Math 1B quiz 3

Sep 16, 2009

Section 105

1. (3 pt) Compute the integral $\int e^{(x+e^x)}\,dx$.

2. (3 pt) Calculate the integral $\int_{-2}^{2} x^2\,dx$ approximately using Midpoint Rule with $n = 4$.

3. (4 pt) Let $f(x) = \dfrac{1}{(x+1)(x^2+1)}$. Does the integral $\int_0^\infty f(x)\,dx$ converge? What about the integral $\int_{-\infty}^{0} f(x)\,dx$? Justify your answers. You do not have to evaluate the integral. (Warning: Comparison Theorem does not work for negative functions. However, $\int_a^b f(x)\,dx$ converges if and only if $\int_a^b -f(x)\,dx$ converges.)

Section 106

1. (3 pt) Compute the integral $\int e^{(e^{-x}-x)}\,dx$.

2. (3 pt) Calculate the integral $\int_{-2}^{2} x^2\,dx$ approximately using Trapezoid Rule with $n = 4$.

3. (4 pt) Let $f(x) = \dfrac{1}{(x+1)(x^2+1)}$. Does the integral $\int_0^\infty f(x)\,dx$ converge? What about the integral $\int_{-\infty}^{0} f(x)\,dx$? Justify your answers. You do not have to evaluate the integral. (Warning: Comparison Theorem does not work for negative functions. However, $\int_a^b f(x)\,dx$ converges if and only if $\int_a^b -f(x)\,dx$ converges.)

Solutions for section 105

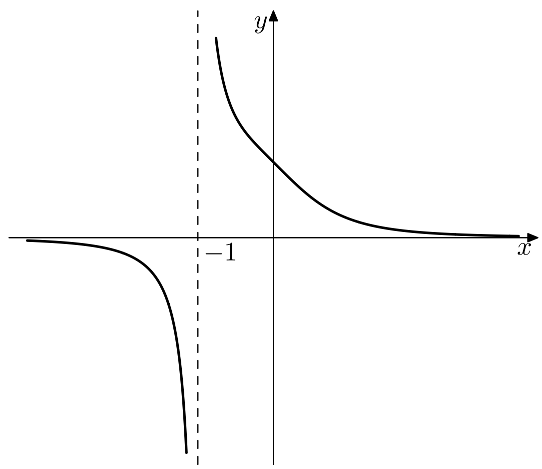
1. Make the substitution $u = e^x$: then $du = e^x\,dx$, and

$$\int e^{x+e^x}\,dx = \int e^x e^{e^x}\,dx = \int e^u\,du$$
$$= e^u + C = e^{e^x} + C.$$

2. The answer is

$$\frac{2-(-2)}{4}\left[\left(-\frac{3}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2\right] = 5.$$

3. (Solution 1) The function $f(x)$ is defined everywhere except at $x = -1$.



For the integral $\int_0^\infty f(x)\,dx$, note that for $x > 0$, $f(x) \leqslant \frac{1}{x^3}$. Now, $\int \frac{dx}{x^3} = -\frac{1}{2x^2} + C$, which has limit 0 as $x \to +\infty$. So the integral $\int_1^\infty \frac{dx}{x^3}$ converges. By Comparison Theorem, the integral $\int_1^\infty f(x)\,dx$ converges. Since $f(x)$ is well-defined and continuous on $[0,1]$, the integral $\int_0^1 f(x)\,dx$ converges. Therefore, $\int_0^\infty f(x)\,dx$ converges. (Note that the integral $\int_0^\infty \frac{dx}{x^3}$ does not converge.) Alternatively, use that $f(x) \leqslant \frac{1}{x^2+1}$ for $x > 0$.

For the integral $\int_{-\infty}^0 f(x)\,dx$, first analyse the integral $\int_{-1}^0 f(x)\,dx$. We have $x^2+1 \leqslant 2$ for $-1 \leqslant x \leqslant 0$, so $f(x) \geqslant \frac{1}{2(x+1)}$. However, $\int \frac{dx}{2(x+1)} = \frac{1}{2}\ln|x+1| + C$, which has limit $-\infty$ as $x \to -1$; therefore, $\int_{-1}^0 \frac{dx}{2(x+1)}$ diverges. By Comparison Theorem, the integral $\int_{-1}^0 f(x)\,dx$ diverges. Therefore, the integral $\int_{-\infty}^0 f(x)\,dx$ diverges.

3. (Solution 2) Find the antiderivative $\int f(x)\,dx = F(x) + C$ by partial fractions:

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1},$$
$$1 = A(x^2+1) + (Bx+C)(x+1) = (A+B)x^2 + (B+C)x + (A+C).$$

Therefore,

$$0 = A + B,$$
$$0 = B + C,$$
$$1 = A + C.$$

We find $A = \frac{1}{2}$, $B = -\frac{1}{2}$, $C = \frac{1}{2}$, so

$$f(x) = \frac{1}{2(x+1)} + \frac{1-x}{2(x^2+1)},$$
$$F(x) = \frac{1}{4}(2\ln|x+1| + 2\arctan x - \ln|x^2+1|).$$

To analyze $\int_0^\infty f(x)\,dx$, it is enough to find

$$\lim_{x\to+\infty} F(x) = \frac{1}{4}\lim_{x\to+\infty} 2\arctan x + \ln\left|\frac{(x+1)^2}{x^2+1}\right|$$
$$= \frac{\pi}{4} + \frac{1}{4}\lim_{x\to+\infty}\ln\left|\frac{(1+(1/x))^2}{1+1/x^2}\right| = \frac{\pi}{4} + \frac{1}{4}\ln 1 = \frac{\pi}{4}$$

Since this is finite, the integral $\int_0^\infty f(x)\,dx$ converges.

For $\int_{-\infty}^0 f(x)\,dx$, we have to find the limits of $F(x)$ at $x = -1$ and $x = -\infty$ to make sure that the integral converges. However,

$$\lim_{x\to-1} F(x) = \lim_{x\to-1} \frac{1}{2}\arctan x - \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\ln|x+1|$$
$$= \frac{1}{2}\arctan(-1) - \frac{1}{4}\ln 2 + \frac{1}{2}\lim_{x\to-1}\ln|x+1| = -\infty.$$

Therefore, the integral $\int_{-\infty}^0 f(x)\,dx$ diverges.

Solutions for section 106

1. Make the substitution $u = e^{-x}$: then $du = -e^{-x}\,dx$, and

$$\int e^{(e^{-x}-x)}\,dx = \int e^{e^{-x}}e^{-x}\,dx = -\int e^u\,du$$
$$= -e^u + C = -e^{e^{-x}} + C.$$

2. The answer is

$$\frac{2-(-2)}{2\cdot 4}[1\cdot(-2)^2+2\cdot(-1)^2+2\cdot 0^2+2\cdot 1^2+1\cdot 2^2]=6.$$

3. See solution to problem 3 for section 105.