Math 1B worksheet

Sep 28–30, 2009

0. Are the following statements true or false?

(a) If the sequence a_n converges to zero, then the series $\sum_{n=1}^{\infty} a_n$ converges.

(b) If the sequence a_n converges to 1, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

(c) If the sequence a_n converges to 7, then the series $\sum_{n=1}^{\infty} a_{n+1} - a_n$ diverges.

(d) If the sequence a_n converges to 7, then the series $\sum_{n=1}^{\infty} a_{n+1} - a_n$ converges and its sum is 7.

1–6. Determine if the following series converge or diverge. For the convergent ones, compute their sum.

$$\sum_{n=1}^{\infty} \frac{e^n - e^{2n}}{e^{3n}}, \tag{1}$$

$$\sum_{n=1}^{\infty} \ln n, \tag{2}$$

$$\sum_{n=1}^{\infty} (-1)^n n, \tag{3}$$

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right), \tag{4}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 3n + 2}, \tag{5}$$

$$\sum_{n=3}^{\infty} \arctan(n+2) - \arctan n. \tag{6}$$

7–8. Determine whether the following series converge or diverge using the integral test. Do not forget to verify the conditions of the theorem! Do not compute the sum.

$$\sum_{n=3}^{\infty} \frac{n^2}{e^n}, \tag{7}$$

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}. \tag{8}$$

9. Consider the series $\sum_{n=1}^{\infty} a_n$, where $a_n = \frac{1}{n^2+n}$.

(a) Prove that the series converge and calculate its sum s.

(b) Let s_n be the sum of the first n terms of the series and put $r_n = s - s_n$. Estimate r_n from above and from below using remainder estimate for the integral test.

(c) Find a formula for r_n and verify explicitly that the estimates obtained in (b) hold.

10. Show that the series $\sum_{n=1}^{\infty} \sin^2(\pi n)$ converges, but the integral $\int_1^{\infty} \sin^2(\pi x)\, dx$ diverges. Does this contradict the integral test?

100* Find a formula for the sequence

$$s_m = \sum_{n=1}^{m} \sin n.$$

Does the series $\sum_{n=1}^{\infty} \sin n$ converge?

Hints and answers

0. (a) False; consider $a_n = \frac{1}{n}$ for a counterexample

(b) True by Divergence Test

(c,d) Both are false. We have $S_m = \sum_{n=1}^{m}(a_{n+1} - a_n) = a_{m+1} - a_1$ (telescoping series); then $\lim_{m\to\infty} S_m = 7 - a_1$.

1. We have

$$\sum_{n=1}^{\infty} \frac{e^n - e^{2n}}{e^{3n}} = \left(\sum_{n=1}^{\infty} e^{-2n} - \sum_{n=1}^{\infty} e^{-n} \right).$$

Both series in RHS are geometric series, but they start from $n = 1$ instead of $n = 0$. They both converge because $|p| < 1$; to compute the sum of the first series, we subtract and add the 0^{th} term:

$$\sum_{n=1}^{\infty} e^{-2n} = \sum_{n=1}^{\infty} \left(\frac{1}{e^2}\right)^n$$

$$= -1 + \sum_{n=0}^{\infty} \left(\frac{1}{e^2}\right)^n = -1 + \frac{1}{1 - e^{-2}} = \frac{e^{-2}}{1 - e^{-2}}.$$

The second series is calculated similarly.

Answer: $\frac{e^{-2}}{1-e^{-2}} - \frac{e^{-1}}{1-e^{-1}}$.

2. Apply the Divergence Test: $\lim_{n\to\infty} \ln n = +\infty$.

Answer: Diverges.

3. Apply the Divergence Test: $\lim_{n\to\infty}(-1)^n n$ does not exist.

Answer: Diverges.

4. Use that $\ln\left(1+\frac{1}{n}\right) = \ln(n+1) - \ln n$. Then $\sum_{n=1}^{m} \ln\left(1+\frac{1}{n}\right) = \ln(m+1)$.

Answer: Diverges.

5. Use that $\frac{1}{n^2+3n+2} = \frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$.

Answer: $\frac{1}{2}$.

6. We may compute

$$\sum_{n=3}^{m} \arctan(n+2) - \arctan n$$

$$= -\arctan 3 - \arctan 4 + \arctan(m+1) + \arctan(m+2)$$

and then take the limit as $m \to \infty$.

Answer: $-\arctan 3 - \arctan 4 + \pi$.

7. Use $f(x) = x^2 e^{-x}$; it is decreasing because $f'(x) \leqslant 0$ for $x \geqslant 3$ and $f(x)$ is nonnegative. The integral $\int_3^{\infty} f(x)\,dx$ converges, say, by explicit antiderivative computation.

Answer: Converges.

8. Use $f(x) = \frac{1}{x^2-1}$; it is nonnegative, decreasing because $f'(x) \leqslant 0$ for $x \geqslant 2$ and has an antiderivative $F(x) = \frac{1}{2}\log\left(\frac{x-1}{x+1}\right)$. This antiderivative has a finite limit (zero) as $x \to \infty$; therefore, the integral converges;

Answer: Converges.

9. Using that $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, we get

$$s_n = 1 - \frac{1}{n+1}, \; s = \lim_{n\to\infty} s_n = 1, \; r_n = \frac{1}{n+1}.$$

Now, put $f(x) = \frac{1}{x^2+x}$; it is nonnegative and decreasing (since $f'(x) \leqslant 0$ for $x \geqslant 1$) and has an antiderivative $F(x) = \ln\left(\frac{x}{x+1}\right)$. We then have the remainder esimates

$$\int_{n+1}^{\infty} f(x)\,dx \leqslant r_n \leqslant \int_{n}^{\infty} f(x)\,dx,$$

which turn into

$$\ln\left(\frac{n+2}{n+1}\right) \leqslant \frac{1}{n+1} \leqslant \ln\left(\frac{n+1}{n}\right).$$

The first of these two inequalities, when exponentiated, becomes

$$e^{\frac{1}{n+1}} \geqslant 1 + \frac{1}{n+1},$$

which is a special case of the inequality

$$e^x \geqslant 1 + x$$

true for all real x. The second inequality can be rewritten is

$$e^{-\frac{1}{n+1}} \geqslant 1 - \frac{1}{n+1},$$

which is again a special case of $e^x \geqslant 1 + x$.

10. The series consists of all zeroes, so it converges to zero. The integral diverges because the limit of the antiderivative as $x \to +\infty$ is infinite. (To prove that, use Squeeze Theorem.) However, this does not contradict the integral test because the function $\sin^2(\pi x)$ is not decreasing.

100. Multiply by $\sin\frac{1}{2}$ and use that $\sin n \sin\frac{1}{2} = \frac{1}{2}\left(\cos\left(n-\frac{1}{2}\right) - \cos\left(n+\frac{1}{2}\right)\right)$.

Answer: $s_m = \frac{1}{2\sin(1/2)}(\cos(1/2) - \cos(m + 1/2))$; diverges.