MATH 54-1, H/W 13

6.1 (4) Divide by $x(x+1)$:

$$y''' - \frac{3}{x+1} y' + \frac{1}{x(x+1)} y = 0.$$

The coefficients are defined & continuous everywhere except $x = 0, -1$. The initial condition is posed at $x = -\frac{1}{2}$; therefore, the solution is defined on the interval $(-1, 0)$.

(20) General solution: $y = x^2 + c_1 + c_2 x + c_3 x^3$.

$$y' = 2x + c_2 + 3c_3 x^2$$

Initial conditions: $y'' = 2 + 6c_3 x$

$$\begin{aligned} 2 &= y(1) = 1 + c_1 + c_2 + c_3 \\ -1 &= y'(1) = 2 + c_2 + 3c_3 \\ -4 &= y''(1) = 2 + 6c_3 \end{aligned} \longrightarrow \begin{aligned} c_1 &= 2 \\ c_2 &= 0 \\ c_3 &= -1 \end{aligned} \longrightarrow y = 2 - x^3 + x^2$$

(26) (a) The Wronskian is $\det \begin{bmatrix} y_1(x_0) & \cdots & y_n(x_0) \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix} =$

$= \det \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & & 1 \end{bmatrix} = 1 \neq 0$; therefore, our solutions are Lin. Ind.

(b) Take $y = \gamma_0 y_1 + \gamma_1 y_2 + \cdots + \gamma_{n-1} y_n$. Then by linearity, y solves the differential equation. It remains to verify that it satisfies the initial conditions: for every $j = 0, 1, \ldots, n-1$, we have

$$\begin{aligned} y^{(j)}(x_0) &= \gamma_0 \cdot y_1^{(j)}(x_0) + \cdots + \gamma_j \cdot y_{j+1}^{(j)}(x_0) + \cdots + \gamma_{n-1} y_n^{(j)}(x_0) = \\ &= \gamma_0 \cdot 0 + \cdots + \gamma_j \cdot 1 + \cdots + \gamma_{n-1} \cdot 0 = \\ &= \gamma_j, \text{ as required.} \end{aligned}$$

6.2 (18) The fundamental system is

$$\{\underbrace{e^x, xe^x, x^2e^x}_{(D-1)^3}, \underbrace{e^{2x}}_{(D-2)}, \underbrace{e^{-\frac{x}{2}}\cos\frac{\sqrt{3}x}{2}, e^{-\frac{x}{2}}\sin\frac{\sqrt{3}x}{2}}_{\substack{(D^2+D+1)\\ \text{roots } -\frac{1}{2}\pm i\frac{\sqrt{3}}{2}}}, \underbrace{e^{-3x}\cos x, xe^{-3x}\cos x, x^2e^{-3x}\cos x, e^{-3x}\sin x, xe^{-3x}\sin x, x^2e^{-3x}\sin x}_{\substack{(D^2+6D+10)^3\\ \text{roots } -3\pm i}}$$

General solution:

$$(A_1+A_2x+A_3x^2)e^x + Be^{2x} + e^{-\frac{x}{2}}\left(C\cos\frac{\sqrt{3}x}{2} + D\sin\frac{\sqrt{3}x}{2}\right) +$$

$$+ e^{-3x}\left[\cos x\,(E_1+E_2x+E_3x^2) + \sin x\,(F_1+F_2x+F_3x^2)\right].$$

9.1 (10)

$$\begin{bmatrix} y \\ y' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ \frac{n^2}{t^2}-1 & -\frac{1}{t} \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix}$$

9.4 (8)

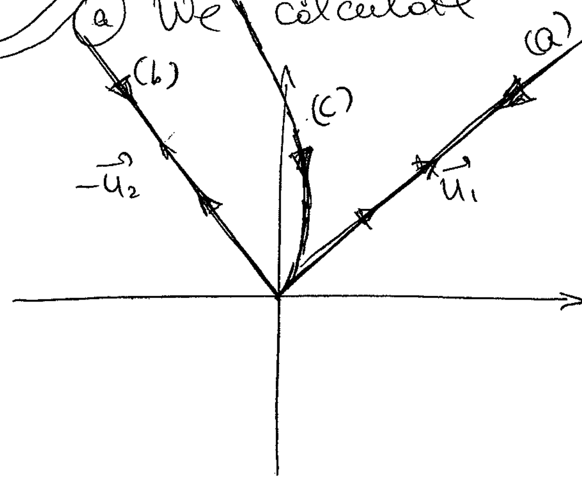
$$\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cos t \end{bmatrix}.$$

9.5 (18)

(a) We calculate $A\vec{u}_1 = \vec{u}_1$, $A\vec{u}_2 = -3\vec{u}_2$.

(b) $e^{-t}\vec{u}_1$ (c) $-e^{-3t}\vec{u}_2$

(d) $e^{-t}\vec{u}_1 - e^{-3t}\vec{u}_2$; the coordinates of this in the basis $\{\vec{u}_1,\vec{u}_2\}$ are (e^{-t}, e^{-3t}).
If $f(t) = e^{-t}$, $g(t) = e^{-3t}$ then $f, g > 0$ and $g(t) = f(t)^3$



(35) (a) Find $\det(A-\lambda I) = \lambda^2 - 2\lambda + 1 = (\lambda-1)^2 \to$
$\to$ eigenvalue -1, multiplicity 2

$\mathrm{Nul}(A-\lambda I) = \mathrm{Nul}\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \to$ the basis is $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$.

(b) Put $\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{x}_1(t) = e^{-t}\vec{u}_1$; then $\vec{x}_1'(t) = -e^{-t}\vec{u}_1 = e^{-t}(A\vec{u}_1) = A\vec{x}_1(t) \to \vec{x}_1' = A\vec{x}_1$.

(c) If $\vec{x}_2(t) = te^{-t}\vec{u}_1 + e^{-t}\vec{u}_2$ then
$\vec{x}_2'(t) = -te^{-t}\vec{u}_1 + e^{-t}(\vec{u}_1 - \vec{u}_2)$;
$A\vec{x}_2(t) = te^{-t}\cdot A\vec{u}_1 + e^{-t}\cdot A\vec{u}_2$.
For $\vec{x}_2' = Ax_2$, we need
$A\vec{u}_1 = -\vec{u}_1$, $A\vec{u}_2 = \vec{u}_1 - \vec{u}_2 \to (A+I)\vec{u}_1 = 0$, $(A+I)\vec{u}_2 = \vec{u}_1 \to$ we can take $\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$,

(d) $(A+I)^2\vec{u}_2 = 0$. In fact, $(A+I)^2 = 0$.