Sets

A *set* is a collection of some objects.

For example,

$\mathbb{R}$ — the set of all real numbers.

A couple more examples:

$\mathcal{C}$ ~~$\mathcal{C}$~~ — the set of all chairs

G — the set of all green chairs.

Every set $\mathcal{A}$ has a corresponding logical statement (in one variable): "~~Does~~ "x" lies in $\mathcal{A}$". In other words, ~~that~~ specifying a set is the same as specifying ~~the~~ which objects lie in this set. (we write $\underline{x \in \mathcal{A}}$ to say that the object x lies in the set $\mathcal{A}$)

Fore example:

'$x \in \mathbb{R}$' means 'x is a real number'

'$x \in \mathcal{C}$' means 'x is a chair'

'$x \in G$' means 'x is a chair and x is green'

Some operations on sets

· If $\mathcal{A}, \mathcal{B}$ are sets, then we say that $\mathcal{A} \subset \mathcal{B}$ if $\mathcal{A}$ is contained in $\mathcal{B}$; or, each element of $\mathcal{A}$ lies in $\mathcal{B}$. Formally: '$\forall x$: if $x \in \mathcal{A}$, then $x \in \mathcal{B}$'.

For example, $G \subset \mathcal{C}$. Indeed, if x is a green chair, then x is a chair.

Some sets can be specified by listing ~~a~~ all their elements. For example, the set $\{1, 2, 7\}$ contains three elements — the numbers $1, 2, 7$.

- We can always specify a set by the corresponding logical statement. For example,

$\mathcal{C} = \{x \mid x \text{ is a chair}\}$.

$\mathbb{R} = \{x \mid x \text{ is a real number}\}$

- If $\mathcal{A}$ is a set and $S(x)$ is a logical statement, then we denote by $\{x \in \mathcal{A} \mid S(x)\}$ the set of all x that lie in $\mathcal{A}$ and for which $S(x)$ holds. In other words, $x \in \{x \in \mathcal{A} \mid S(x)\}$ means $x \in \mathcal{A}$ and $S(x)$ is true.

For example, $G = \{x \in \mathcal{C} \mid x \text{ is green}\}$.

Here are some examples of sets and the corresponding logical statements:

Set $\mathcal{A}$	$\vec{y} \in \mathcal{A}$ if and only if ...
Solution set of the equation $Ax = b$	$A\vec{y} = b$
Set of all $\vec{b}$ for which the equation $Ax = \vec{b}$ has a solution $\parallel$ Span of the columns of A $\parallel$ $\{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\} = \text{Col } A$	$\exists \vec{x}: A\vec{x} = \vec{y}$
Nul A (by definition) $\parallel$ $\{\vec{x} \mid A\vec{x} = \vec{0}\}$	$A\vec{y} = \vec{0}$

Set of all roots of the equation $x^2+x-1=0$	$y^2+y-1=0$
Set of all invertible matrices	$\exists B: \quad Y \cdot B = B \cdot Y = I$
$\text{Span}\{\vec{v}_1, \dots, \vec{v}_n\}$ $\shortparallel$ $\{c_1\vec{v}_1 + \dots + c_n\vec{v}_n \mid c_1, \dots, c_n \in \mathbb{R}\}$	$\exists c_1, \dots, c_n \in \mathbb{R}:$ $\vec{y} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$
$\{(a,b,c) \mid a+b+c=0\}$	$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \ y_1+y_2+y_3=0$
$\left\{ \begin{bmatrix} a-b \\ b-c \\ c-a \end{bmatrix} \middle\vert\ a,b,c \in \mathbb{R} \right\}$	$\cancel{\vec{y}=}$ $\exists a,b,c \in \mathbb{R}: \vec{y} = \begin{bmatrix} a-b \\ b-c \\ c-a \end{bmatrix}$
$\{1, 3, 7\}$	$\vec{y}=1$ or $\vec{y}=3$ or $\vec{y}=7$

<u>Sample problem</u>: Consider the linear transformation

$T: \mathbb{P}_3 \to \mathbb{P}_3$ given by $T(f) = f''$, for all $f \in \mathbb{P}_3$.
second derivative of f

Define $\quad \text{Ker}\, T = \{ f \in \mathbb{P}_3 \mid T(f) = 0 \}$

$\text{Ran}\, T = \{ T(f) \mid f \in \mathbb{P}_3 \}$.

(a) Explain what it means for f to lie in $\text{Ker}\, T$ and what it means for f to lie in $\text{Ran}\, T$

(b) Describe $\text{Ker}\, T$ and $\text{Ran}\, T$

Solution on the next page

Solution (a)

$f \in \operatorname{Ker} T$ means '$f \in \mathbb{P}_3$ and $f''=0$'

$f \in \operatorname{Ran} T$ means 'There exists $g \in \mathbb{P}_3$ such that $g''=f$'.

(b) $f \in \operatorname{Ker} T \iff f \in \mathbb{P}_3$ and $f''=0 \iff$

$\iff f$ ~~is~~ has the form $c_1 + c_2 t$ for some $c_1, c_2 \in \mathbb{R}$

$\iff f \in \mathbb{P}_1$. So, $\operatorname{Ker} T = \mathbb{P}_1$.

$f \in \operatorname{Ran} T \iff f = g''$ for some $g \in \mathbb{P}_3$.

Take $g = a + bt + ct^2 + dt^3 \in \mathbb{P}_3$; then $g'' = 2c + 6dt$

We now prove that $\operatorname{Ran} T$ is also equal to $\mathbb{P}_1$:

(1) Assume that $f \in \operatorname{Ran} T$; we will prove that $f \in \mathbb{P}_1$. Indeed, if $f \in \operatorname{Ran} T$, then $\exists g \in \mathbb{P}_3$: $f = g''$. If $g = a + bt + ct^2 + dt^3$, then $f = g'' = 2c + 6d \cdot t$ is a polynomial of degree ≤ 1.

(2) Assume that $f \in \mathbb{P}_1$; we prove that $f \in \operatorname{Ran} T$. Write $f = a + bt$ and put $g = \frac{at^2}{2} + \frac{bt^3}{6}$; then $g \in \mathbb{P}_3$ and $g'' = f$. This proves that $f \in \operatorname{Ran} T$.