Logic

A logical statement is a statement that is true or false.

Example: 'This room has a window' — False

A logical statement can depend on free variables. In order to ~~specif~~ find whether it is true or false, you need to specify ~~(these variables)~~ the values of these variables.

Example: 'Room A has a window' — A is the free variable

Another example: 'The equation $A\vec{x}=\vec{b}$ has a solution' — the free variables are A and $\vec{b}$, but not $\vec{x}$!

One can get rid of free variables by applying quantifiers. There are two of them:

- $\forall$ = 'for each'. For example, 'Everyone thinks that Spain will win the World Cup' is the same as '$\forall$ person X, X thinks that Spain will win the World Cup'

To prove an $\forall X (\dots)$ statement, you need to verify $(\dots)$ for every X. For example, to prove the statement above, you need to ask every person on Earth (and on the orbit) what they think about Spain winning. If at least one does not think so, the whole statement if false

- $\exists$ = 'exists'. For example,
'Someone thinks that Spain will win' means
'$\exists$ ~~x~~ person X: X thinks that Spain will win'.

To prove an $\exists X(\ldots)$ statement, you need to provide an example of X for which it is true. Say, "Simon, Math 54 GSI, thinks that Spain will win".

- The negation of a $\forall(\ldots)$ statement is a $\exists(\ldots)$ statement. For example, the negative of
(1) 'Everyone thinks that Spain will win' is
(2) 'Someone thinks that Spain will lose'.
So, to disprove (1), you need to provide an example of the person who thinks that Spain will lose. This is called a counter example.

Side note, needed for midterm: set inclusion

If S and T are sets, we say that $S \subset T$, if every element of S lies in T. ~~For~~ So:

$$\forall \vec{x} \in S, \ \vec{x} \in T.$$

To prove a statement like that, you have to verify that for each $\vec{x}$ ~~lie~~, if $\vec{x} \in S$, then $\vec{x} \in T$.

Example: '{People} $\subset$ {Animals}'. To prove, pick each person and verify that they are an animal.

Back to 54-related statements

We say	We mean	Free variables
The vector $\vec{x}$ solves the equation $A\vec{x}=\vec{b}$	$A\vec{x}=\vec{b}$	$A, \vec{b}, \vec{x}$
The equation $A\vec{x}=\vec{b}$ has a solution	$\exists \vec{x}: A\vec{x}=\vec{b}$	$A, \vec{b}$
The equation $A\vec{x}=\vec{b}$ has a solution for each right hand side	$\forall \vec{b}\ \exists \vec{x}: A\vec{x}=\vec{b}$	A
The equation $A\vec{x}=\vec{0}$ has only the trivial solution $\vec{x}=\vec{0}$	$\forall \vec{x}$: if $A\vec{x}=\vec{0}$, then $\vec{x}=0$	A
W is a subspace of V (by definition)	$\vec{0}\in W$ AND $\forall \vec{u}\in W\ \forall \vec{v}\in W: \vec{u}+\vec{v}\in W$ AND $\forall \vec{u}\in W\ \forall c\in\mathbb{R}: c\cdot\vec{u}\in W$ AND $W\subset V$.	W, V
$\vec{x}\in \mathrm{Span}\{\vec{a}_1, \vec{a}_2\}$ (by definition)	$\exists c_1, c_2\in\mathbb{R}: \vec{x}=c_1\vec{a}_1+c_2\vec{a}_2$	$\vec{x}, \vec{a}_1, \vec{a}_2$
A is invertible (by definition)	$\exists B: A\cdot B=I$ and $BA=I$	

Example of $\forall / \exists$ reasoning

Given: matrix A, matrix B: $AB = I$.

Fact 1: the equation $A\vec{x} = \vec{b}$ has a solution for each $\vec{b}$ ($\forall \vec{b}\ \exists \vec{x}: A\vec{x} = \vec{b}$).

Proof: Take an arbitrary $\vec{b}$. Then

$$AB = I \Rightarrow (AB)\vec{b} = \vec{b} \Rightarrow A(B\vec{b}) = \vec{b}.$$

Thus, $\vec{x} = B\vec{b}$ provides an example for the statement $\exists \vec{x}: A\vec{x} = \vec{b}$. This statement is therefore true. Since we could pick $\vec{b}$ arbitrarily, the statement $\forall \vec{b}\, \exists \vec{x}: A\vec{x} = \vec{b}$ is true.

Fact 2: the equation $B\vec{x} = \vec{0}$ has only the zero solution. ($\forall \vec{x}$: if $B\vec{x} = \vec{0}$, then $\vec{x} = \vec{0}$).

Proof: Pick an arbitrary $\vec{x}$ and assume that $B\vec{x} = \vec{0}$. Since $AB = I$, $\vec{x} = (AB)\vec{x} = A(B\vec{x}) = A\vec{0} = \vec{0}$. We proved that "if $B\vec{x} = \vec{0}$, then $\vec{x} = \vec{0}$". Since we did this for arbitrary $\vec{x}$, we proved that

$$\forall \vec{x}: \text{ if } B\vec{x} = \vec{0}, \text{ then } \vec{x} = \vec{0}.$$