Math 1B quiz 4

Sep 23, 2009

Section 105

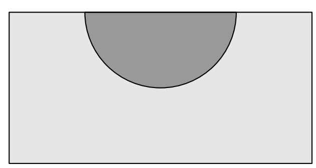
1. (3 pt) Find the length of the curve given by the equation $y = \frac{1}{3}\sqrt{x}(x-3)$, $1 \leqslant x \leqslant 9$.

2. (3 pt) Set up the integrals representing the following values, but do not compute them:

   (a) (1 pt) area of the surface obtained by rotating the curve $x^2 = y^3$, $0 \leqslant x \leqslant 8$, around the x axis;

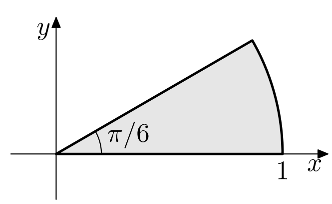
   (b) (1 pt) area of the surface obtained by rotating the same curve as in (a), but around the y axis;

   (c) (1 pt) hydrostatic force against one side of a vertical plate in shape of a half-disc of radius 1 submerged in a liquid of density ρ so that it touches the surface of the liquid:

   The functions you integrate should be written as explicit expressions depending only on the variable of integration (not on other variables, or, say, on $f(x)$)!

3. (4 pt) Compute the x coordinate of the centroid of a sector of disc of radius 1 and angle $\pi/6$:

Section 106

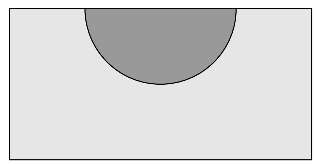
1. (3 pt) Find the length of the curve given by the equation $y = \frac{1}{3}\sqrt{x}(x-3)$, $1 \leqslant x \leqslant 9$.

2. (3 pt) Set up the integrals representing the following values, but do not compute them:

   (a) (1 pt) area of the surface obtained by rotating the curve $x^3 = y^2$, $0 \leqslant x \leqslant 4$, $y > 0$, around the x axis;

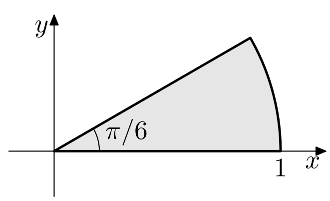
   (b) (1 pt) area of the surface obtained by rotating the same curve as in (a), but around the y axis;

   (c) (1 pt) hydrostatic force against one side of a vertical plate in shape of a half-disc of radius 1 submerged in a liquid of density ρ so that it touches the surface of the liquid:

   The functions you integrate should be written as explicit expressions depending only on the variable of integration (not on other variables, or, say, on $f(x)$)!

3. (4 pt) Compute the y coordinate of the centroid of a sector of disc of radius 1 and angle $\pi/6$:

Solutions for section 105

1. We have $f(x) = \frac{1}{3}\sqrt{x}(x-3) = \frac{1}{3}x^{3/2} - x^{1/2}$; therefore, $f'(x) = \frac{1}{2}(x^{1/2} - x^{-1/2})$ and

$$\begin{aligned} 1 + (f'(x))^2 &= \frac{4 + (x^{1/2} - x^{-1/2})^2}{4} \\ &= \frac{x + x^{-1} + 2}{4} = \frac{(x^{1/2} + x^{-1/2})^2}{4}. \end{aligned}$$

Therefore, the length of the curve is

$$\begin{aligned} \int_1^9 \sqrt{1 + (f'(x))^2}\,dx &= \frac{1}{2}\int_1^9 x^{1/2} + x^{-1/2}\,dx \\ &= \left(\frac{x^{3/2}}{3} + x^{1/2}\right)\Big|_{x=1}^9 = \frac{32}{3}. \end{aligned}$$

2. (a) We have $y = x^{2/3} = f(x)$, $0 \leqslant x \leqslant 8$, and $f'(x) = \frac{2}{3}x^{-1/3}$, so the answer is

$$2\pi\int_0^8 f(x)\sqrt{1 + (f'(x))^2}\,dx = 2\pi\int_0^8 x^{2/3}\sqrt{1 + \frac{4}{9}x^{-2/3}}\,dx.$$

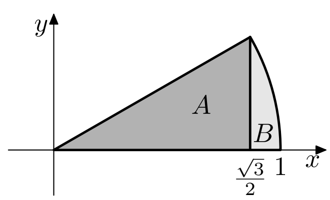
(b) We have $x = y^{3/2} = g(y)$, $0 \leqslant y \leqslant 4$, and $g'(y) = \frac{3}{2}y^{1/2}$, so the answer is

$$2\pi\int_0^4 g(y)\sqrt{1 + (g'(y))^2}\,dy = 2\pi\int_0^4 y^{3/2}\sqrt{1 + \frac{9}{4}y}\,dy.$$

(c) At depth y, $0 \leqslant y \leqslant 1$, the pressure is $\rho g y$, the element of the area of our plate is $2\sqrt{1 - y^2}\,dy$, so the answer is

$$2\rho g\int_0^1 y\sqrt{1 - y^2}\,dy.$$

3. Cut our shape into two regions, A and B:



The region A is the shape under the graph of $y = f(x) = x/\sqrt{3}$ for $0 \leqslant x \leqslant \sqrt{3}/2$, so the moments of A are

$$\begin{aligned} M_y(A) &= \int_0^{\sqrt{3}/2} xf(x)\,dx = \frac{1}{\sqrt{3}}\int_0^{\sqrt{3}/2} x^2\,dx = \frac{1}{8}, \\ M_x(A) &= \frac{1}{2}\int_0^{\sqrt{3}/2} f(x)^2\,dx = \frac{1}{6}\int_0^{\sqrt{3}/2} x^2\,dx = \frac{1}{16\sqrt{3}}. \end{aligned}$$

(Note that M_y corresponds to x-coordinate of the centroid, while M_x corresponds to its y-coordinate.)

The region B is the shape under the graph of $y = g(x) = \sqrt{1-x^2}$ for $\frac{\sqrt{3}}{2} \leqslant x \leqslant 1$, so the moments of B are

$$M_y(B) = \int_{\sqrt{3}/2}^{1} xf(x)\,dx = \int_{\sqrt{3}/2}^{1} x\sqrt{1-x^2}\,dx = \frac{1}{2}\int_0^{1/4} \sqrt{z}\,dz = \frac{1}{24},$$
$$M_x(B) = \frac{1}{2}\int_{\sqrt{3}/2}^{1} f(x)^2\,dx = \frac{1}{2}\int_{\sqrt{3}/2}^{1} 1-x^2\,dx = \frac{1}{3} - \frac{3\sqrt{3}}{16}.$$

(We used the substitution $z = 1 - x^2$ in the first integral above.) The area of the whole shape is $S = \frac{\pi}{12}$, so the coordinates of the centroid are

$$x_c = \frac{M_y(A) + M_y(B)}{S} = \frac{2}{\pi},$$
$$y_c = \frac{M_x(A) + M_x(B)}{S} = \frac{1}{\pi}(4 - 2\sqrt{3}).$$

Solutions for section 106

1. See the solution for problem 1 of section 105.

2. (a) We have $y = x^{3/2} = f(x)$, $0 \leqslant x \leqslant 4$, and $f'(x) = \frac{3}{2}x^{1/2}$, so the answer is

$$2\pi\int_0^4 f(x)\sqrt{1+(g'(x))^2}\,dx = 2\pi\int_0^4 x^{3/2}\sqrt{1+\frac{9}{4}x}\,dx.$$

(b) We have $x = y^{2/3} = g(y)$, $0 \leqslant y \leqslant 8$, and $g'(y) = \frac{2}{3}y^{-1/3}$, so the answer is

$$2\pi\int_0^8 f(y)\sqrt{1+(g'(y))^2}\,dy = 2\pi\int_0^8 y^{2/3}\sqrt{1+\frac{4}{9}y^{-2/3}}\,dy.$$

(c) See the solution for problem 2 (c) of section 105.

3. See the solution for problem 3 of section 105.