MATH 279 HOMEWORK 2

1. Consider the Gamma function

$$\Gamma(s+1) = \int_0^\infty x^s e^{-x}\,dx, \quad s \geq 0.$$

(Recall that $\Gamma(n+1) = n!$ for $n \in \mathbb{N}_0$.) Using a version of the method of stationary phase, show Stirling's expansion:

$$\Gamma(s+1) \sim \left(\frac{s}{e}\right)^s \sqrt{2\pi s}(1 + c_1 s^{-1} + c_2 s^{-2} + \cdots) \quad \text{as} \quad s \to \infty$$

where $c_1, c_2, \dots$ are some real coefficients. (Hint: make the change of variables $x = sy$. Analyze the resulting integral using the method of stationary phase with $h := \frac{1}{s}$. Here we have an integral of the form $\int e^{\varphi(y)/h} a(y)\,dy$ rather than $\int e^{i\varphi(y)/h} a(y)\,dy$ so one needs to revisit the proof of stationary phase.)

2. This exercise is an 'extension' of Exercise 4 in the previous homework to pseudo-differential operators with compactly supported symbols. Define

$$u(x;h) := e^{i\Phi(x)/h} b(x)\,dx \tag{0.1}$$

where $\Phi \in C^\infty(\mathbb{R}^n;\mathbb{R})$ and $b \in C_c^\infty(\mathbb{R}^n;\mathbb{C})$.

Take $a \in C_c^\infty(\mathbb{R}^{2n};\mathbb{C})$ and consider the h-dependent family of operators $\mathrm{Op}_h(a) : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ defined by

$$\mathrm{Op}_h(a)u(x) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}\langle x-y,\xi\rangle} a(x,\xi) u(y)\,dy d\xi.$$

Using the method of stationary phase, show that for u given by (0.1) we have

$$\mathrm{Op}_h(a)u(x;h) = e^{i\Phi(x)/h} c(x;h)$$

where c has the asymptotic expansion as $h \to 0$

$$c(x;h) \sim \sum_{k=0}^\infty h^k c_k(x)$$

and $c_0(x) = a(x, \nabla\Phi(x))b(x)$. (Very determined students are welcome to compute c_1 as well.)