Worksheet 14: Dimension and linear transformations

1. Lay, 4.5.13.

Answer: The dimension of $\operatorname{Col} A$ is 3, the dimension of $\operatorname{Nul} A$ is 2.

2. Lay, 4.5.19.

Answers: (a) True (b) False (does not need to pass through the origin) (c) False (the dimension is 5, as a basis is given by $\{1, t, t^2, t^3, t^4\}$) (d) True (e) True

3.* Lay, 4.5.27.

Solution: Assume that $\mathbb{P}$ is finite dimensional and $\dim \mathbb{P} = n < \infty$. Then the space $\mathbb{P}_n$ of all polynomials of degree $\leq n$ is $n+1$-dimensional, as it has basis $\{1, t, \dots, t^n\}$; however, $\mathbb{P}_n$ is a subspace of $\mathbb{P}$, which would imply by Theorem 4.5.11 that $\dim \mathbb{P}_n \leq \dim \mathbb{P}$, a contradiction with the fact that $\dim \mathbb{P} = n$.

4. Using the definition of a linear transformation, prove that the transformation $T : \mathbb{P} \to \mathbb{P}$ given by $T(f) = f'$ is linear. (Here $\mathbb{P}$ is the space of all polynomials and f' is the derivative of a polynomial f.)

Solution: We need to verify that for any $c, d \in \mathbb{R}$ and $f, g \in \mathbb{P}$, $T(cf + dg) = cT(f) + dT(g)$. Recalling the definition of T, this turns into

$$(cf + dg)' \equiv cf' + dg'.$$

This follows from the properties of differentiation.

5. Using the definition of a linear transformation, prove that the transformation $S : \mathbb{P} \to \mathbb{P}$ given by $S(f)(t) = tf(t)$ is linear.

Solution: We need to verify that for any $c, d \in \mathbb{R}$ and $f, g \in \mathbb{P}$, $S(cf + dg) = cS(f) + dS(g)$. Recalling the definition of S, this turns into

$$t(cf(t) + dg(t)) \equiv c(tf(t)) + d(tg(t)),$$

a true identity.

6. Find the matrix of the linear transformation $T : \mathbb{P}_1 \to \mathbb{P}_2$ given by $T(f)(t) = (t+1)f(t)$ in the bases $\{1, t\}$ of $\mathbb{P}_1$ and $\{1, t, t^2\}$ of $\mathbb{P}_2$. Is T 1-to-1? Is it onto?

Solution: We have

$$T(1) = 1 + t = 1 \cdot 1 + 1 \cdot t + 0 \cdot t^2,$$
$$T(t) = t + t^2 = 0 \cdot 1 + 1 \cdot t + 1 \cdot t^2.$$

Therefore, the matrix is

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

A has a pivot in each column, so T is 1-to-1; A does not have a pivot in each row, so T is not onto.

7. Find the matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{P}_2$ given by $T(x_1, x_2)(t) = x_1 t + x_2(1+t^2)$ in the bases $\{(1, 1),\ (1, -1)\}$ of $\mathbb{R}^2$ and $\{1, t, t^2\}$ of $\mathbb{P}_2$.

Solution: We have

$$T(1, 1) = 1 + t + t^2 = 1 \cdot 1 + 1 \cdot t + 1 \cdot t^2,$$
$$T(1, -1) = t - 1 - t^2 = (-1) \cdot 1 + 1 \cdot t + (-1) \cdot t^2.$$

Therefore, the matrix is

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

8. Prove that the transformation from problem 4 is onto, but not 1-to-1; find its kernel.

Solution: Onto: we need to prove that for each $g \in \mathbb{P}$, there exists $f \in \mathbb{P}$ such that $f' = g$. This is true since every polynomial has an antiderivative, which is also a polynomial.

Not 1-to-1: a polynomial f is in the kernel of T if and only if $f' = 0$. Therefore, the kernel of T consists of constant polynomials (or we can say that it is spanned by 1).

9. Prove that the transformation from problem 5 is 1-to-1, but not onto; find its range.

Solution: 1-to-1: We need to prove that for every $f \in \mathbb{P}$, if $tf(t) \equiv 0$, then $f \equiv 0$. This is true, as $tf(t) \equiv 0$ implies that $f(t) = 0$ for all $t \neq 0$; thus, f has infinitely many roots, which can only happen when $f \equiv 0$.

Not onto: a polynomial f is in the range of S if and only if $f = tg$ for some polynomial g. Therefore, the range of S consists of polynomials with zero constant term.

10. Given the transformations T from problem 4 and S from problem 5, find $T \circ S$ and $S \circ T$. (Recall that $T \circ S$ is the composition of T and S, defined by $(T \circ S)(f) = T(S(f))$.)

Solution: We find

$$(T \circ S)f(t) = (tf(t))' = f(t) + tf'(t),$$
$$(S \circ T)f(t) = tf'(t).$$

Note that $T \circ S \neq S \circ T$.

100.* (Lagrange interpolation) (Do not attempt this problem in section; however, you might want to come back to it later.) Let $\mathbb{P}_n$ be the space of polynomials of degree no more than n, and assume that $t_0, \dots, t_n \in \mathbb{R}$ are $n+1$ distinct points. (We fix these points from now on.)

(a) Consider the transformation $T : \mathbb{P}_n \to \mathbb{R}^{n+1}$ given by

$$T(f) = (f(t_0), \dots, f(t_n)).$$

Prove that it is linear.

(b) Prove that T is 1-to-1. (Hint: a nontrivial polynomial of degree no more than n can have at most n roots.)

(c) Use part (b) and IMT to prove that T is onto. (Hint: take some bases of $\mathbb{P}_n$ and $\mathbb{R}^{n+1}$ and consider the matrix of T in these bases; why is it square?)

(d) Reformulate part (c) as follows: **for every $s_0, \dots, s_n \in \mathbb{R}^n$, there exists a unique polynomial f of degree $\leq n$ such that $f(t_0) = s_0, \dots, f(t_n) = s_n$.**

(e) Try to think what the statement of (d) means for $n = 0, 1, 2$ in terms of the graph of f.