Math 1B worksheet

Nov 30, 2009

1. Consider the equation

$$y'' - y' = f(x). \tag{1}$$

(a) Find the general solution of the homogeneous equation.

(b) Find the general solution of the inhomogeneous equation if $f(x) = \sin x$.

(c) Write the trial solutions for the method of undetermined coefficients for the following functions, but do not determine the coefficients:

$$f(x) = e^x \cos(2x),$$
$$f(x) = (x^2 + x + 1)\sin(4x),$$
$$f(x) = 5x.$$

2. Consider the equation

$$x''(t) + cx'(t) + 4x(t) = 0. \tag{2}$$

(a) Solve the equation for $c = 0$. Find the solution satisfying $x(0) = 0$ and $x'(0) = 1$.

(b) Find two linearly independent solutions for the equation, depending on $c \geqslant 0$. Explain when we have overdamping, critical damping, and underdamping.

3–5. Write the solutions of the following initial value problems using power series. Find the radii of convergence.

$$y'' - y = 1,\ y(0) = 0,\ y'(0) = 0; \tag{3}$$
$$y'' + xy = 0,\ y(0) = 0,\ y'(0) = 1; \tag{4}$$
$$y'' + \frac{2y'}{x} = 0,\ y(1) = 1, y'(1) = -1. \tag{5}$$

Hints and answers

1. (a) $y = c_1 + c_2 e^x$.

(b) $y = \frac{1}{2}(\cos x - \sin x) + c_1 + c_2 e^x$.

(c) $y = Ae^x \cos(2x) + Be^x \sin(2x)$; $y = (Ax^2 + Bx + C)\sin(4x) + (Dx^2 + Ex + F)\cos(4x)$; $y = (Ax + B)x$.

2. (a) The general solution is $x = c_1 \cos(2t) + c_2 \sin(2t)$; the solution to the initial value problem is $x = \frac{1}{2}\sin(2t)$.

(b) For $0 \leqslant c < 4$, we have underdamping: $x_1(t) = e^{-c/2}\cos(\beta t)$, $x_2(t) = e^{-c/2}\sin(\beta t)$, where $\beta = \frac{1}{2}\sqrt{16 - c^2}$.

For $c = 4$, we have critical damping: $x_1(t) = e^{-2t}$, $x_2(t) = te^{-2t}$.

For $c > 4$, we have overdamping: $x_1(t) = e^{r_1 t}$, $x_2(t) = e^{r_2 t}$, where r_1 and r_2 are the two real roots to the equation $r^2 + cr + 4 = 0$; both r_1 and r_2 are negative.