Worksheet 19: Change of basis

Assume that V is some vector space and $\dim V = n < \infty$. Let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ and $\mathcal{C} = \{\vec{c}_1, \dots, \vec{c}_n\}$ be two bases of V. For any vector $\vec{v} \in V$, let $[\vec{v}]_\mathcal{B}$ and $[\vec{v}]_\mathcal{C}$ be its coordinate vectors with respect to the bases $\mathcal{B}$ and $\mathcal{C}$, respectively. These vectors are related by the formula

$$[\vec{v}]_\mathcal{C} = \mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}}[\vec{v}]_\mathcal{B}. \tag{1}$$

Here $\mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ is the **change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$**, given by

$$\mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} [\vec{b}_1]_\mathcal{C} & \dots & [\vec{b}_n]_\mathcal{C} \end{bmatrix}. \tag{2}$$

If $T : V \to V$ is a linear transformation, then recall that its matrix in the basis $\mathcal{B}$ is given by

$$[T]_\mathcal{B} = \begin{bmatrix} [T(\vec{b}_1)]_\mathcal{B} & \dots & [T(\vec{b}_n)]_\mathcal{B} \end{bmatrix}. \tag{3}$$

It is related to T by the formula

$$[T(\vec{v})]_\mathcal{B} = [T]_\mathcal{B}[\vec{v}]_\mathcal{B} \text{ for all } \vec{v} \in V. \tag{4}$$

The matrix of T in the basis $\mathcal{B}$ and its matrix in the basis $\mathcal{C}$ are related by the formula

$$[T]_\mathcal{C} = \mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}}[T]_\mathcal{B}\mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}}^{-1}. \tag{5}$$

We see that the matrices of T in two different bases are **similar**.

In particular, if $V = \mathbb{R}^n$, $\mathcal{C}$ is the canonical basis of $\mathbb{R}^n$ (given by the columns of the $n \times n$ identity matrix), T is the matrix transformation $\vec{v} \mapsto A\vec{v}$, and $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis of $\mathbb{R}^n$ composed of eigenvectors of A: $A\vec{v}_j = \lambda_j\vec{v}_j$, $j = 1, \dots, n$, $\lambda_j \in \mathbb{R}$, then the change of coordinates matrix $P = \mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ has the form

$$P = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}. \tag{6}$$

Then if D is the diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$, we know that

$$A = PDP^{-1}. \tag{7}$$

On the other hand, (5) gives

$$[T]_{\mathcal{C}} = P[T]_{\mathcal{B}}P^{-1}. \tag{8}$$

This is the same as (7), if we notice that $[T]_{\mathcal{C}} = A$ (since $\mathcal{C}$ is the canonical basis) and $[T]_{\mathcal{B}} = D$ (since $\mathcal{B}$ is composed of eigenvalues of A).

1. Lay, 4.7.9. (Use (2) or the method of Example 4.8.3.)
Solution: Solving two vector equations, we find

$$[\vec{b}_1]_{\mathcal{C}} = \begin{bmatrix} 9 \\ -4 \end{bmatrix}, \ [\vec{b}_2]_{\mathcal{C}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

So,

$$\mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}.$$

Next,

$$\mathcal{P}_{\mathcal{B}\leftarrow\mathcal{C}} = \mathcal{P}^{-1}_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}.$$

2. Let $\mathcal{B}$ and $\mathcal{C}$ be the bases of $\mathbb{R}^2$ from problem 1. If $[\vec{x}]_{\mathcal{B}} = (1, 1)$, use (1) to find $[\vec{x}]_{\mathcal{C}}$.
Solution: We have

$$[\vec{x}]_{\mathcal{B}} = \mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}}[\vec{x}]_{\mathcal{C}} = \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}.$$

3. In $\mathbb{P}_1$, find the change of coordinates matrix $\mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}}$, where $\mathcal{B} = \{(1 - t, 1 + t)\}$ and $\mathcal{C} = \{1, t\}$.
Solution: We have

$$[1 - t]_{\mathcal{C}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ [1 + t]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix};$$

therefore,

$$\mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

4. For $f = 1 + 2t$ and $\mathcal{B}, \mathcal{C}$ as in problem 3, find the coordinate vector $[f]_{\mathcal{C}}$. Then, solve the equation (1) to find $[f]_{\mathcal{B}}$.

Solution: We have $[f]_{\mathcal{C}} = (1, 2)$. Then, by (1)

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} [f]_{\mathcal{B}}.$$

Solving this equation, we find

$$[f]_{\mathcal{B}} = \begin{bmatrix} -1/2 \\ 3/2 \end{bmatrix}.$$

We can also verify that

$$1 + 2t = -\frac{1}{2}(1 - t) + \frac{3}{2}(1 + t).$$

5. For $\mathcal{B}, \mathcal{C}$ as in problem 3 and the linear transformation $T : \mathbb{P}_1 \to \mathbb{P}_1$ defined by $(T(f))(t) = (t+1)f'(t)$, use (3) to find its matrix $[T]_{\mathcal{C}}$. Then, use (5) to find $[T]_{\mathcal{B}}$.

Solution: We have

$$T(1) = 0, \; T(t) = 1 + t;$$

therefore,

$$[T(1)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; [T(t)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$[T]_{\mathcal{C}} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Now, (5) yields

$$[T]_{\mathcal{B}} = \mathcal{P}^{-1}_{\mathcal{C}\leftarrow\mathcal{B}}[T]_{\mathcal{C}}\mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}.$$

We can verify that

$$T(1-t) = -(1+t) = 0 \cdot (1-t) + (-1) \cdot (1+t),$$
$$T(1+t) = 1+t = 0 \cdot (1-t) + 1 \cdot (1+t).$$

6. Diagonalize the matrix A and find a basis of $\mathbb{R}^2$ in which the matrix of the transformation $\vec{x} \mapsto A\vec{x}$ is diagonal:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}.$$

Solution: The diagonalization is given by $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \ D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Therefore, the matrix of the transformation $\vec{x} \mapsto A\vec{x}$ in the basis $\{(1,0),(2,1)\}$ is equal to D (and thus is diagonal).

7. Lay, 5.4.19.

Solution: Since B is similar to A, there exists an invertible matrix P such that $B = P^{-1}AP$. Each of the matrices P^{-1}, A, P is invertible; therefore, B is invertible as their product and $B^{-1} = P^{-1}A^{-1}P$. This implies that B^{-1} is similar to A^{-1}.

8. Lay, 5.4.21.

Solution: Since B is similar to A, there exists an invertible matrix P such that $B = PAP^{-1}$. Since C is similar to A, there exists an invertible matrix Q such that $C = QAQ^{-1}$. Multiplying the latter by Q^{-1} to the left and by Q to the right, we get $A = Q^{-1}CQ$; substituting this into the equation for B, we get

$$B = P(Q^{-1}CQ)P^{-1} = (PQ^{-1})C(QP^{-1}) = RCR^{-1},$$

where $R = PQ^{-1}$ is invertible. Thus, B is similar to C.