Math 54, Section 214 Your name: Key
Quiz 8, April 9, 2010

Please write your name on each sheet. Show your work clearly and in order, including intermediate steps in the solutions and the final answer.

1. (10 pt) Let P_2 be the space of polynomials of degree no more than 2 with the inner product given by

$$\langle p, q\rangle = p(0)q(0) + p(1)q(1) + p(2)q(2).$$

Find a unit vector in P_2 that is orthogonal to $1+t$ and $1-t$.

We have: $\text{span}\{1+t, 1-t\} = \text{Span}\{1, t\}$, so it is enough to find a vector orthogonal to 1 and t.

A general polynomial in P_2 has the form $p = a + bt + ct^2$. Then, we need

$\cancel{0 = \langle p, 1\rangle = (b+c) + a + (b}$

$0 = \langle p, 1\rangle = a + (a+b+c) + (a+2b+4c) = 3a+3b+5c$

$0 = \langle p, t\rangle = a+b+c + 2(a+2b+4c) = 3a+5b+9c.$

So, $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \text{Nul}\begin{bmatrix} 3 & 3 & 5 \\ 3 & 5 & 9 \end{bmatrix} = \text{Nul}\begin{bmatrix} 3 & 3 & 5 \\ 0 & 2 & 4 \end{bmatrix} =$

$= \text{Nul}\begin{bmatrix} 3 & 3 & 5 \\ 0 & 1 & 2 \end{bmatrix}$; we can put $c = -1$, $b = 2$, $a = -\frac{1}{3}$.

Or, $a = 1$, $b = -6$, $c = 3$ $\rightarrow$ $p = 1 \cancel{t} - 6t + 3t^2$.

Now, $\langle p, p\rangle = 1 + 4 + 1 = 6$ $\rightarrow$ the answer is

$$\boxed{\frac{1}{\sqrt{6}}(1 - 6t + 3t^2).}$$

2. (10 pt) Consider the matrix

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}.$$

(a) Prove that

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

are eigenvectors of A, and find the corresponding eigenvalues.

(b) Find a diagonal matrix D and an orthogonal matrix P such that $A = PDP^{-1}$.

(a) We have $A\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \cdot \vec{v}_1 \rightarrow \lambda = 0$

$A\vec{v}_2 = \begin{bmatrix} 3 \\ 3 \\ -6 \end{bmatrix} = -3\vec{v}_2 \Longrightarrow \lambda = -3$

(b) Since A is symmetric, its third eigenvector $\vec{v}_3$ has to be orthogonal to $\vec{v}_1$ and $\vec{v}_2$. So,

$\vec{v}_3 \in \text{Col}\begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}^{\perp} = \text{Nul}\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 2 \end{bmatrix} = \text{Nul}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$;

we can take $\vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $A\vec{v}_3 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} = 3\vec{v}_3 \Rightarrow \lambda = 3.$

Normalizing, we get $A = PDP^{-1}$, where $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

and $P = \begin{bmatrix} \frac{\vec{v}_1}{|\vec{v}_1|} & \frac{\vec{v}_2}{|\vec{v}_2|} & \frac{\vec{v}_3}{|\vec{v}_3|} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 2/\sqrt{6} & 0 \end{bmatrix}$