# Math 1B quiz 6

Oct 7, 2009

## Section 105

1. (5 pt) Does the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$ converge absolutely, converge conditionally, or diverge? If it converges, estimate the error $|s - s_n|$, where $s$ is the sum of the series and $s_n$ is the sum of the first $n$ terms.

2. (5 pt) Consider the series $\sum_{n=1}^{\infty} \frac{(2n)!c^n}{(n!)^2}$, where $c > 0$ is a constant parameter. For which values of $c$ does the Ratio Test guarantee convergence of the series? For which values does it imply divergence? For which $c$ is the test inconclusive?

## Section 106

1. (5 pt) Does the series $\sum_{n=1}^{\infty} (-1)^{n} \frac{n}{n^2+1}$ converge absolutely, converge conditionally, or diverge? If it converges, estimate the error $|s - s_n|$, where $s$ is the sum of the series and $s_n$ is the sum of the first $n$ terms.

2. (5 pt) Consider the series $\sum_{n=1}^{\infty} \frac{(n!)^2 b^n}{(2n)!}$, where $b > 0$ is a constant parameter. For which values of $b$ does the Ratio Test guarantee convergence of the series? For which values does it imply divergence? For which $b$ is the test inconclusive?

## Solutions for section 105

1. Put $a_n = (-1)^{n+1}\frac{n}{n^2+1}$. First, we study the series of absolute values $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n}{n^2+1}$. We have

$$\lim_{n\to\infty} \frac{a_n}{1/n} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1;$$

since the p-series $\sin_{n=1}^{\infty} \frac{1}{n}$ diverges, by the Limit Comparison Test the series $\sum_{n=1}^{\infty} |a_n|$ diverges. Therefore, the series $\sum_{n=1}^{\infty} a_n$ is not absolutely convergent.

Now, we study the convergence of the series $\sum_{n=1}^{\infty} a_n$ itself. We have $a_n = (-1)^{n+1} b_n$, where $b_n = \frac{n}{n^2+1}$ is positive; we may apply the Alternating Series Test to conclude that $\sum_{n=1}^{\infty} a_n$ converges. Indeed,

$$\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1}{n+\frac{1}{n}} = 0.$$

It remains to verify that the sequence $b_n$ is decreasing. For that, it is enough to prove that the function $f(x) = \frac{x}{x^2+1}$ is decreasing for $x \geqslant 1$. This in turn follows from the inequality

$$f'(x) = \frac{1-x^2}{(x^2+1)^2} \leqslant 0 \text{ for } x \geqslant 1.$$

Since the series $\sum_{n=1}^{\infty} a_n$ converges, but the series $\sum_{n=1}^{\infty} |a_n|$ diverges, the series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent.

Finally, the we use the error estimate for alternating series to get

$$|s - s_n| \leqslant b_{n+1} = \frac{n+1}{(n+1)^2+1}.$$

2. We put $a_n = \frac{(2n)!c^n}{(n!)^2}$ and compute

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)c}{(n+1)^2} = \lim_{n\to\infty} \frac{\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)c}{\left(1+\frac{1}{n}\right)^2} = 4c.$$

Therefore, the series is convergent for $0 < c < \frac{1}{4}$, divergent for $c > \frac{1}{4}$; the test is inconclusive for $c = \frac{1}{4}$.

## Solutions for section 106

1. See the solution for problem 1 in section 105.

2. We put $a_n = \frac{(n!)^2 b^n}{(2n)!}$ and compute

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2 b}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^2 b}{\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)} = \frac{b}{4}.$$

Therefore, the series is convergent for $0 < b < 4$, divergent for $b > 4$; the test is inconclusive for $b = 4$.