18.156, SPRING 2017, PROBLEM SET 4

1.* Assume that $\Omega \subset \mathbb{C}$ is connected and $\lambda \in \Omega \mapsto A(\lambda)$ is a holomorphic family of Fredholm operators $\mathcal{H}_1 \to \mathcal{H}_2$ on some Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$. Assume also that $A(\lambda)$ is invertible for some value of λ. By Analytic Fredholm Theory, we know that $\lambda \mapsto A(\lambda)^{-1}$ is a meromorphic family of bounded operators $\mathcal{H}_2 \to \mathcal{H}_1$ with poles of finite rank. Fix a pole $\lambda_0 \in \Omega$ of $A(\lambda)^{-1}$ and consider the Laurent expansion (with A_0 holomorphic near λ_0 and $A_1, \dots, A_J$ of finite rank)

$$A(\lambda)^{-1} = A_0(\lambda) + \sum_{j=1}^{J} \frac{A_j}{(\lambda - \lambda_0)^j}, \quad A_J \neq 0.$$

Put $N := \dim \ker A(\lambda_0) = \dim \ker A(\lambda_0)^*$ and take some bases

$$u_1, \dots, u_N \in \ker A(\lambda_0), \quad v_1, \dots, v_N \in \ker A(\lambda_0)^*.$$

Consider the $N \times N$ matrix with entries

$$a_{jk} = \langle \partial_\lambda A(\lambda_0) u_j, v_k \rangle_{\mathcal{H}_2}, \quad j, k = 1, \dots, N \tag{1}$$

(a) Assume that (1) is the identity matrix. Show that $J = 1$ and

$$A_1 f = \sum_{j=1}^{N} \langle f, v_j \rangle_{\mathcal{H}_2} \cdot u_j, \quad f \in \mathcal{H}_2.$$

(Hint: augment $A(\lambda)$ by $\partial_\lambda A(\lambda_0) u_j$ and $\partial_\lambda A(\lambda_0)^* v_k$ to obtain an invertible Grushin problem at λ_0 and obtain a Taylor expansion for the terms in the inverse of the Grushin operator. Then use Schur's complement formula.)

(b) Show that $J = 1$ (i.e. λ_0 is a semisimple pole) if and only if the matrix (1) is invertible. (Hint: assuming $J = 1$, consider the constant term in the Laurent expansion of the identity $\|w\|_{\mathcal{H}_2}^2 = \langle A(\lambda) A(\lambda)^{-1} w, w \rangle_{\mathcal{H}_2}$ for $w \in \ker A(\lambda_0)^*$.)

2. Assume that $u \in H^2_{\mathrm{loc}}(\mathbb{R}^n)$ is outgoing at some $\lambda \in \mathbb{C}$, that is $u = R_0(\lambda) g$ for some $g \in L^2_{\mathrm{comp}}(\mathbb{R}^n)$. Define $f := (P_V - \lambda^2) u$. Show that $f \in L^2_{\mathrm{comp}}(\mathbb{R}^n)$ and $u = R_V(\lambda) f$. (Hint: use analytic continuation but be careful.)

3.* Assume that $\lambda_0 \in \mathbb{C} \setminus \{0\}$ is a resonance which is semisimple, that is, with $B_0(\lambda)$ holomorphic near λ_0,

$$R_V(\lambda) = B_0(\lambda) + \frac{\Pi_{\lambda_0}}{\lambda^2 - \lambda_0^2}.$$

Show that the range of Π_{λ_0} is given by

$$\Pi_{\lambda_0}(L^2_{\mathrm{comp}}(\mathbb{R}^n)) = \{u \in H^2_{\mathrm{loc}}(\mathbb{R}^n) \mid (P_V - \lambda_0^2) u = 0,\ u \text{ is outgoing at } \lambda_0\}. \tag{2}$$

If the semisimplicity condition is not satisfied, can you prove that the range of Π_{λ_0} consists of outgoing functions? If not, what goes wrong? (Hint: use the identity $R_V(\lambda) = R_0(\lambda)(I + VR_0(\lambda)\rho)^{-1}(I - VR_0(\lambda)(1-\rho))$ and express $(I + VR_0(\lambda)\rho)^{-1}$ via $R_V(\lambda)$ to make sure that it has a semisimple pole at λ_0. For the other direction, note that every function on the right-hand side of (2) has the form $R_0(\lambda_0)g$ for some $g \in L^2_{\mathrm{comp}}(\mathbb{R}^n)$ with $g = -VR_0(\lambda_0)g$.)

4. Assume that $\lambda_0 \in \mathbb{C} \setminus \{0\}$ is a resonance and R_V has the Laurent expansion (with $B_0(\lambda)$ holomorphic near λ_0)

$$R_V(\lambda) = B_0(\lambda) + \sum_{j=1}^{J} \frac{(P_V - \lambda_0^2)^{j-1}\Pi_{\lambda_0}}{(\lambda^2 - \lambda_0^2)^j}.$$

Show that for each $f \in L^2_{\mathrm{comp}}(\mathbb{R}^n)$, the function $u := (P_V - \lambda_0^2)^{J-1}\Pi_{\lambda_0} f$ is an outgoing at λ_0 solution to the equation $(P_V - \lambda_0^2)u = 0$. (The first part of the hint to Exercise 3 is useful here as well.)