Qualifying Examination Syllabus for Semyon Dyatlov

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Major Topic: Scattering Theory (Classical Analysis)

General tools. Analytic Fredholm theory. Holomorphic continuation of the free resolvent in odd and even dimensions. Meromorphic continuation of the perturbed resolvent in the context of black box scattering. [Sjöstrand, 2.1 and 2.3][Taylor, 9.7]

One-dimensional scattering by a compactly supported potential. Free resolvent. Meromorphic continuation of the perturbed resolvent. Plane waves and spectral measure of the perturbed operator. Scattering matrix. Wave operators. Expansion of solutions of the wave equation in terms of resonances. Distribution of resonances. [TZ, 1.1–1.4]

Three-dimensional scattering by a smooth obstacle:

The scattering problem. Incoming and outgoing radiation conditions. Uniqueness theorem. Existence theorem (limiting absorption principle). Plane waves. Green's function. Asymptotic behaviour of solutions; scattering amplitudes. [Taylor, 9.1]

Spectral measure. Spectral measure of $\sqrt{-\Delta}$ in terms of plane waves. Scattering analogues of the Fourier transform, their unitarity. [Taylor, 9.2]

Scattering operator. Scattering operator via scattering amplitudes. Unitarity and optical theorem. Neumann operator and reciprocity relation. [Taylor, 9.3]

Connections with the wave equation. Wave group and its spectral representations. Wave operators, their unitarity. Local energy decay. Translation representations, Radon and Lax-Philips transforms. [Taylor, 9.4–9.6]

Meromorphic continuation. Meromorphic continuation of the resolvent, scattering amplitudes, and scattering operator via black box scattering. Infinite number of resonances on the imaginary axis. [Sjöstrand, 2.3][Taylor, 9.7]

Trace formulas. Trace $\mathrm{Tr}(\phi(\sqrt{-\Delta}) - \phi(\sqrt{-\Delta_0}))$ and the scattering phase. [Taylor, 9.8]

Major Topic: Microlocal Analysis (Classical Analysis)

Fundamentals. Symbols, their basic properties, asymptotic sums. Phase functions. Oscillatory integrals, basic regularity properties, operators. Method of stationary phase. [GS, Chapters 1–2]

Pseudodifferential operators. Properly supported operators, the complete symbol. Adjoints, products of operators and changes of variables. Construction of a parametrix for an elliptic pseudodifferential operator. L^2 and H^s-regularity. Distributions, pseudodifferential operators, elliptic operators, and H^s-regularity on compact manifolds. Cotlar-Stein lemma and Calderón-Vaillancourt theorem. Sharp Gårding inequality. [GS, Chapters 3–4]

Local symplectic geometry. Canonical symplectic structure on the cotangent bundle. Lagrangian submanifolds and local existence of solutions for Hamilton-Jacobi equations. Darboux theorem. Canonical transformations. [GS, Chapters 5 and 9]

Wavefront sets. Definition, basic properties, and behaviour under pseudodifferential operators. Operations: inner product, tensor product, distribution kernels, pullback, multiplication. Wavefront sets of oscillatory integrals. [GS, Chapter 7]

WKB approximation for solutions of PDEs. Construction of a local parametrix for the Cauchy problem for a strictly hyperbolic PDE; the singular support of its kernel. Propagation of singularities for operators of real principal type. [GS, Chapters 6 and 8]

Minor Topic: Riemannian Geometry (Geometry)

Smooth manifolds. Basic structures: morphisms, orientation, local coordinates, partitions of unity. Vector bundles. Tensor bundles and tensor fields. Tangent and cotangent bundles, differential forms. [Lee, Chapter 2]

Riemannian metrics. Existence. Elementary constructions: raising and lowering indices, inner product on tensors, integration. Model examples: Euclidean space, spheres, hyperbolic models (hyperboloid, Poincare ball, half-space). [Lee, Chapter 3]

Connections and geodesics. Affine connections, Levi-Civita connection. Existence and uniqueness of geodesics, the exponential map, normal neighborhoods. Length of curves, Riemannian distance function. First variation formula. Gauss Lemma, minimal geodesics and Hopf-Rinow Theorem. [Lee, Chapters 4–6]

Curvature. Riemannian curvature tensor, symmetries, Ricci, scalar, and sectional curvature. [Lee, Chapter 7]

Major theorems. Gauss-Bonnet Theorem. Classification of constant curvature metrics. [Lee, Chapter 9] [Petersen, 5.6 and 6.1]

References

[GS] A. Grigis and J. Sjöstrand, *Microlocal Analysis for Differential Operators.* Cambridge University Press, 1994.

[Lee] J. M. Lee, *Riemannian Manifolds: An Introduction to Curvature.* Springer, 1997.

[Petersen] P. Petersen, *Riemannian Geometry.* Springer, 1998.

[Sjöstrand] J. Sjöstrand, *Lectures on resonances.* Lecture notes.

[Taylor] M. E. Taylor, *Partial Differential Equations II. Qualitative studies of linear equations.* Springer, 1996.

[TZ] S.-H. Tang and M. Zworski, *Potential Scattering on the Real Line.* Lecture notes.