MATH 279 HOMEWORK 8

1. Let $P := p^{\mathrm{w}}(x, hD_x)$ where $p = p_0 + \mathcal{O}(h)$, $p \in C_{\mathrm{c}}^\infty(\mathbb{R}^{2n})$ is bounded uniformly in h and supported in an h-independent compact set, and $p_0 \in C_{\mathrm{c}}^\infty(\mathbb{R}^{2n})$ is real-valued and h-independent. Let $U(t) = e^{-itP/h} : L^2 \to L^2$ which is well-defined since P is a bounded operator on L^2 and the function $e^{-it\lambda/h}$ is entire.

(a) Using that $P - P^* = \mathcal{O}(h)_{L^2\to L^2}$ show the bound

$$\|U(t)\|_{L^2\to L^2} \leq e^{C|t|}.$$

(Hint: differentiate $\|U(t)u\|_{L^2}^2$ in t.)

(b) Explain why the proof of Egorov's Theorem still applies, giving that for each $a \in C_{\mathrm{c}}^\infty(\mathbb{R}^{2n})$ there exists $a_t \in C_{\mathrm{c}}^\infty(\mathbb{R}^{2n})$ such that

$$U(-t)a^{\mathrm{w}}(x, hD_x)U(t) = a_t^{\mathrm{w}}(x, hD_x) + \mathcal{O}(h^\infty)_{L^2\to L^2},$$
$$a_t = a \circ e^{tH_{p_0}} + \mathcal{O}(h), \quad \operatorname{supp} a_t \subset e^{-tH_{p_0}}(\operatorname{supp} a).$$

In particular, where does the subprincipal part $p - p_0$ come up in the construction of a_t?

2. Let $P_0 := p_0^{\mathrm{w}}(x, hD_x)$, $P := p^{\mathrm{w}}(x, hD_x)$ where $p = p_0 - ihq$, $p_0, q \in C_{\mathrm{c}}^\infty(\mathbb{R}^{2n})$ are h-independent and p_0 is real-valued. Define $U(t) := e^{-itP/h}$, $U_0(t) := e^{-itP_0/h}$.

(a) Show that

$$U_0(-t)U(t) = b_t^{\mathrm{w}}(x, hD_x) + \mathcal{O}(h)_{L^2\to L^2}$$

where $b_t \in C^\infty(\mathbb{R}^{2n})$, $b_t - 1 \in C_{\mathrm{c}}^\infty$, is the attenuation coefficient defined by

$$b_t(x, \xi) := \exp\Big(-\int_0^t q(e^{sH_{p_0}}(x, \xi))\, ds\Big).$$

(This has applications to the study of damped waves, with $U(t)$ being the damped propagator.)

(b) Use part (a) to show that for each $a \in C_{\mathrm{c}}^\infty(\mathbb{R}^{2n})$ we have

$$e^{a^{\mathrm{w}}(x, hD_x)} = (e^a)^{\mathrm{w}}(x, hD_x) + \mathcal{O}(h)_{L^2\to L^2}.$$