Math 1B worksheet

Oct 21, 2009

1–4. Find the Taylor series for the following functions centered at the given point a. (Assume that f has a power series expansion.) Do not use the formulas on page 743 for problems 1–3. For problem 4, use the binomial series.

$$f(x) = (x+1)^2, \;\; a = 1, \tag{1}$$
$$f(x) = \sin(\pi x), \;\; a = 0, \tag{2}$$
$$f(x) = \frac{1}{x}, \;\; a = 3, \tag{3}$$
$$f(x) = x\sqrt{1+x^2}, \;\; a = 0. \tag{4}$$

5–6. Calculate the following limits using power series. What does this imply for absolute convergence of the series $\sum_{n=1}^{\infty} f\left(\frac{1}{n}\right)$?

$$f(x) = e^x - 1 - \sin x, \;\; \lim_{x\to 0} \frac{f(x)}{x^2}, \tag{5}$$
$$f(x) = \ln(1+2x), \;\; \lim_{x\to 0} \frac{f(x)}{x}. \tag{6}$$

7–9. Use formulas on page 743 and/or multiplication/division of power series to find the first three nonzero terms in the Maclaurin series for the function:

$$f(x) = \cos^2 x, \tag{7}$$
$$f(x) = e^{x^2} \arctan x, \tag{8}$$
$$f(x) = \frac{e^x}{1-x}. \tag{9}$$

10–11. Approximate the following functions near $x = 0$ by their Taylor polynomials (with the given number of terms). Estimate the error (depending on x) using Taylor's inequality or alternating series remainder estimate. Find how small x has to be so that the error is less than 0.01:

$$f(x) = \cos x, \;\; T_2, \tag{10}$$
$$f(x) = e^x, \;\; T_3. \tag{11}$$

Hints and answers

1. $f(x) = 4 + 4(x-1) + (x-1)^2$.
2. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1} x^{2n+1}}{(2n+1)!}$.
3. $f(x) = \sum_{n=0}^{\infty} (-1)^n 3^{-n-1} x^n$.
4. $f(x) = \sum_{n=0}^{\infty} \binom{1/2}{n} x^{2n+1}$.
5. We find $f(x) = \frac{x^2}{2} + \frac{x^3}{3} + \cdots$.

Answer: 1/2; converges absolutely.

6. We find $f(x) = 2x - 2x^2 + \cdots$.

Answer: 2; does not converge absolutely.

7. $f(x) = 1 - x^2 + \frac{1}{3}x^4 + \cdots$.
8. $f(x) = x + \frac{2}{3}x^3 + \frac{11}{30}x^5 + \cdots$.
9. $f(x) = 1 + 2x + \frac{5}{2}x^2 + \cdots$.
10. $T_2(x) = 1 - \frac{1}{2}x^2 = T_3(x)$; $|f(x) - T_2(x)| \leqslant \frac{x^4}{24}$.
11. $T_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$; $|f(x) - T_3(x)| \leqslant \max(1, e^x)\frac{x^4}{24}$.