Worksheet 1: systems of linear equations

1–2. Write the augmented matrix for the following system. Then, solve the system using elementary operations. Finally, draw the solution set of each of two equations in the system and indicate the solution set of the system.

$$\begin{cases} x_1 + 2x_2 &= 0, \\ 2x_1 + x_2 &= -3; \end{cases} \tag{1}$$

$$\begin{cases} x_1 + 2x_2 &= 1, \\ 2x_1 + 4x_2 &= 0. \end{cases} \tag{2}$$

Solution to problem 1:

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -3 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & -3 \end{bmatrix}$$
$$\xrightarrow{R_2 \leftarrow -\frac{1}{3}R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}.$$

Therefore, the only solution is $x_1 = -2$, $x_2 = 1$. The picture should contain two lines intersecting at the point $(-2, 1)$.

Solution to problem 2:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

We see that the second equation is now $0 \cdot x_1 + 0 \cdot x_2 = -2$, which has no solutions. Therefore, the system is inconsistent. The picture should contain two parallel lines.

3. Row reduce the following augmented matrix first to REF, then to RREF. Then, solve the corresponding system of linear equations:

$$\begin{bmatrix} 1 & 2 & 0 & 7 \\ 1 & 3 & -1 & 10 \\ 0 & 1 & -1 & 3 \end{bmatrix}. \tag{3}$$

Solution:

$$\begin{bmatrix} 1 & 2 & 0 & 7 \\ 1 & 3 & -1 & 10 \\ 0 & 1 & -1 & 3 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 1 & 2 & 0 & 7 \\ 0 & 1 & -1 & 3 \\ 0 & 1 & -1 & 3 \end{bmatrix}$$
$$\xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{bmatrix} 1 & 2 & 0 & 7 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is an REF. There are two pivot positions (row 1, column 1 and row 2, column 2). Then, we proceed to the RREF:

$$\begin{bmatrix} 1 & 2 & 0 & 7 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is the RREF. Then, x_3 is the only free variable, and the values of x_1 and x_2 are given by

$$x_1 = 1 - 2x_3, \;\; x_2 = 3 + x_3.$$

4–5. Solve the system of linear equations with the given RREF of the augmented matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4}$$

$$\begin{bmatrix} 0 & 1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{5}$$

Solution to problem 4: The system is inconsistent because the last column is a pivot column.

Solution to problem 5: The system is consistent since the last column is not a pivot column. The pivot columns are 2, 4, and 5; therefore, the free variables are x_1 and x_3. The values of x_2, x_4, x_5 are given by

$$x_2 = 2 + x_3, \;\; x_4 = 1, \;\; x_5 = 0.$$

6–8. Given the REF, decide whether the corresponding system is consistent and if it is, whether the solution is unique. (Here $\blacksquare$ denotes a nonzero number, while $*$ denotes any number.)

$$\begin{bmatrix} \blacksquare & * \\ 0 & 0 \end{bmatrix}, \tag{6}$$

$$\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \end{bmatrix}, \tag{7}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{8}$$

Solutions: (6) Consistent, unique solution (7) Inconsistent (8) Consistent, solution not unique.

9. Write out all possible forms of a 3×3 RREF. For each of them, decide whether the corresponding system is consistent and whether the solution is unique. In each case, try to sketch the set of solutions of each equation on the plane and then explain the result. (You may replace all $*$ symbols by 0 for the last part.)

Solution: Here are the possibilities: (INC = inconsistent, CU = consistent, unique solution, CNU = consistent, solution not unique)

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ (CNU, each solution set is the whole plane)},$$

$$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ (CNU, one line)},$$

$$\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix} \text{ (CU, two lines intersecting at one point)},$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ (INC)}.$$