18.156, SPRING 2017, PROBLEM SET 3

1. Let $U \subset \mathbb{R}^n$ be an open set and

$$\Phi \in C^\infty(U;\mathbb{R}), \quad a \in C_c^\infty(U;\mathbb{C})$$

be two functions. Consider the following oscillatory integral:

$$I(h) = \int_U e^{i\Phi(x)/h} a(x)\, dx, \quad 0 < h < 1. \tag{1}$$

Assume that Φ has no critical points on the support $\operatorname{supp} a$:

$$\nabla\Phi \neq 0 \quad \text{on } \operatorname{supp} a. \tag{2}$$

Show that $I(h) = \mathcal{O}(h^\infty)$, that is for each N there exists a constant C_N such that for all $h \in (0,1)$, $|I(h)| \leq C_N h^N$. This statement is often called the *method of nonstationary phase* because (2) states that the phase function Φ is not stationary on the support of the amplitude a. (Hint: use a partition of unity and a change of variables to reduce to the case when $\Phi(x_1,\dots,x_n) = x_1$. Then repeatedly integrate by parts in x_1.)

For the next exercise, we use the following definition of semiclassical wavefront set WF_h. Assume that $u = u(h) \in L^2(\mathbb{R}^n)$ is a family of functions depending on a parameter $h \in (0,1)$. We also assume that u is bounded in L^2_{comp} uniformly in h, that is for each $\chi \in C_c^\infty(\mathbb{R}^n)$ there exists a constant C_χ such that $\|\chi u\|_{L^2} \leq C_\chi$ for all $h \in (0,1)$. Define the wavefront set $\mathrm{WF}_h(u) \subset \mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_\xi$ as follows: $(x_0,\xi_0) \in \mathbb{R}^{2n}$ does **not** lie in $\mathrm{WF}_h(u)$ if and only if there exists $\chi \in C_c^\infty(\mathbb{R}^n)$, $\chi(x_0) \neq 0$, and an h-independent neighborhood $W \subset \mathbb{R}^n$ of ξ_0 such that

$$\sup_{\xi \in W} |\widehat{\chi u}(\xi/h)| = \mathcal{O}(h^\infty).$$

Here $\widehat{\chi u}$ is the Fourier transform:

$$\widehat{\chi u}(\eta) = \int_{\mathbb{R}^n} e^{-i\langle x,\eta\rangle} \chi(x) u(x)\, dx.$$

2. Assume that $U \subset \mathbb{R}^n \times \mathbb{R}^m$ is an open set and we are given

$$\varphi \in C^\infty(U;\mathbb{R}), \quad a \in C_c^\infty(U;\mathbb{C}).$$

Define the following *oscillatory integral* $I(x;h)$:

$$u(x;h) = \int_{\mathbb{R}^m} e^{i\varphi(x,\theta)/h} a(x,\theta)\, d\theta, \quad x \in \mathbb{R}^n,\ 0 < h < 1. \tag{3}$$

Show that $\mathrm{WF}_h(u) \subset \Lambda_\varphi$, where $\Lambda_\varphi \subset \mathbb{R}^{2n}$ is defined by

$$\Lambda_\varphi = \{(x, \partial_x \varphi(x,\theta)) \mid (x,\theta) \in U,\ \partial_\theta \varphi(x,\theta) = 0\}.$$

(Hint: for each $\chi \in C_c^\infty(\mathbb{R}^n)$, the Fourier transform $\widehat{\chi u}(\xi/h)$ has the form (1). For which χ and ξ can we apply the method of nonstationary phase?)

Expressions of the form (3) are the bread and butter of semiclassical analysis. We will soon meet one of the main examples of these, semiclassical pseudodifferential operators. (If you are familiar with symplectic geometry, you might be amused to learn that under a mild nondegeneracy assumption on φ, the set Λ_φ is actually a Lagrangian submanifold of $\mathbb{R}^{2n}$ with the symplectic form $\sum_{j=1}^n d\xi_j \wedge dx_j$.)

3.* This exercise is a special case of the *method of stationary phase* which is the main tool for obtaining asymptotic expansions in semiclassical analysis. This method extends nonstationary phase by allowing nondegenerate critical points of Φ. Instead of $\mathcal{O}(h^\infty)$, the resulting integral has an expansion in powers of h, with terms coming from the critical points of Φ. See for instance §§3.4–3.5 in Zworski's book for details.

Let $a \in C^\infty(\mathbb{R})$ be a Schwartz function on $\mathbb{R}$ (that is, $x^j \partial_x^k a(x)$ is bounded for all $j,k \in \mathbb{N}_0$) and consider the integral

$$I_a(h) = \int_{\mathbb{R}} e^{ix^2/h} a(x)\, dx.$$

We will show the one-term asymptotic expansion

$$I(h) = h^{1/2} \cdot \sqrt{\pi} e^{i\pi/4} a(0) + \mathcal{O}(h) \quad \text{as } h \to 0. \tag{4}$$

(a) Integrate by parts to show that

$$I_{xa}(h) = \frac{ih}{2} I_{a'}(h).$$

Conclude that when $a(0) = 0$, we have $I_a(h) = \mathcal{O}(h)$.

(b) For the special case $a(x) = e^{-x^2}$, compute

$$I(h) = \sqrt{\pi}\Big(1 - \frac{i}{h}\Big)^{-1/2} = \sqrt{\pi} e^{i\pi/4} h^{1/2} + \mathcal{O}(h^{3/2}).$$

(Hint: first compute $\int_{\mathbb{R}} e^{-sx^2}\, dx$ for $s > 0$. Using analytic continuation, compute this integral for $s \in \mathbb{C}$, $\operatorname{Re} s > 0$.)

(c) Combine parts (a) and (b) above to obtain (4).