# 18.156, SPRING 2017, PROBLEM SET 5, SOLUTIONS

We denote by $\mathrm{Diff}_h$ the space of all semiclassical differential operators acting on $C^\infty(\mathbb{R}^n)$ with smooth coefficients which are polynomial in $h$, that is operators which have the following form for some $N$:

$$\sum_{|\alpha|\leq N}\sum_{j=0}^{N} h^j a_{\alpha j}(x)(hD_x)^\alpha \tag{1}$$

where $a_{\alpha j}\in C^\infty(\mathbb{R}^n)$. It is easy to see that the class Diff is closed under compositions and adjoints. We also have the following

**Lemma 0.1.** *1. If $a\in \mathrm{Poly}^k_h$, then $A:=\mathrm{Op}_h(a)$ lies in $\mathrm{Diff}_h$ and for all $(x,\eta)\in\mathbb{R}^{2n}$,*

$$\big(Ae^{\frac{i}{h}\langle\bullet,\eta\rangle}\big)(x)=e^{\frac{i}{h}\langle x,\eta\rangle}a(x,\eta). \tag{2}$$

*2. If $A\in\mathrm{Diff}_h$ and (2) holds for some $a\in\mathrm{Poly}^k_h$, then $A=\mathrm{Op}_h(a)$.*

*Proof.* 1. It suffices to consider the case when $a(x,\xi)=a_\alpha(x)\xi^\alpha$ for some multiindex $\alpha$. Then

$$\big(Ae^{\frac{i}{h}\langle\bullet,\eta\rangle}\big)(x)=a_\alpha(x)(hD_x)^\alpha e^{\frac{i}{h}\langle x,\eta\rangle}=e^{\frac{i}{h}\langle x,\eta\rangle}a_\alpha(x)\eta^\alpha.$$

2. We write $A$ in the form (1) and compute

$$a(x,\eta)=e^{-\frac{i}{h}\langle x,\eta\rangle}\big(Ae^{\frac{i}{h}\langle\bullet,\eta\rangle}\big)(x)=\sum_{|\alpha|\leq N}\sum_{j=0}^{N}h^j a_{\alpha j}(x)\eta^\alpha.$$

For $x,h$ fixed, both sides of this equation are polynomials in $\eta$. Then $a_{\alpha j}(x)$ are uniquely determined by $a$ and we get $A=\mathrm{Op}_h(a)$. □

**1.** We first note that for each $\eta\in\mathbb{R}^n$,

$$e^{-\frac{i}{h}\langle x,\eta\rangle}\,\mathrm{Op}_h(a)e^{\frac{i}{h}\langle x,\eta\rangle}=\mathrm{Op}_h(a_\eta),\quad a_\eta\in\mathrm{Poly}^k_h,\quad a_\eta(x,\xi;h)=a(x,\xi+\eta;h).$$

Indeed, it suffices to consider the case $a(x,\xi)=a_\alpha(x)\xi^\alpha$. This case follows by noting that $a_\alpha(x)$ commutes with $e^{\frac{i}{h}\langle x,\eta\rangle}$ (both being multiplication operators) and

$$e^{-\frac{i}{h}\langle x,\eta\rangle}(hD_{x_r})e^{\frac{i}{h}\langle x,\eta\rangle}=hD_{x_r}+\eta_r.$$

Next, we have for $q\in C^\infty(\mathbb{R}^n)$

$$\mathrm{Op}_h(a)q(x)=\sum_{j=0}^{\infty}h^j\sum_{|\beta|=j}\frac{1}{\beta!}\partial_\xi^\beta a(x,0)D_x^\beta q(x) \tag{3}$$

where again it suffices to consider the case $a(x,\xi)=a_\alpha(x)\xi^\alpha$. Applying (3) to the symbol $a_\eta$, we finish the proof.

**2.** We compute for each $\eta\in\mathbb{R}^n$, using (2) and Exercise 1,

$$\big(\operatorname{Op}_h(a)\operatorname{Op}_h(b)e^{\frac{i}{h}\langle\bullet,\eta\rangle}\big)(x)=\big(\operatorname{Op}_h(a)(e^{\frac{i}{h}\langle\bullet,\eta\rangle}b(\bullet,\eta))\big)(x)=e^{\frac{i}{h}\langle x,\eta\rangle}c(x,\eta),$$

$$c(x,\xi;h)=\sum_{j=0}^\infty h^j c_j(x,\xi),\quad c_j(x,\xi;h)=\sum_{|\alpha|=j}\frac{1}{\alpha!}\partial_\xi^\alpha a(x,\xi;h)\cdot D_x^\alpha b(x,\xi;h).$$

We have $c_j(x,\xi;h)\in\operatorname{Poly}_h^{k+\ell-j}$, since $\partial_\xi^\alpha a\in\operatorname{Poly}_h^{k-j}$ and $D_x^\alpha b\in\operatorname{Poly}_h^\ell$. Then $c\in\operatorname{Poly}_h^{k+\ell}$. Now part 2 of Lemma 0.1 gives $\operatorname{Op}_h(a)\operatorname{Op}_h(b)=\operatorname{Op}_h(c)$. The formulas for the principal parts of $\operatorname{Op}_h(a)\operatorname{Op}_h(b)$ and $[\operatorname{Op}_h(a),\operatorname{Op}_h(b)]$ follow immediately from the expansion for $c$.

**3.** It suffices to consider the case $a(x,\xi)=a_\beta(x)\xi^\beta$. Integrating by parts, we have for all $u,v\in C_c^\infty(\mathbb{R}^n)$ (recalling that $D=-i\partial$)

$$\int_{\mathbb{R}^n}\overline{v(x)}\cdot(hD_x)^\beta u(x)\,dx=\int_{\mathbb{R}^n}u(x)\cdot\overline{(hD_x)^\beta v(x)}\,dx,$$

that is $\big((hD_x)^\beta\big)^*=(hD_x)^\beta$. Since $a_\beta(x)^*=\overline{a_\beta(x)}$, we have

$$\operatorname{Op}_h(a)^*=\big(a_\beta(x)(hD_x)^\beta\big)^*=(hD_x)^\beta\overline{a_\beta(x)}=\operatorname{Op}_h(\xi^\beta)\operatorname{Op}_h(\overline{a_\beta(x)})$$

and the latter product is computed by Exercise 2.

**4.** It suffices to consider the case $a(x,\xi)=a_\alpha(x)\xi^\alpha$. We have

$$e^{-i\varphi/h}\operatorname{Op}_h(a)e^{i\varphi/h}=a_\alpha(x)e^{-i\varphi/h}(hD_{x_1})^{\alpha_1}\cdots(hD_{x_n})^{\alpha_n}e^{i\varphi/h}.$$

Since

$$e^{-i\varphi/h}(hD_{x_r})e^{i\varphi/h}=hD_{x_r}+\varphi'_{x_r}=\operatorname{Op}_h(\xi_r+\varphi'_{x_r})$$

we have

$$e^{-i\varphi/h}\operatorname{Op}_h(a)e^{i\varphi/h}=a_\alpha(x)\operatorname{Op}_h(\xi_1+\varphi'_{x_1})^{\alpha_1}\cdots\operatorname{Op}_h(\xi_n+\varphi'_{x_n})^{\alpha_n}.$$

By Exercise 2 this is equal to

$$\operatorname{Op}_h\big(a_\alpha(x)(\xi_1+\varphi'_{x_1})^{\alpha_1}\cdots(\xi_n+\varphi'_{x_n})^{\alpha_n}+h\operatorname{Poly}_h^{k-1}\big)$$

and it remains to note that

$$a_\alpha(x)(\xi_1+\varphi'_{x_1})^{\alpha_1}\cdots(\xi_n+\varphi'_{x_n})^{\alpha_n}=a(x,\xi+\nabla\varphi(x)).$$

**5.** It suffices to consider the case when $a(x,\xi)=a_\alpha(x)\xi^\alpha$. By the Fourier inversion formula and a change of variables, we have

$$u(x)=(2\pi)^{-n}\int_{\mathbb{R}^n}e^{i\langle x,\eta\rangle}\hat{u}(\eta)\,d\eta=(2\pi h)^{-n}\int_{\mathbb{R}^n}e^{\frac{i}{h}\langle x,\xi\rangle}\hat{u}(\xi/h)\,d\xi.$$

Differentiating under the integral sign, we obtain

$$(hD_x)^\alpha u(x) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x,\xi\rangle} \xi^\alpha \hat{u}(\xi/h)\, d\xi.$$

It follows that

$$\mathrm{Op}_h(a)u(x) = a_\alpha(x)(hD_x)^\alpha u(x) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x,\xi\rangle} a(x,\xi) \hat{u}(\xi/h)\, d\xi.$$