Worksheet 2: vectors

1. Given the vectors

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

find the vectors $\vec{u} + 2\vec{v}$ and $3\vec{u} - \vec{v}$. Draw them on the plane using the parallelogram rule.

Answer:

$$\vec{u} + 2\vec{v} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}, \ 3\vec{u} - \vec{v} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

2. Let $\vec{u}$ and $\vec{v}$ be as in Exercise 1, and put

$$\vec{b} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Using the system of linear equations written in class, find x_1 and x_2 such that

$$x_1\vec{u} + x_2\vec{v} = \vec{b}.$$

Solution: x_1 and x_2 have to solve the system with the augmented matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix}.$$

Using row reduction, we find $x_1 = -1$ and $x_2 = 1$.

3. Let

$$\vec{a}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \ \vec{a}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Write the system equivalent to the vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b}$$

and solve it. Does $\vec{b}$ lie in Span$(\vec{a}_1, \vec{a}_2)$?

Solution: The augmented matrix of the system is

$$\begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & 0 \end{bmatrix}.$$

After row reduction, we get an REF

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 4 \end{bmatrix}.$$

Since the last column is pivot, the system is inconsistent. Therefore, $\vec{b}$ does not lie in Span$(\vec{a}_1, \vec{a}_2)$.

4. Give the geometric description of Span$(\vec{a}_1, \vec{a}_2)$ for $\vec{a}_1$ and $\vec{a}_2$ from Exercise 3.

Answer: The line passing through $\vec{a}_1$ and the origin.

5. Lay, 1.3.7.

Answer: $\vec{a} = \vec{u} - 2\vec{v}$, $\vec{b} = 2\vec{u} - 2\vec{v}$, $\vec{c} = 2\vec{u} - 3.5\vec{v}$, $\vec{d} = 3\vec{u} - 4\vec{v}$.

6. Prove the property (vi) in the gray box on Lay, p. 32, by analogy with what we did in class.

Solution: Let

$$\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix};$$

then

$$(c+d)\vec{u} = \begin{bmatrix} (c+d)u_1 \\ \vdots \\ (c+d)u_n \end{bmatrix} = \begin{bmatrix} cu_1 + du_1 \\ \vdots \\ cu_n + du_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix} + \begin{bmatrix} du_1 \\ \vdots \\ du_n \end{bmatrix} = c\vec{u} + d\vec{u}.$$

7. Lay, 1.3.24.

Solution: (a) True, see the beginning of the subsection 'Vectors in $\mathbb{R}^n$'.

(b) True, use Fig. 7 to draw the parallelogram determined by $\vec{u} - \vec{v}$ and $\vec{v}$.

(c) False; in fact, the linear combination with all coefficients zero will be the zero vector.

(d) True; see the statement that refers to Fig. 11.

(e) True; see the paragraph following the definition of Span.

8.* Let $\vec{u}_1, \vec{u}_2, \vec{u}_3$ be three vectors in $\mathbb{R}^2$. Prove that the intersection point of the medians of the triangle with vertices $\vec{u}_1, \vec{u}_2, \vec{u}_3$ is given by the linear combination

$$\frac{1}{3}(\vec{u}_1 + \vec{u}_2 + \vec{u}_3).$$

(Hint: start with the formula for the midpoint of a segment we studied in class.)

Solution: The midpoint of the interval with endpoints $\vec{u}_2$ and $\vec{u}_3$ is given by $\vec{v} = \frac{1}{2}(\vec{u}_2 + \vec{u}_3)$. Now, we use the fact that the intersection point of the medians lies two thirds of the way between $\vec{u}_1$ and $\vec{v}$; therefore, it is given by

$$\vec{u}_1 + \frac{2}{3}(\vec{v} - \vec{u}_1) = \frac{1}{3}(\vec{u}_1 + \vec{u}_2 + \vec{u}_3).$$

Alternatively, the calculation above shows that $\vec{w} = \frac{1}{3}(\vec{u}_1+\vec{u}_2+\vec{u}_3)$ lies on the segment connecting $\vec{u}_1$ and $\vec{v}$; that is, on one of the medians of our triangle. Since the expression for $\vec{w}$ stays the same if we permute $\vec{v}_1, \vec{v}_2, \vec{v}_3$, $\vec{w}$ lies on each median of our triangle. As a byproduct, we recover the geometric fact that the medians of a triangle intersect at one point.