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Quasi-normal modes
for Kerr–de Sitter black holes

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Motivation

Gravitational waves

- Gravitational waves are perturbations of the curvature of the space-time, just like light waves are perturbations of the electromagnetic field
- They can be used to detect and study black holes
- There are numerous gravitational wave detectors: GEO 600, LIGO, MiniGRAIL, VIRGO, . . .
- Quasi-normal modes are the complex frequencies of gravitational waves; they characterize a black hole much like the EM spectrum characterizes a star
- No gravitational waves coming from outer space have been detected on Earth so far; however, there is indirect evidence that they exist (Hulse–Taylor binary system)

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Mathematics of black holes

Scattering theory on black holes

There are many works by physicists on quasi-normal modes; however, there have been only a handful of attempts to put these works on a mathematical foundation: Bachelot '91, Bachelot–Motet-Bachelot '93, Sá Barreto–Zworski '97, Bony–Häfner '07, Melrose–Sá Barreto–Vasy '08, Vasy '10, . . .

The mathematical definition of quasi-normal modes comes from scattering theory:

- Start with a Lorentzian metric on a 4D space-time and the corresponding wave equation $\Box u = 0$. In the simplest (linear scalar) case, gravitational waves are modeled as solutions to this equation, and we are interested in their behavior for large time.

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Mathematics of black holes

Scattering theory on black holes

- Take the Fourier transform in time: $\Box u = 0$ becomes

$$P(\omega)\hat{u}(\omega) = f(\omega), \ \omega \in \mathbb{R}$$

  where $P(\omega)$ is a certain operator on the space slice, and f depends on the initial conditions.

- Prove the existence of a meromorphic family $R(\omega), \ \omega \in \mathbb{C}$, of operators on the space slice, such that

$$\hat{u}(\omega) = R(\omega)f(\omega).$$

  This family is called the scattering resolvent. It is a right inverse to $P(\omega)$, with outgoing boundary conditions.

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Mathematics of black holes

Scattering theory on black holes

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Mathematics of black holes

Scattering theory on black holes

- Use contour deformation and estimates on $R(\omega)$ in the strip $\{\operatorname{Im}\omega > -\nu\}$ to obtain the asymptotic resonance expansion as $t \to +\infty$:

$$u(t,x) = \sum_{\operatorname{Im}\hat{\omega} > -\nu} e^{-it\hat{\omega}} \sum_j t^j u_{\hat{\omega},j}(x) + O(e^{-\nu t}).$$

  Here $\hat{\omega}$ are resonances, the poles of $R(\omega)$.

- Conclude that Quasi-Normal Modes = Resonances
- In case we only have a small resonance free strip $\{\operatorname{Im}\omega > -\varepsilon\}$, resonance expansion takes the form of exponential decay of linear waves, modulo resonance at 0.

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Related work

Decay of waves on black hole backgrounds

There are numerous results on decay of linear waves on both spherically symmetric and rotating black holes using physical space methods:

Andersson–Blue '09, Bony–Häfner '10, Blue–Sterbenz '05,
Dafermos–Rodnianski '07, '08, '09, '10,
Donninger–Schlag–Soffer '09,
Finster–Kamran–Smoller–Yau '09, Luk '09, '10,
Marzuola–Metcalfe–Tataru–Tohaneanu '08,
Tataru '09, Tataru–Tohaneanu '08. . .

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Marzuola–Metcalfe–Tataru–Tohaneanu '08,
Tataru '09, Tataru–Tohaneanu '08...

However, most of these results deal with the case of zero cosmological constant, when there is an asymptotically flat infinity. In this case, the global meromorphy of the scattering resolvent $R(\omega)$ is unlikely, and the rate of decay is only polynomial in time.

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Related work

Black holes with positive cosmological constant

Schwarzschild–de Sitter (spherically symmetric)

- Sá Barreto–Zworski '97: constructed the scattering resolvent and showed that QNMs approximately lie on a lattice.
- Bony–Häfner '07: proved the resonance expansion.

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Related work

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Kerr–de Sitter (rotating)

- Vasy '10: constructed the scattering resolvent
- Together with earlier results of Wunsch–Zworski '10 and Datchev–Vasy '10, this gives an alternative proof for some results presented here, such as exponential decay of linear waves. However, quantization condition and resonance expansion cannot be recovered this way.

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Overview

The presented research concentrates on the slowly rotating Kerr–de Sitter black hole. The main results are:

- meromorphy of the scattering resolvent in the entire complex plane;
- resonance free strip and exponential decay of linear waves;
- resonance expansion;
- semiclassical approximation (quantization condition) for QNMs in any strip of fixed width;
- comparison of the semiclassical approximation with QNMs computed by physicists.

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Kerr–de Sitter metric

$$g = -\rho^2 \Big(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\Big)$$

$$-\frac{\Delta_\theta \sin^2\theta}{(1+\alpha)^2 \rho^2}(a\,dt - (r^2 + a^2)\,d\varphi)^2$$

$$+\frac{\Delta_r}{(1+\alpha)^2 \rho^2}(dt - a\sin^2\theta\,d\varphi)^2.$$
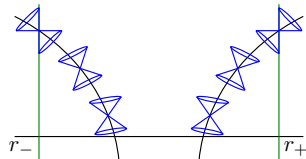
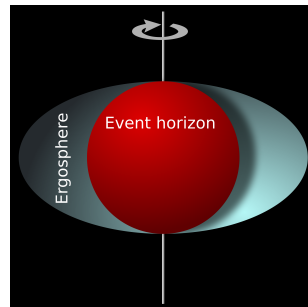Two radial timelike geodesics, with light cones shown

Here a is the angular momentum; $\rho(r,\theta)$ and $\Delta_\theta(\theta)$ are nonzero functions, and $\Delta_r(r)$ is a fourth degree polynomial. The metric is defined on $\mathbb{R}_t \times (r_-, r_+) \times \mathbb{S}^2_{\theta,\varphi}$, where $r_\pm$ are two roots of the equation $\Delta_r = 0$. The surfaces $\{r = r_\pm\}$ are event horizons.

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Features of the metric

- Two event horizons. Each of them is an apparent singularity, which can be removed by a change of variables $(t, r, \theta, \varphi) \rightarrow (t^*, r, \theta, \varphi^*)$, with $t^* \sim t + c_\pm \ln |r - r_\pm|$ near $r = r_\pm$.
- Symmetries: both ∂_t and ∂_φ are Killing, and the geodesic flow is completely integrable
- Inside the two ergospheres, located close to the event horizons, the field ∂_t is spacelike and thus the operator $P(\omega)$ is not elliptic

Picture courtesy of Wikipedia.

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Theorem 1 [D Fix]

'10 a compact region $K \subset (r_-, r_+) \times \mathbb{S}^2$. If the angular momentum a is small enough, depending on K, then:

- The cutoff scattering resolvent $\mathbf{1}_K R_g(\omega) \mathbf{1}_K$ is meromorphic on the entire $\mathbb{C}$ and holomorphic in $\{\operatorname{Im} \omega \geq 0, \ \omega \neq 0\}$.
- There is a resonance free strip $\{-\varepsilon < \operatorname{Im} \omega < 0\}$.

Combining Theorem 1 with red-shift effect, we get

Theorem 2 [D '10]

For $a, \varepsilon > 0$ small enough and u any solution to the wave equation on Kerr–de Sitter with sufficiently regular initial data, there exists a constant u_0 such that

$$\|u(t^*) - u_0\|_{H^1} \leq Ce^{-\varepsilon t^*}.$$

Here $t^* \sim t + c_\pm \ln |r - r_\pm|$ near $r = r_\pm$.

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Resonance expansion

Theorem 3 [D '11]

Let $\nu > 0$. Then any solution $u(t^*)$ to the wave equation $\Box u = 0$ with $u|_{t^*=0} = f_0 \in H^s(X)$, $u_t|_{t^*=0} = f_1 \in H^{s-1}(X)$ and s large enough satisfies

$$u(t^*) = \sum_{\mathrm{Im}\,\hat{\omega} > -\nu} e^{-i\hat{\omega}t^*} \sum_{0 \le j < J_{\hat{\omega}}} (t^*)^j \Pi_{\hat{\omega},j}(f_0, f_1) + O_{H^1(X)}(e^{-\nu t^*}),$$

as $t^* \to +\infty$. The sum is over QNMs $\hat{\omega}$, and each $\Pi_{\hat{\omega},j} : H^s \oplus H^{s-1} \to H^1$ is a finite dimensional operator; $X = (r_- - \delta, r_+ + \delta) \times \mathbb{S}^2$ is the whole space slice.

Theorem 3 justifies interpretation of quasi-normal modes as complex frequencies of (linear scalar) gravitational waves.

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Quantization condition

Theorem 4 [D '11]

Fix $\nu > 0$. Then quasi-normal modes with $\text{Im}\,\omega > -\nu$, $\text{Re}\,\omega \gg 1$ are simple and given modulo $O(|\omega|^{-\infty})$ by
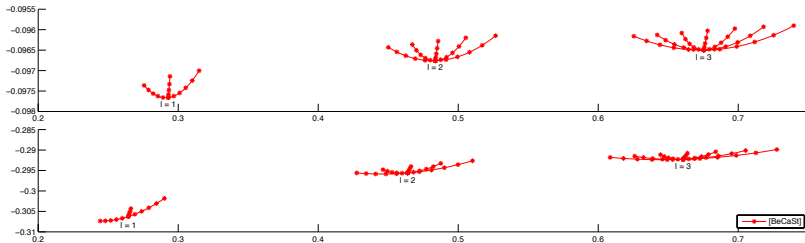
$$\omega = \mathcal{F}(m, l, k), \ m, l, k \in \mathbb{Z}, \ 0 \leq m \leq C_m, \ |k| \leq l, \ l \gg 1.$$

Here $\mathcal{F}$ is a classical symbol in the (l, k) variables and

$$\mathcal{F}(m, l, k)|_{a=0} = \frac{\sqrt{1 - 9\Lambda M^2}}{3\sqrt{3}M}[(l + 1/2) - i(m + 1/2)] + O(l^{-1}),$$

$$(\partial_k \mathcal{F})|_{a=0} = \frac{2 + 9\Lambda M^2}{27M^2}a + O(a^2 + l^{-1}).$$

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Comparison to exact QNMs

- m corresponds to the depth of the resonance ($m = 0$ for the top picture and $m = 1$ for the bottom one)
- l is the index of the angular eigenvalue
- k is the angular momentum
- Each line on the picture corresponds to QNMs for fixed values of m, l, k, and $a = 0, 0.05, \ldots, 0.25$. The lines for different k all start at the same Schwarzschild–de Sitter QNM; this splitting is analogous to Zeeman effect.

- One can follow the proof of Theorem 4 and extract an explicit construction for the asymptotic series of the quantization symbol $\mathcal{F}$.

- We have computed the series for $\mathcal{F}$ numerically for the case $l - |k| = O(1)$, using bottom of the well asymptotics.

- In the following slides, we compare the semiclassical approximation to QNMs, given by $\mathcal{F}$, to exact QNMs as computed by Berti–Cardoso–Starinets '09 using Leaver's continued fraction method.

- Berti–Cardoso–Starinets consider the case $\Lambda = 0$ of the Kerr black hole. Our theorems do not apply; however, the reason is the asymptotically flat infinity, while QNMs are generated by trapping, located in a compact set. Therefore, we can still carry out the computations.

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Comparison to exact QNMs
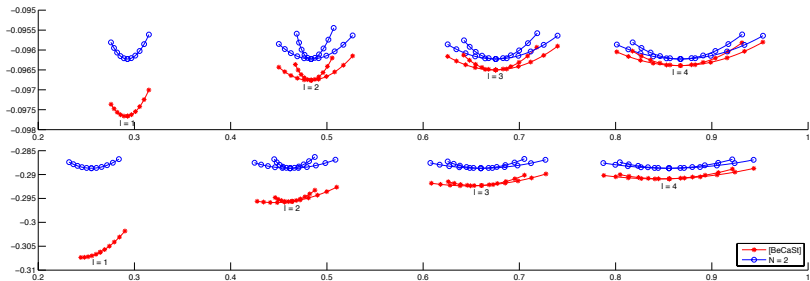
For $m = 0, 1$, $l = 1, \ldots, 4$, $l - |k| \leq 1$, and $a = 0, 0.05, \ldots, 0.25$, we compare order 2 semiclassical approximation to QNMs (that is, sum of the first 3 terms in the asymptotic series for $\mathcal{F}$) with QNMs computed by Berti–Cardoso–Starinets. We see that semiclassical approximation gets better when m is small and $l \to +\infty$.

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Comparison to exact QNMs
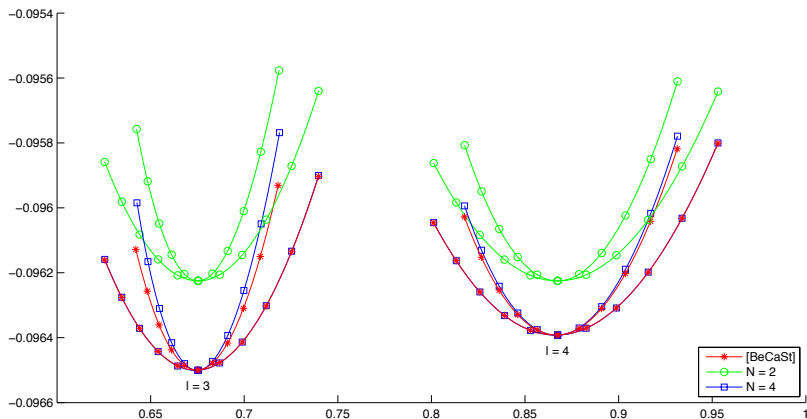
For $m = 0$, $l = 3, 4$, $l - |k| \leq 1$, and $a = 0, 0.05, \ldots, 0.25$, we compare order 2 semiclassical approximation to QNMs, order 4 approximation, and data of Berti–Cardoso–Starinets. We see that the higher order approximation is considerably better.

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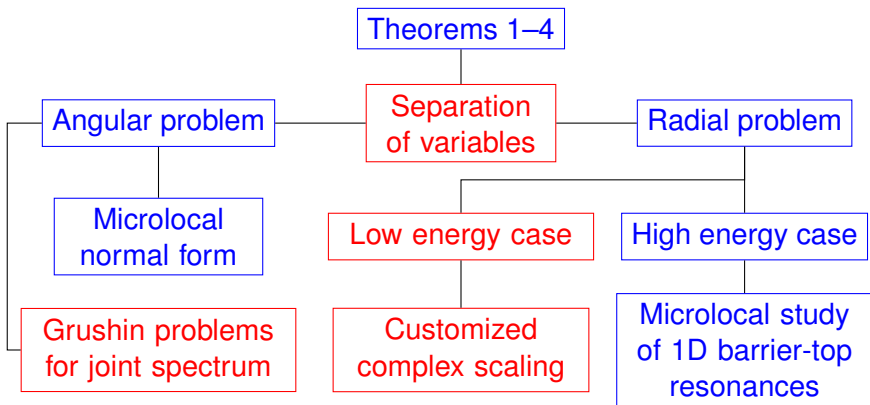
The items in red contain new (or at least less standard)
methods and will be elaborated on in the following slides.

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Separation of variables

Separation of variables

The operator $P(\omega)$ is invariant under the axial rotation
$\varphi \mapsto \varphi + s$. Take $k \in \mathbb{Z}$ and let $\mathcal{D}'_k = \text{Ker}(D_\varphi - k)$ be the space
of functions with angular momentum k; then

$$\rho^2 P(\omega)|_{\mathcal{D}'_k} = P_r(\omega, k) + P_\theta(\omega)|_{\mathcal{D}'_k},$$

where P_r is a differential operator in r and P_θ is a differential
operator on $\mathbb{S}^2$. For $a = 0$, P_θ is independent of ω and is just the
negative Laplace–Betrami operator on the round sphere.

$$P_r(\omega, k) = D_r(\Delta_r D_r) - \frac{(1+\alpha)^2}{\Delta_r}((r^2 + a^2)\omega - ak)^2,$$

$$P_\theta(\omega) = \frac{1}{\sin\theta} D_\theta(\Delta_\theta \sin\theta D_\theta) + \frac{(1+\alpha)^2}{\Delta_\theta \sin^2\theta}(a\omega \sin^2\theta - D_\varphi)^2.$$

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Separation of variables

Separation of variables

We need to invert the operator $\rho^2 P(\omega)|_{\mathcal{D}'_k} = P_r(\omega, k) + P_\theta(\omega)|_{\mathcal{D}'_k}$

Problems for $a \neq 0$

- P_θ is not self-adjoint $\rightarrow$ complete system of eigenfunctions?
- P_θ depends on ω

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Separation of variables

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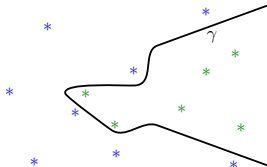
Problems for $a \neq 0$

- P_θ is not self-adjoint $\rightarrow$ complete system of eigenfunctions?
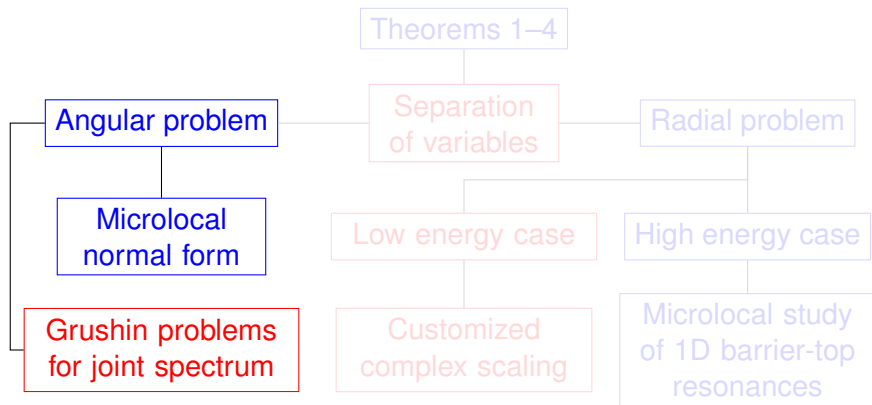- P_θ depends on ω

Solution (see also Ben-Artzi–Devinatz '83, Mazzeo–Vasy '02)

For each $\lambda \in \mathbb{C}$, construct $R_r(\omega, k, \lambda) = (P_r(\omega, k) + \lambda)^{-1}$ and $R_\theta(\omega, \lambda) = (P_\theta(\omega) - \lambda)^{-1}$ and write

$$R(\omega)|_{\mathcal{D}'_k} = \frac{1}{2\pi i} \int_\gamma R_r(\omega, k, \lambda) \otimes R_\theta(\omega, \lambda)|_{\mathcal{D}'_k} \, d\lambda.$$

Here γ is a contour separating the poles of R_r from those of R_θ.

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Angular problem

Theorems 1–4

Angular problem

Separation
of variables

Radial problem

Microlocal
normal form

Low energy case

High energy case

Grushin problems
for joint spectrum

Customized
complex scaling

Microlocal study
of 1D barrier-top
resonances

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Angular problem

We study poles of the angular resolvent $(P_\theta(\omega) - \lambda)^{-1}|_{\mathcal{D}'_k}$.

Semiclassical problem

- Equivalent to finding joint spectrum of (P_1, P_2), with P_1 an $O(h)$ perturbation of a self-adjoint operator and $P_2 = hD_\varphi$
- The principal symbols (p_1, p_2) form an integrable system

Microlocal normal form

Microlocally near a Liouville torus for (p_1, p_2), we can write

$$BP_1B^{-1} = f(hD_x, hD_y; h), \ BP_2B^{-1} = hD_y$$

- B is a Fourier integral operator;
- D_x, D_y are operators on $\mathbb{T}^2 = \mathbb{R}^2_{x,y}/(2\pi\mathbb{Z}^2)$;
- the joint pseudospectrum is given by $(f(h\mathbb{Z}, h\mathbb{Z}), h\mathbb{Z})$.

Similar methods are used in Hitrik–Sjöstrand '03.

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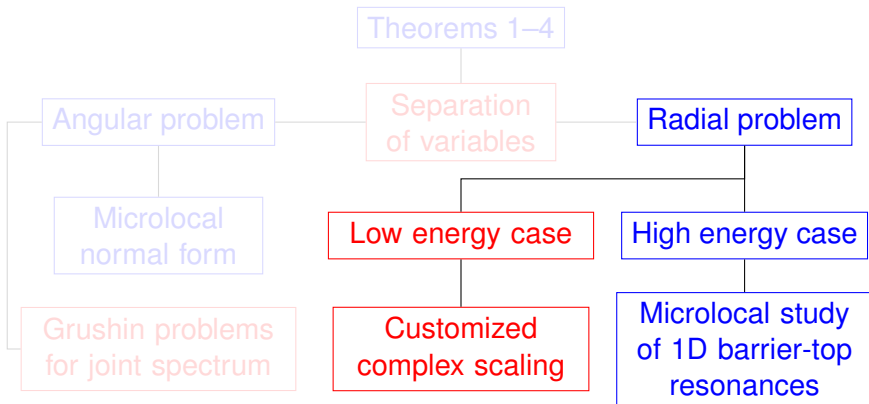Angular problem

Grushin problems for joint spectrum

Let P_1, P_2 be h-semiclassical differential operators on a compact manifold. Given microlocal normal form for P_1, P_2, how to obtain information on their joint spectrum?

Lemma

Assume that $[P_1, P_2] = 0$ and there exist operators $A_1, A_2 : L^2 \to L^2$, $S_1 : \mathbb{C} \to L^2$, $S_2 : L^2 \to \mathbb{C}$, such that:

- every two of the operators P_1, P_2, A_1, A_2 commute modulo $O(h^\infty)$;
- (almost joint eigenfunction) $S_2 S_1 = 1 + O(h^\infty)$ and $(*)S_1, S_2(*) = O(h^\infty)$, with $(*)$ any of P_1, P_2, A_1, A_2;
- (joint invertibility) $P_1 A_1 + P_2 A_2 = 1 - S_1 S_2 + O(h^\infty)$.

Then there exists an element of the joint spectrum of P_1, P_2 within $O(h^\infty)$ of the origin.

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Radial problem

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Radial problem

After a Regge–Wheeler change of variables $r \to x$ mapping $r_{\pm} \mapsto \pm\infty$, the operator $P_r + \lambda$ is roughly equivalent to

$$P_x = D_x^2 + V(x; \omega, \lambda, k), \ V(x) \sim (\omega - ak)^2 - \lambda e^{\mp x}, \ \pm x \gg 1.$$

For fixed ω, λ, k, the radial resolvent R_r is constructed using 1D potential scattering. The hard part is then to obtain uniform estimates on R_r in the following two cases:
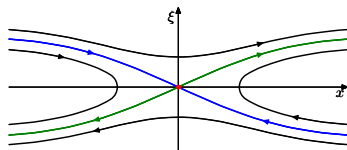
Trapping for P_x

- Low energy regime: $|\lambda| \gg |\omega|^2 + |ak|^2$;
- High energy regime: $|\lambda| + |k|^2 = O(|\omega|^2)$. Here, trapping comes into play; we use previous research on barrier-top resonances (Gérard–Sjöstrand '87, Sjöstrand '87, Ramond '96), with microlocal analysis near the trapped set of Colin de Verdière–Parisse '94. (One can also use Wunsch–Zworski '10 for a resonance free strip.)

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Radial problem

Customized complex scaling

We study scattering for $P_x = D_x^2 + V(x)$, in the low energy regime $|\lambda| \gg |\omega|^2 + |ak|^2$; $V(x) \sim (\omega - ak)^2 - \lambda e^{\mp x}$ for $\pm x \gg 1$.

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Radial problem

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- Standard complex scaling fails (no ellipticity near $x = \pm\infty$).
- Let u be an outgoing solution to $P_x u = f \in L^2_{\text{comp}}$; extend it analytically to a neighborhood of $\mathbb{R}$ in $\mathbb{C}$.
- Use semiclassical analysis on two circles to get control on u at two distant, but fixed, points $z_\pm \in \mathbb{C}$.
- Formulate a BVP for the restriction of u to a certain contour between z_- and z_+, and get $\|u\| \lesssim |\lambda|^{-1}\|f\|$.

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Radial problem

Customized complex scaling

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- Use semiclassical analysis on two circles to get control on
  u at two distant, but fixed, points $z_\pm \in \mathbb{C}$.
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Radial problem

Customized complex scaling

We study scattering for $P_x = D_x^2 + V(x)$, in the low energy regime $|\lambda| \gg |\omega|^2 + |ak|^2$; $V(x) \sim (\omega - ak)^2 - \lambda e^{\mp x}$ for $\pm x \gg 1$.

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Thank you for your attention!