Resonances in the semiclassical limit

$P_V = -\partial_x^2 + V$. Resonances near the real line?

We know as $|\mathrm{Re}\,\lambda| \to \infty$, $|\mathrm{Im}\,\lambda| \le C_0$, there are no resonances (spectral gap...)

The reason is that V is a lower order perturbation, i.e. for a fn. $e^{\pm i\lambda x}$, $\mathrm{Re}\,\lambda >> 1$, $\partial_x^2 e^{i\lambda x}$ has size λ^2 but $Ve^{i\lambda x}$ has size 1...

We want to have an operator where we can have resonances in the high frequency regime.

The simplest case is the <u>semiclassically rescaled operator</u>

$$P_V(h) = -h^2\partial_x^2 + V, \quad V \in C_c^\infty(\mathbb{R};\mathbb{R}),$$

$h>0$ small parameter (semiclassical constant)

Note: equivalent to taking potential $h^{-2}V$.

Now imagine we have something oscillating at frequency λ: $e^{i\lambda x}$. Then $h^2\partial_x^2 e^{i\lambda x} = -(h\lambda)^2 e^{i\lambda x}$.

To be comparable with the effects ~~of~~ V, we should take $\lambda = \omega/h$, $\omega \sim 1$.

Note: $(P_V(h) - \omega^2)u = 0$ does have solutions $e^{\pm i\omega/h}$ when $|x| >> 1$.

How does $P_V(h)$ act on high frequency functions? Imagine $\xi \in \mathbb{R}$, $u = e^{\frac{i\xi x}{h}} a(x)$, a is slowly varying as $h \to 0$

Then $P_V(h) u = (\xi^2 + V(x)) u = p(x,\xi) u$
$+ O(h)$ $\qquad + O(h)$

where $p(x,\xi) = \xi^2 + V(x)$ is the semiclassical principal symbol of $P_V(h)$

Later in the course we will have an overview of <u>semiclassical analysis</u> (interested? look at Zworski's book for now)

and will discover:

- For each $u = u(x;h)$, we can get
  $WF_h(u) \subset \mathbb{R}^2_{x,\xi}$: ~~where~~
  $(x,\xi) \in WF_h(u)$ means that u ~~lives~~ is present at position x & frequency ξ/h (= semiclassical frequency ξ)
- If u solves $(P_V(h) - \omega^2)u = 0$ $\quad (*)$ + ($\operatorname{Im} \omega = O(h)$)
  then $WF_h(u) \subset \{(x,\xi) \in \mathbb{R}^2 : p(x,\xi) = \operatorname{Re} \omega^2\}$
  energy ~~surface~~
- Moreover if u solves $(*)$, + ($\operatorname{Im}\omega = O(h)$) then $WF_h(u)$ is invariant under the Hamiltonian flow e^{tH_p}.

Here $H_p = (\partial_\xi p) \cdot \partial_x - (\partial_x p) \cdot \partial_\xi$.

I.e. $(x(t), \xi(t)) = e^{tH_p}(x_0, \xi_0)$ solves the ODE

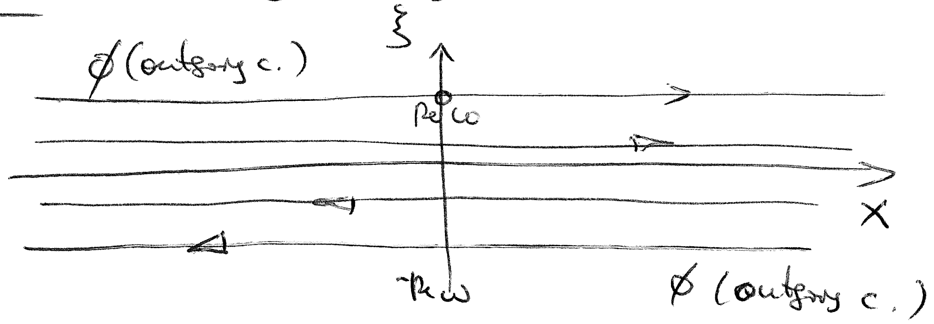
$\dot{x} = \partial_\xi p(x,\xi) = 2\xi$
$\dot{\xi} = -\partial_x p(x,\xi) = -\cancel{2} V'(x)$

with initial conditions $x(0) = x_0$, $\xi(0) = \xi_0$.

• If u is outgoing, i.e. $u(x) \sim e^{\pm\frac{i\omega x}{h}}$, $\pm x \gg 1$,
then $WF_h(u) \cap \{\pm x \gg 1\} \subset \{\xi = \pm \mathrm{Re}\,\omega\}$.

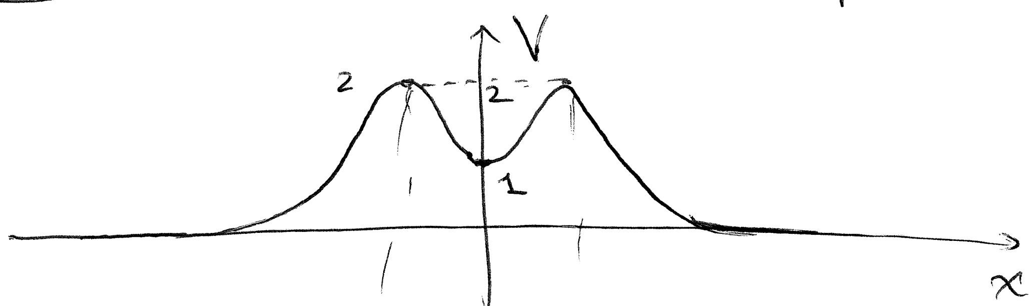
<u>Example 1</u> $V \equiv 0$. Get $\dot{x} = 2\xi$, $\dot{\xi} = 0$

flow lines of e^{tH_p}

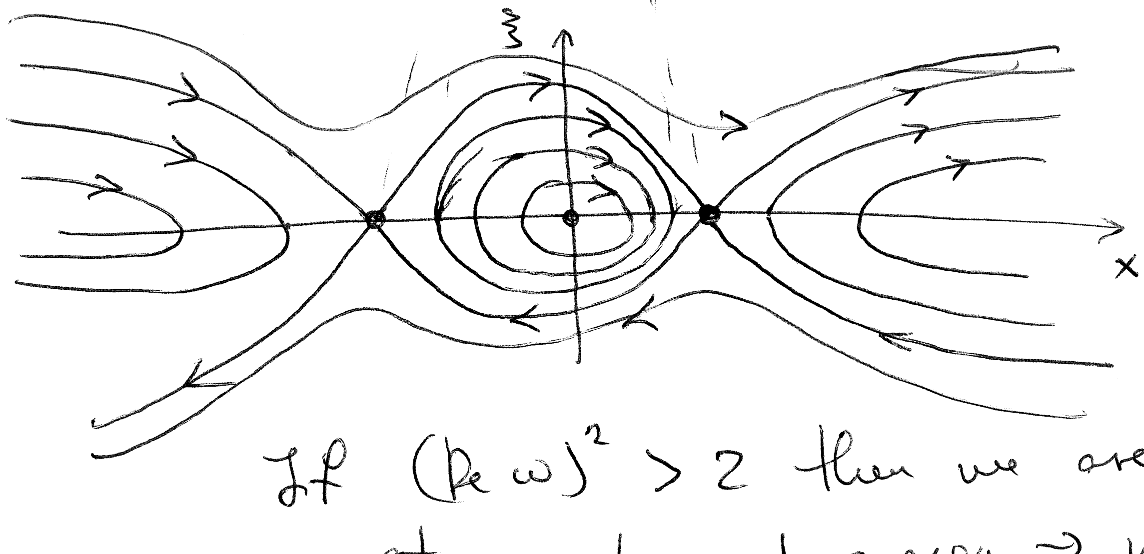


+ propagation $\Rightarrow$ no resonances as soon as $\mathrm{Re}\,\omega > 0$.

<u>Example 2</u> V looks like a well potential:



Flow lines of e^{tH_p}: look at level sets of $p = \xi^2 + V(x)$:

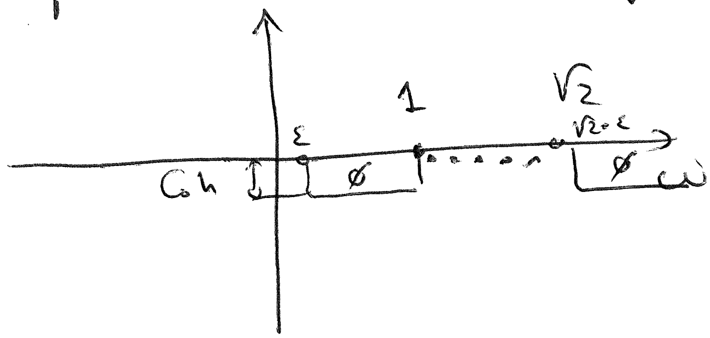


If $(\mathrm{Re}\,\omega)^2 > 2$ then we are at nontrapped energy $\to$ no resonances with $\mathrm{Re}\,\omega > \sqrt{2} + \varepsilon$, $\mathrm{Im}\,\omega = O(h)$.

On the other hand, have strong trapping in $(\mathrm{Re}\,\omega)^2 \in [1, 2)$

And again no trapping at $0 < (\mathrm{Re}\,\omega)^2 < 1$.

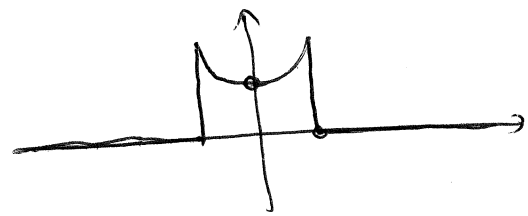
So, expect resonances of $P_V(h)$ to look like: $\forall \varepsilon > 0$, $h \ll 1$



Note: λ a resonance $\Leftrightarrow -\bar{\lambda}$ a resonance (resonant state $\bar{u}$)
So enough to do $\operatorname{Re} \lambda > 0$

<u>A "basic" example</u>

$$V(x) = \begin{cases} x^2+1, & |x| \le 1 \\ 0, & |x| > 1. \end{cases}$$



What does it mean for u to be a resonant state for $P_V(h)$ at ω? Note V is discontinuous. ($V \in L^\infty_{\text{comp}}$)

Need $(P_V(h) - \omega^2)u = 0$ in distributions ($u \in H^2 \ldots$)

Which is same as $(P_V(h) - \omega^2)u = 0$ on $(-1,1)$, $u \in C^\infty(-1,1)$

and $u(x) = C_\pm e^{\pm \frac{i\omega}{h} x}$, $\pm x > 1$

& u, u' continuous at $x = \pm 1$.

i.e. $u'(\pm 1) \mp \frac{i}{h} \omega u(\pm 1) = 0$.

We have an approximate resonant state

$\tilde{u}(x) = e^{-\frac{x^2}{2h}}$ with $(-h^2 \partial_x^2 + x^2 + 1 - (1+h))\tilde{u} = 0$

This is the ground state of quantum harmonic oscillator $-h^2\partial_x^2 + x^2$. Check: $h\partial_x \tilde{u} = -x\tilde{u}$, so

$(-h^2\partial_x^2)\tilde{u} = h\partial_x(x\tilde{u}) = -x^2\tilde{u} + h\tilde{u}$.

<u>Thm</u>. For h small enough, $P_V(h)$ has a resonance of the form $\sqrt{z}$, $z = 1 + h - \frac{4i}{\pi} h^{1/2} e^{-1/h}(1 + O(h))$.

<u>Proof</u>. Will not give it here. See §2.8 in the book. $\square$