# 18.156, SPRING 2017, PROBLEM SET 1

Exercises 1–3 concern the Cauchy problem for a one-dimensional wave equation

$$\begin{aligned} w_{tt} - w_{xx} + V(x)w &= g, \\ w|_{t=0} = f_0(x), \quad w_t|_{t=0} &= f_1(x) \end{aligned} \tag{1}$$

where the potential $V$ is real-valued, smooth, and compactly supported, denoted $V \in C_c^\infty(\mathbb{R};\mathbb{R})$. We assume that $f_0, f_1 \in C_c^\infty(\mathbb{R})$ and $g \in C_c^\infty((0,\infty)_t \times \mathbb{R}_x)$, then (1) has a unique solution $w \in C^\infty(\mathbb{R}^2_{t,x})$.

**1.** Using d'Alembert's formula (moving $Vw$ to the right-hand side), show that:

**(a)** for each $t \geq 0$, the function $w(t, \bullet)$ is compactly supported;

**(b)** if $r_0 > 0$ is such that $\operatorname{supp} V, \operatorname{supp} f_0, \operatorname{supp} f_1, \operatorname{supp} g \subset \{|x| < r_0\}$ then

$$w(t,x) = w_\pm(x \mp t) \quad \text{for } \pm x \geq r_0,\ t \geq 0$$

for some smooth ($C^\infty$) functions $w_\pm$.

**2.** Consider the following energy quantities associated to $w$:

$$\mathcal{E}(t) := \frac{1}{2}\int_{\mathbb{R}} |w_t(t,x)|^2 + |w_x(t,x)|^2 + V(x)|w(t,x)|^2\,dx,$$

$$\mathcal{E}_0(t) := \frac{1}{2}\int_{\mathbb{R}} |w_t(t,x)|^2 + |w_x(t,x)|^2 + |w(t,x)|^2\,dx.$$

**(a)** Show that for $t \geq 0$,

$$\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t \int_{\mathbb{R}} \operatorname{Re}(\overline{w_t(s,x)}g(s,x))\,dxds.$$

Use this to show that when $V \geq 0$, we have $\mathcal{E}_0(t) \leq C(1+t)^2$ for some constant $C$ and all $t \geq 0$.

**(b)** In the case of general $V$, show that for some constant $C_V$ depending only on $V$

$$\mathcal{E}_0'(t) \leq C_V\mathcal{E}_0(t) + \int_{\mathbb{R}} |g(t,x)|^2\,dx.$$

Use this to show that $\mathcal{E}_0(t) \leq Ce^{C_V t}$ for some constant $C$ and all $t \geq 0$.

**3*.** This exercise gives a better condition on the support of $w$ than the one you found in Exercise 1(a). Consider some $(t_0, x_0)$ such that $t_0 \geq 0$ and

$$f_0(x) = f_1(x) = 0 \quad \text{for } |x - x_0| \leq t_0;$$

$$g(t,x) = 0 \quad \text{for } 0 \leq t \leq t_0, \quad |x - x_0| \leq t_0 - t.$$

Show that $w(t_0, x_0) = 0$. (Hint: argue similarly to Exercise 2(b) but change the definition of $\mathcal{E}_0$ to only integrate over the interval $\{|x - x_0| \le t_0 - t\}$.)

Exercises 4–7 concern the Schrödinger operator on $C^\infty(\mathbb{R})$,

$$P_V = D_x^2 + V = -\partial_x^2 + V, \quad V \in C_c^\infty(\mathbb{R}; \mathbb{R}).$$

They use the *Wronskian function* of two functions $u, v \in C^1(\mathbb{R})$, defined as

$$W(u, v) = u \cdot v_x - u_x \cdot v.$$

Note that for any $u, v$ and $\lambda \in \mathbb{C}$

$$\partial_x W(u, v) = v \cdot (P_V - \lambda^2)u - u \cdot (P_V - \lambda^2)v.$$

Also, for any $u, v_1, v_2$ such that $W(v_1, v_2) \ne 0$, we have

$$\begin{pmatrix} u \\ u' \end{pmatrix} = \frac{1}{W(v_1, v_2)} \begin{pmatrix} -v_2 & v_1 \\ -v_2' & v_1' \end{pmatrix} \begin{pmatrix} W(u, v_1) \\ W(u, v_2) \end{pmatrix}. \tag{2}$$

Define the unique solutions $e_\pm(x; \lambda)$ to the equation $(P_V - \lambda^2)e_\pm = 0$ which satisfy

$$e_\pm(x) = e^{\pm i\lambda x} \quad \text{for } \pm x \gg 1,$$

note that they are smooth in $x$ and holomorphic in $\lambda$, and $\mathbf{W}(\lambda) := W(e_+, e_-)$ is constant in $x$.

**4.** Assume that $\lambda$ is such that $\mathbf{W}(\lambda) \ne 0$. Show that for $f \in C_c^\infty(\mathbb{R})$ the unique solution $u$ to the problem

$$(P_V - \lambda^2)u = f; \quad u(x) = c_\pm e^{\pm i\lambda x} \quad \text{for } \pm x \gg 1 \text{ and some } c_\pm \in \mathbb{C}$$

has the form $u = R_V(\lambda)f$ where

$$\begin{aligned} R_V(\lambda)f(x) &= \int_{\mathbb{R}} R_V(x, y; \lambda) f(y)\, dy, \\ R_V(x, y; \lambda) &= \frac{e_+(x)e_-(y)[x > y] + e_-(x)e_+(y)[x < y]}{\mathbf{W}(\lambda)}. \end{aligned} \tag{3}$$

**5.** Show that

$$W(e_+, e^{i\lambda x})(x) = \int_x^\infty V(y) e_+(y) e^{i\lambda y}\, dy,$$
$$W(e_+, e^{-i\lambda x})(x) = -2i\lambda + \int_x^\infty V(y) e_+(y) e^{-i\lambda y}\, dy.$$

Estimating the right-hand sides by (2), show that for $|\operatorname{Im}\lambda| \le C_0$, we have as $|\lambda| \to \infty$

$$W(e_+, e^{i\lambda x})(x) = \mathcal{O}(1), \quad W(e_+, e^{-i\lambda x})(x) = -2i\lambda + \mathcal{O}(1)$$

uniformly in $x$, where the constant in $\mathcal{O}(\cdot)$ depends only on $C_0$ and $V$. Using the identity (2), show that for $x$ in any fixed compact set,

$$\begin{aligned} e_+(x) &= e^{i\lambda x} + \mathcal{O}(|\lambda|^{-1}), \quad e_+'(x) = i\lambda e^{i\lambda x} + \mathcal{O}(1); \\ e_-(x) &= e^{-i\lambda x} + \mathcal{O}(|\lambda|^{-1}), \quad e_-'(x) = -i\lambda e^{-i\lambda x} + \mathcal{O}(1). \end{aligned} \tag{4}$$

**6. (a)** Show that $\mathbf{W}(\lambda)^{-1}$ is meromorphic in $\lambda$ and conclude that $R_V(\lambda)$ defined in (3) is a meromorphic family of operators $C_c^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$.

**(b)** Using Exercise 5, show that for $|\operatorname{Im}\lambda| \le C_0$ and $|\operatorname{Re}\lambda|$ large enough, $\lambda$ is not a pole of $R_V$ and we have the high frequency bound for all $\chi \in C_c^\infty(\mathbb{R})$, with the constant $C$ depending only on $V, C_0, \chi$,

$$\|\chi R_V(\lambda)\chi\|_{L^1\to L^\infty} \le \frac{C}{|\lambda|}.$$

**7*.** This exercise is a basic case of the WKB (Wentzel–Kramers–Brillouin) approximation, giving an expansion of $e_\pm(x;\lambda)$ in nonpositive integer powers of $\lambda$ which improves over (4). We will assume that $|\operatorname{Im}\lambda| \le C_0$ for some constant $C_0$ and study the limit $|\operatorname{Re}\lambda| \to \infty$.

**(a)** Construct a sequence of smooth functions $a_\pm^{(n)}(x)$ such that $a_\pm^{(n)}(x)$ are locally constant for $|x| \gg 1$, $a_\pm^{(n)}(x) = \delta_{n0}$ for $\pm x \gg 1$, and for each $N$ the function

$$e_\pm^{(N)}(x) := e^{\pm i\lambda x}\sum_{n=0}^{N} \lambda^{-n} a_\pm^{(n)}(x)$$

satisfies uniformly in $x$

$$(P_V - \lambda^2)e_\pm^{(N)}(x) = \mathcal{O}(|\lambda|^{-N}).$$

**(b)** Using Wronskians, show that locally uniformly in $x$

$$e_\pm(x) = e_\pm^{(N)}(x) + \mathcal{O}(|\lambda|^{-N-1}), \quad \partial_x e_\pm(x) = \partial_x e_\pm^{(N)}(x) + \mathcal{O}(|\lambda|^{-N}).$$

**(c)** Show that the scattering matrix $S(\lambda)$ admits an asymptotic expansion in nonpositive integer powers of $\lambda$, and write out this expansion modulo $\mathcal{O}(|\lambda|^{-2})$.