} + C\|Qu\|_{\mathcal{H}^r}$$

- Radial estimate [Melrose '94], $\mathcal{H}^r \sim H^r$ near E_s^*
- Propagation of singularities
- Dual radial estimate, $\mathcal{H}^r \sim H^{-r}$ near E_u^*

The microlocal picture, Anosov case

$P(\lambda) := \frac{h}{i}(X + \lambda) = \mathrm{Op}_h(p)$,
$p(x, \xi) = \langle \xi, X(x) \rangle$
$e^{tH_p} : (x, \xi) \mapsto (\varphi^t(x), (d\varphi^t(x))^{-T}\xi)$

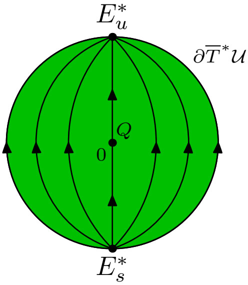


$$\|\mathrm{Op}_h(a)u\|_{\mathcal{H}^r} \leq Ch^{-1}\|Pu\|_{\mathcal{H}^r} + C\|Qu\|_{\mathcal{H}^r}$$

- Radial estimate [Melrose '94], $\mathcal{H}^r \sim H^r$ near E_s^*
- Propagation of singularities
- Dual radial estimate, $\mathcal{H}^r \sim H^{-r}$ near E_u^*

The microlocal picture, Anosov case

$P(\lambda) := \frac{h}{i}(X+\lambda) = \mathrm{Op}_h(p)$,
$p(x,\xi) = \langle \xi, X(x)\rangle$
$e^{tH_p} : (x,\xi) \mapsto (\varphi^t(x), (d\varphi^t(x))^{-T}\xi)$



$$\|\mathrm{Op}_h(a)u\|_{\mathcal{H}^r} \leq Ch^{-1}\|Pu\|_{\mathcal{H}^r} + C\|Qu\|_{\mathcal{H}^r}$$

- Radial estimate [Melrose '94], $\mathcal{H}^r \sim H^r$ near E_s^*
- Propagation of singularities
- Dual radial estimate, $\mathcal{H}^r \sim H^{-r}$ near E_u^*

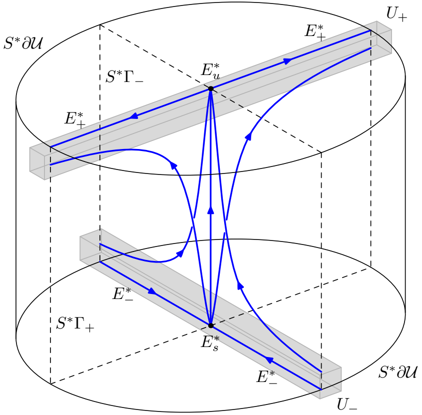
The microlocal picture, general case

Problem: singularities escape both to infinite frequencies and through the spatial boundary

More general propagation estimates needed. . .

The microlocal picture, general case

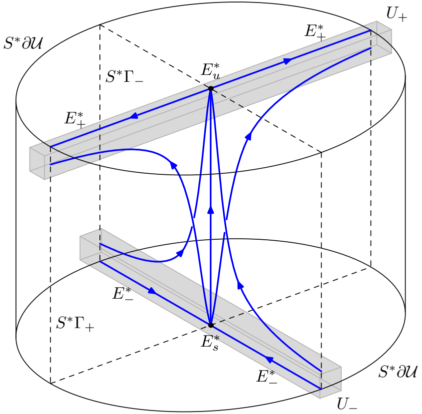
Problem: singularities escape both to infinite frequencies and through the spatial boundary

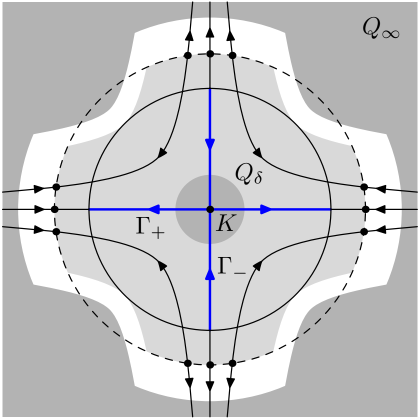


More general propagation estimates needed. . .

The microlocal picture, general case

Problem: singularities escape both to infinite frequencies and through the spatial boundary

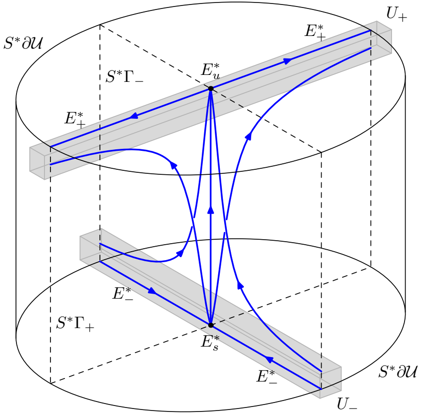


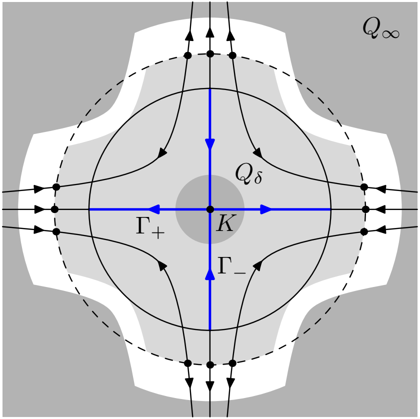


More general propagation estimates needed. . .

The microlocal picture, general case

Problem: singularities escape both to infinite frequencies and through the spatial boundary





More general propagation estimates needed...

The zeta function

$$\zeta_R(\lambda) = \prod_{\gamma^\sharp} (1 - e^{-\lambda T_{\gamma^\sharp}}), \quad R(\lambda) = \int_0^\infty e^{-\lambda t} (\varphi^{-t})^* \, dt$$

Theorem 2

$\zeta_R(\lambda)$ extends meromorphically to $\lambda \in \mathbb{C}$.

The zeta function

$$\zeta_R(\lambda) = \prod_{\gamma^\sharp}(1 - e^{-\lambda T_{\gamma^\sharp}}), \quad R(\lambda) = \int_0^\infty e^{-\lambda t}(\varphi^{-t})^* \, dt$$

Theorem 2

$\zeta_R(\lambda)$ extends meromorphically to $\lambda \in \mathbb{C}$.

- $\dfrac{\zeta_R'(\lambda)}{\zeta_R(\lambda)} \sim \sum_\gamma \dfrac{T_\gamma^\sharp e^{-\lambda T_\gamma}}{|\det(I - \mathcal{P}_\gamma)|}$, $\quad \operatorname{tr}^\flat(\varphi^{-t})^* = \sum_\gamma \dfrac{T_\gamma^\sharp \delta(t - T_\gamma)}{|\det(I - \mathcal{P}_\gamma)|}$
  [Ruelle, Atiyah–Bott–Guillemin]
- $\zeta_R'(\lambda)/\zeta_R(\lambda) \sim \operatorname{tr}^\flat(R(\lambda)(\varphi^{-t_0})^*)$, but what is $\operatorname{tr}^\flat$?
- The proof of Theorem 1 gives a wavefront set condition, which makes it possible to take the flat trace

The zeta function

$$\zeta_R(\lambda) = \prod_{\gamma^\sharp}(1 - e^{-\lambda T_{\gamma^\sharp}}), \quad R(\lambda) = \int_0^\infty e^{-\lambda t}(\varphi^{-t})^* \, dt$$

Theorem 2

$\zeta_R(\lambda)$ extends meromorphically to $\lambda \in \mathbb{C}$.

- $$\frac{\zeta_R'(\lambda)}{\zeta_R(\lambda)} \sim \sum_\gamma \frac{T_\gamma^\sharp e^{-\lambda T_\gamma}}{|\det(I - \mathcal{P}_\gamma)|}, \quad \operatorname{tr}^\flat(\varphi^{-t})^* = \sum_\gamma \frac{T_\gamma^\sharp \delta(t - T_\gamma)}{|\det(I - \mathcal{P}_\gamma)|}$$
  [Ruelle, Atiyah–Bott–Guillemin]
- $\zeta_R'(\lambda)/\zeta_R(\lambda) \sim \operatorname{tr}^\flat(R(\lambda)(\varphi^{-t_0})^*)$, but what is $\operatorname{tr}^\flat$?
- The proof of Theorem 1 gives a wavefront set condition, which makes it possible to take the flat trace

The zeta function

$$\zeta_R(\lambda)=\prod_{\gamma^\sharp}(1-e^{-\lambda T_{\gamma^\sharp}}),\quad R(\lambda)=\int_0^\infty e^{-\lambda t}(\varphi^{-t})^*\,dt$$

Theorem 2

$\zeta_R(\lambda)$ extends meromorphically to $\lambda\in\mathbb{C}$.

- $\dfrac{\zeta_R'(\lambda)}{\zeta_R(\lambda)}\sim\sum_\gamma\dfrac{T_\gamma^\sharp e^{-\lambda T_\gamma}}{|\det(I-\mathcal{P}_\gamma)|},\quad \operatorname{tr}^\flat(\varphi^{-t})^*=\sum_\gamma\dfrac{T_\gamma^\sharp\delta(t-T_\gamma)}{|\det(I-\mathcal{P}_\gamma)|}$ [Ruelle, Atiyah–Bott–Guillemin]
- $\zeta_R'(\lambda)/\zeta_R(\lambda)\sim\operatorname{tr}^\flat(R(\lambda)(\varphi^{-t_0})^*)$, but what is $\operatorname{tr}^\flat$?
- The proof of Theorem 1 gives a wavefront set condition, which makes it possible to take the flat trace

The zeta function

$$\zeta_R(\lambda) = \prod_{\gamma^\sharp} (1 - e^{-\lambda T_{\gamma^\sharp}}), \quad R(\lambda) = \int_0^\infty e^{-\lambda t} (\varphi^{-t})^* \, dt$$

Theorem 2

$\zeta_R(\lambda)$ extends meromorphically to $\lambda \in \mathbb{C}$.

- $$\frac{\zeta_R'(\lambda)}{\zeta_R(\lambda)} \sim \sum_\gamma \frac{T_\gamma^\sharp e^{-\lambda T_\gamma}}{|\det(I - \mathcal{P}_\gamma)|}, \quad \operatorname{tr}^\flat (\varphi^{-t})^* = \sum_\gamma \frac{T_\gamma^\sharp \delta(t - T_\gamma)}{|\det(I - \mathcal{P}_\gamma)|}$$
  [Ruelle, Atiyah–Bott–Guillemin]
- $\zeta_R'(\lambda)/\zeta_R(\lambda) \sim \operatorname{tr}^\flat(R(\lambda)(\varphi^{-t_0})^*)$, but what is $\operatorname{tr}^\flat$?
- The proof of Theorem 1 gives a wavefront set condition, which makes it possible to take the flat trace

Further results

- [Faure–Tsujii '13, D–Faure–Guillarmou '14] Bands of resonances for certain contact Anosov flows and the constant curvature case
- [Tsujii '12, Nonnenmacher–Zworski '13] Essential spectral gaps of optimal size (Anosov)
- [Faure–Sjöstrand '11, Datchev–D–Zworski '12] Upper bounds on the number of resonances in strips (Anosov)
- [D–Zworski '14] Stochastic definition and stability of resonances: the eigenvalues of $-X + \varepsilon\Delta$ converge to resonances as $\varepsilon \to 0+$ (Anosov)
- [Guillarmou '14] Lens rigidity: a negatively curved metric on a domain with boundary is (locally) uniquely determined by its exit times and scattering relation. (A surprising application of [D–G '14]; unlike previous results in the field it handles trapping situations)
- [Jin–Zworski '14] There exists a strip with infinitely many resonances for any Anosov flow

Thank you for your attention!