Math 54, final exam

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Name: ______________________________ SID: ____________________

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Problem 7: ________ / 11

Problem 8: ________ / 11

Total: ________ / 75

- Write your solutions in the space provided. Do not use your own paper. I can give you extra paper if needed. Indicate clearly where your answer is.
- Explain your solutions as clearly as possible. This will help me find what you did right and what you did wrong, and award partial credit if possible.
- Justify all your steps. (Problem 5 is exempt from this rule.) A correct answer with no justification will be given 0 points. Pictures without explanations are not counted as justification. You may cite a theorem from the book by stating what it says.
- No calculators or notes are allowed on the exam, except for a single two-sided A4/Letter sized sheet of hand-written notes. Cheating will result in academic and/or disciplinary action. Please turn off cellphones and other electronic devices.

1. Find the limit as $t \to +\infty$ of the formal solution to the following initial/boundary value problem for the heat equation:

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t), \; 0 < x < \pi, \; t > 0;$$
$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0, \; t > 0;$$
$$u(x,0) = \begin{cases} 1, & 0 < x < \pi/2, \\ 0, & \pi/2 \leq x < \pi. \end{cases}$$

(You can take the $t \to \infty$ limit inside series without providing justification.)

2. Compute the Fourier sine series for the function

$$f(x) = \pi - x, \ 0 < x < \pi.$$

Your answer should be in the form $f(x) \sim \sum_{\dots} (\dots) \sin((\dots)x)$. Describe the function to which this series converges, and sketch its graph.

3. Let V be the vector space of all solutions to the differential equation

$$y'' + 4y = 0.$$

(a) Find a basis for V. (You do not need to prove that what you found is actually a basis for V, as long as you find it correctly.)
(b) Let $T : V \to V$ be the linear transformation given by the formula $T(y) = y', \ y \in V$. Find the matrix of T in the basis of V from part (a).

4. Find all least squares solutions to the equation

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Find the least squares error.

5. Given the characteristic polynomial $P(\lambda)$ of some unknown square matrix A, decide:

(a) whether A is certainly diagonalizable, certainly not diagonalizable, or can be diagonalizable or not diagonalizable depending on the dimensions of eigenspaces;

(b) whether A is invertible or not invertible.

No justification is required. For each of the questions (a), you get 2 points if you answer it correctly, -1 point if you answer incorrectly, and 0 points if you decide not to answer. For each of the questions (b), you get 1 point if you answer it correctly, -1 point if you answer it incorrectly, and 0 points if you decide not to answer. If your total score for this problem is negative, it will be replaced by zero.

(1) $P(\lambda) = -\lambda(\lambda^2 + 2\lambda + 1)$

(a) Diagonalizable Not diagonalizable Can be either

(b) Invertible Not invertible

(2) $P(\lambda) = \lambda^2 + 4\lambda + 3$

(a) Diagonalizable Not diagonalizable Can be either

(b) Invertible Not invertible

6. Find the basic solutions to the following boundary value problem for the **damped wave equation** obtained using the method of separation of variables (in other words, solutions that have the form $X(x)T(t)$). You can use the information we have already gathered about the solutions of the corresponding boundary/eigenvalue problem on $X(x)$.

$$\frac{\partial^2 u}{\partial t^2}(x,t) + 2\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t),\ 0 < x < \pi,\ t > 0;$$
$$u(0,t) = u(\pi,t) = 0,\ t > 0.$$

7. Consider the system of linear equations

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

where a, b, c, d are given integers and $ad - bc = 1$. Prove that this system has a unique solution (x_1, x_2) and x_1, x_2 are both integers.

8. Using Cauchy–Schwarz inequality, prove that for every continuous function f on the interval $[0,1]$,

$$\left(\int_0^1 xf(x)\,dx\right)^2 \leq \frac{1}{3}\int_0^1 f(x)^2\,dx$$

Make sure to specify what functional space and what inner product you are using. (You do not have to verify that the properties of vector space and inner product hold.)