Math 1B worksheet

Aug 31, 2009

Please split into groups of 2–4 (preferrably three) people and solve the problems on the board. Please mark on the top of your portion of the board the problems you have attempted with a tick if you have done them or a question mark if you have questions or could use a hint. Many of the problems here come from the textbook; feel free to look up the answers after you have done them.

Problems marked with an asterisk (*) are harder problems which may go well beyond the material of the course (and unlikely to appear on the exam); please only attempt them if you have already done all other problems and you feel bored.

1–7. Find the following integrals:

$$\int_0^1 t \cosh t \, dt, \tag{1}$$

$$\int_0^1 \frac{r^3}{\sqrt{4+r^2}} \, dr, \tag{2}$$

$$\int e^{2\theta} \sin 3\theta \, d\theta, \tag{3}$$

$$\int \frac{\ln x}{x^2} \, dx, \tag{4}$$

$$\int_0^\infty \frac{\arctan x \, dx}{(1+x^2)\sqrt{(\pi/2)^2 - (\arctan x)^2}}, \tag{5}$$

$$\int \ln(x^2+4x+3) \, dx, \tag{6}$$

$$\int \frac{dx}{2+2x+x^2}. \tag{7}$$

8. Use integration by parts to prove the reduction formula

$$\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx.$$

9. Use integration by parts to calculate

$$\int_0^\infty x^n e^{-x} \, dx$$

for any positive integer n.

10*. Let $f(x)$ be a function defined on the interval $a \leqslant x \leqslant b$ and assume that it has derivatives of all orders. (Such functions are called **smooth**.)

(a) Use integration by parts to prove that, for $a \leqslant x \leqslant b$,

$$f(x) = f(a) + f'(a)(x-a) + \int_a^x (x-t)f''(t)\,dt.$$

(b) Let $n \geqslant 0$ be an integer. Using repeated integration by parts, express $f(x)$ in terms of $f(a)$, the first n derivatives of f at the point a, and a certain integral expression which involves $f^{(n+1)}$ (the $n+1$-st derivative of f). Does the result remind you of any formula studied before?

(c) Assume that $|f^{(n+1)}(x)| \leqslant C$ for $a \leqslant x \leqslant b$, where C is a constant. Estimate the integral expression you obtained in (b).

(d) Use (c) to approximate the function $\sin x$ by a polynomial of degree 5 and estimate the error depending on x. Does this approximation work well for large x? Using the fact that $\sin x$ is periodic, make an algorithm to compute it for any x with an estimate of the error independent of x.

Hints and answers

Please do not read these until you have attempted to solve the problems. Remember, you cannot learn mathematics by just looking at the solutions!

1. Integrate by parts. For hyperbolic functions, see Stewart 3.11. If you are unfamiliar with these, first imagine sin in place of cosh and see Stewart 7.1, Example 1. Then put cosh back and use derivatives for hyperbolic functions. Alternatively, write cosh as the sum of two exponents by defintion and reason similarly to Stewart 7.1, Example 3. Be sure to simplify your answer.

Answer: $1 - 1/e$.

2. Do a substitution $u = r^2 + 4$. Alternatively, first notice that we can write $r^3\,dr = r^2 r\,dr = \frac{1}{2} r^2\,d(r^2)$ and the rest of the expression depends only on r^2, so we may use the substitution $v = r^2$. After doing it, the linear substitution $u = v + 4$ makes the square root simpler.

Answer: $\frac{1}{3}(16 - 7\sqrt{5})$.

3. Integrate by parts twice; see Stewart 7.1, Example 4.

Answer: $\frac{1}{13} e^{2\theta}(2\sin 3\theta - 3\cos 3\theta) + C$.

4. Integrate by parts using $u = \ln x$, $dv = \frac{1}{x^2}\,dx$.

Answer: $\frac{1}{2}(1 - \ln 2)$.

5. First do a substitution (hint: what is $\frac{dx}{1+x^2}$?). Then see problem 2.

Answer: $\pi/2$.

6. Recall that $\ln(ab) = \ln a + \ln b$ and see Stewart 7.1, Example 2.

Answer: $(x+1)(\ln(x+1)-1)+(x+3)(\ln(x+3)-1)+C$.

7. Use that $2+2x+x^2=(x+a)^2+b^2$ for some constants a and b (figure out what they are!). Then use a linear substitution.

Answer: $\arctan(x+1)+C$.

8. Take $u=x^n$, $dv=e^x\,dx$.

9. Let I_n be the integral in question. Using the previous problem, prove the recursive formula $I_k=kI_{k-1}$ for any positive integer k. Calculate I_0 explicitly and use the recursive formula n times (each time for a different value of k). For example,

$$I_4=4I_3=4\cdot 3I_2=4\cdot 3\cdot 2I_1=4\cdot 3\cdot 2\cdot I_0=24.$$

Answer: $n!$, the product of all integers from 1 to n.