$P_h = -h^2\Delta_g + h^2V^2$, $p(x,\xi) = \sum_{j,k} g^{jk}(x)\,\xi_j\,\xi_k$
Ψ_h^2

$\varphi_t = \exp(tH_p) : T^*M \circlearrowleft$

Gap of size β: if

$(P_h - \omega^2)u = f$, $\operatorname{supp} f \subset \{r \le r_1\}$,

u outgoing at ω/h,

and $\operatorname{Re}\omega = 1$, $-\beta h \le \operatorname{Im}\omega \le h$, then $\forall \chi$

$\|\chi u\|_{L^2} \le C_\chi \|f\|_{L^2}$.

How about $WF_h(u)$? What if $u = u(h)$, $f = f(h)$

& we normalize u so that $\|\chi_1 u\|_{H_h^1} = 1$

for some appropriately chosen $\chi_1 \in C_c^\infty(M)$.

Then $\forall \chi \in C_c^\infty(M)$, $\|\chi u\|_{L^2} \le C\|\chi_1 u\|_{H_h^1}$, (proved last time)

so $u(h)$ is h-tempered.

Free resolvent case: $(-h^2\Delta - \omega^2)u = f \in L^2_{comp}$, $u = R_{0,h}(\omega)f$.

- Ellipticity: $WF_h(u) \setminus S^*\mathbb{R}^n \subset WF_h(f)$
- Propagation of singularities: if $(x,\xi) \in WF_h(u) \cap S^*\mathbb{R}^n$

& $p_0 = |\xi|^2$, $\varphi_t^0 = e^{tH_{p_0}}$, $\varphi_t^0(x,\xi) = (x + 2t\xi, \xi)$, then

either (a) $\exists\ t \le 0$ s.t. $\varphi_t^0(x,\xi) \in WF_h(f)$

or (b) $\forall\ t \le 0$, $\varphi_t^0(x,\xi) \in WF_h(u)$.

Actually, case (b) does not happen.

E.g. in dim $n=3$, $u(x) = \frac{1}{4\pi h^2}\int_{\mathbb{R}^3} e^{\frac{i}{h}|x-y|}\, a(x,y)\, f(y)\, dy$

Assume $\operatorname{supp} f \subset \{|y| < r_1\}$ but we're interested in $u(x)$ where $|x| > r_1$,

So that $\underset{|y|<r_1}{x \neq y}$ in the integral.

Then $\forall y$, the function

$$v_y : x \mapsto \frac{1}{4\pi h^2} e^{\frac{i}{h}|x-y|} a(x,y)$$

has the form (power of h). $e^{\frac{i}{h}\varphi_y(x)} \tilde{a}_y(x)$

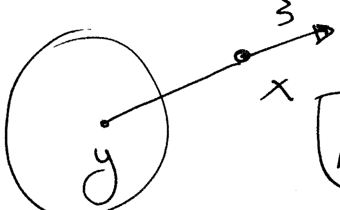
where $\varphi_y \in C^\infty$, $\tilde{a}_y \in C^\infty$ (near x of interest)

So $WF_h(v_y) \cap \{ |x| > r_1 \}$

$$\subset \{ (x, \nabla_x \varphi_y(x)) : |x| > r_1 \}$$

$$\subset \left\{ \left(x, \frac{x-y}{|x-y|}\right) : |x| > r_1 \right\} \subset \left\{ (x,\xi) : \begin{array}{l} |x| > r_1 \\ \langle x, \xi \rangle > 0 \end{array} \right\}$$

"outgoing set"



$$\boxed{\text{And } WF_h(u) \subset \bigcup_{|y|<r_1} WF_h(v_y) \text{ since } u = \int v_y f(y)\,dy}$$

But if case (b) happened, then for large $|t|$, $t<0$, we have

$$\varphi_{-t}(x,\xi) \in \{ |x| > r_1, \ \langle x, \xi \rangle < 0 \} \leftarrow \text{"incoming set"}$$

So case (b) cannot happen.

In other words, in the free case

$$WF_h(u) \subset WF_h(f) \cup \left(\bigcup_{t \geq 0} \varphi_t^0 (WF_h(f) \cap S^*\mathbb{R}^n) \right)$$

General case but $f \equiv 0$:

$(P_h - \omega^2) u = 0 \Rightarrow$

- by elliptic estimate, $WF_h(u) \subset S^*M$
- by propagation of singularities,

$\forall (x_0, \xi_0) \in WF_h(u)$, $\varphi_t(x, \xi) \in WF_h(u)$ $\forall t$ (ASM)

2 cases here:

(a) $(x_0, \xi_0) \in \Gamma_+ \cap S^*M$, so that $\varphi_t(x_0, \xi_0)$ stays bdd as $t \to -\infty$

(b) $\varphi_t(x_0, \xi_0) \to \infty$ as $t \to -\infty$

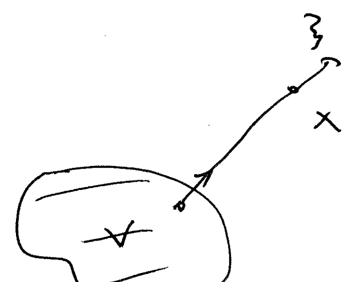
Again, <u>case (b) cannot happen</u>:

for some $\psi \in C_c^\infty(M)$, can write

$(1-\psi) u = R_{0,h}(\omega) v$ for some h-tempered v

So by the free resolvent case,

$WF_h(u) \cap S^*M \subset \bigcup_{t \geq 0} \varphi_{t,0}(WF_h(v))$



For large $|x|$ & $(x, \xi) \in WF_h(u)$,

we see that (x, ξ) should be <u>outgoing</u>: $\langle x, \xi \rangle > 0$

but in case (b), for $|t| \gg 1$, $t < 0$,

$\varphi_t(x_0, \xi_0) \in \{\langle x, \xi \rangle < 0\}$.

Which cannot be in $WF_h(u)$...

<u>Conclusion</u>: $WF_h(u) \subset \Gamma_+ \cap S^*M$ for a resonant state u.

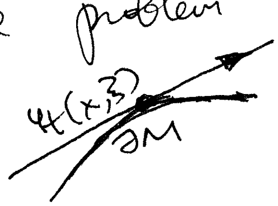
So in particular,
if there is no trapping ($K = \emptyset$) then $\Gamma_+ = \emptyset$
so $WF_h(u) = \emptyset$. But we assumed that

$$\|\chi_1 u\|_{L^2} = 1, \quad \underline{\text{cannot be}}.$$

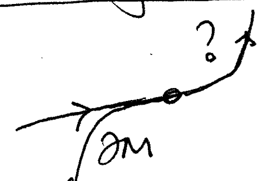
So there are no resonant states $\Rightarrow$
$\Rightarrow$ ω/h not a resonance & we recover
a ~~basic case of~~ weaker form of the nontrapping theorem from last time.

<u>What if $\partial M \neq \emptyset$?</u> ($\partial M$ is C^∞, Dirichlet boundary conditions)

One needs a <u>revised propagation of singularities</u>, with φ_t replaced by the billiard ball flow:
flow along e^{tH_p} (geodesic flow) until we hit ∂M; then reflect off the boundary and keep propagating.
The problem is with <u>glancing trajectories</u>:



or even worse,



Need to define φ_t carefully, not always uniquely defined
& not a smooth flow.

<u>Melrose–Sjöstrand</u> 1982, <u>Ivrii</u> 1980
see [Hörmander, Vol. III, Thm 24.5.3]
propagation of singularities works. Gives gap of any size for nontrapping obstacles...

An important special case is when ∂M is strictly concave (e.g. $M = \mathbb{R}^n \setminus \Omega$ where Ω is strictly convex)

~~For~~ (case of ∂M strictly <u>convex</u> also worked out)

There is a <u>Melrose-Taylor parametrix</u>, describing the structure of all solutions to $(P_h - \omega^2)u = 0$ microlocally near glancing points.

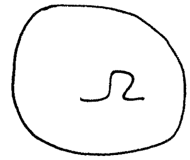
<u>Melrose's view of ~~boundary~~ billiard flow:</u>

will do the basic case when $M = \mathbb{R}^2 \setminus \Omega$, $\Omega \subset \mathbb{R}^2$ strictly convex.

Can write $\Omega = \{q < 0\}$ where $q : \mathbb{R}^2 \to \mathbb{R}$ is C^∞,

Correspondingly $M = \{q \geq 0\}$

e.g. $q(x,y) = x^2 + y^2 - 1$ for the disk.

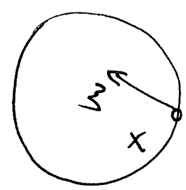


Take some $(x,\xi) \in S^*M = \{p = 1\}$.

If $q(x) > 0$ then can propagate by e^{tH_p} i.e. $e^{tH_p}(x,\xi) \in M$ ~~as~~ for some time, until we hit ∂M.

What if $q(x) = 0$, i.e. <u>we're on the boundary</u>?

Say ξ points inward$_\Omega$ i.e. <u>we just got to the boundary</u>.



This is same as saying that

$\boxed{H_p q(x,\xi) < 0}$ ($H_p q > 0$ corresponds to pointing outside of Ω, $H_p q = 0$ corresponds to glancing)

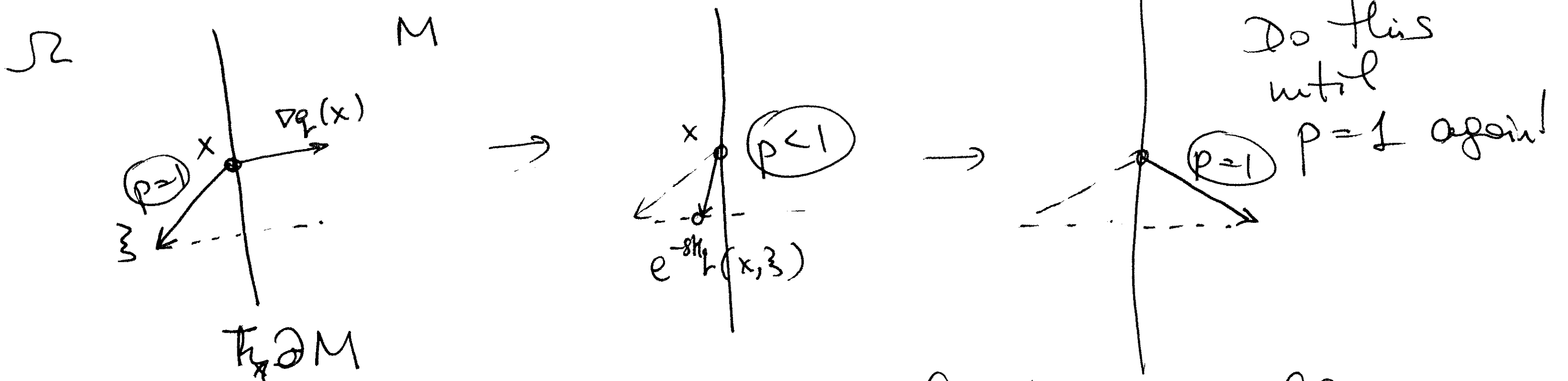
How to find the reflected vector?

Start with

$(x,\xi) \in \{p=1\} \cap \{q=0\} \cap \{H_p q < 0\}$.

Use the Hamiltonian flow e^{-sH_q}: $\quad (q(x,\xi) := q(x))$

$$e^{-sH_q}(x,\xi) = (x, \xi + s\nabla q(x)).$$



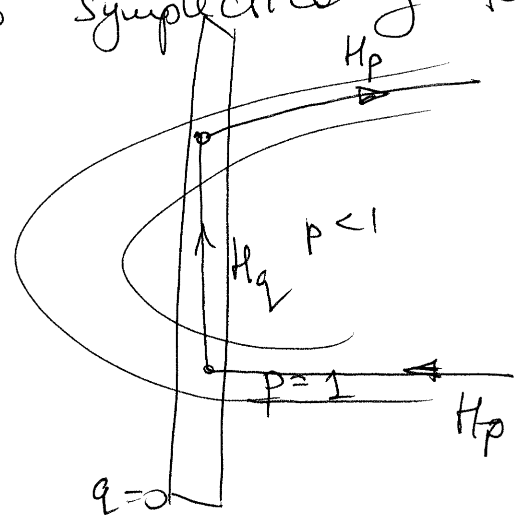
So a symplectic description of bouncing off the boundary is: start with sth. on

$\{p=1\} \cap \{q=0\} \cap \{H_p q < 0\}$

& propagate it along e^{-sH_q} until we get to a point on $\{p=1\} \cap \{q=0\} \cap \{H_p q > 0\}$.

Then can move forward along e^{tH_p} again...

So symplectically the picture is:



So what's the picture when we have glancing?

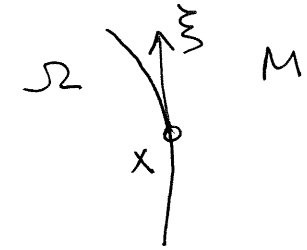
The glancing set is

$$\{p=1\}\cap\{q=0\}\cap\{H_p q=0\}.$$

(codimension 3)

In terms of the Poisson bracket $\{\cdot,\cdot\}$:

$p=1,\ q=0,\ \{p,q\}=0 \quad (1)$



Ω is convex (∂M concave) $\Rightarrow$

$\Rightarrow$ if we propagate a bit forward along H_p

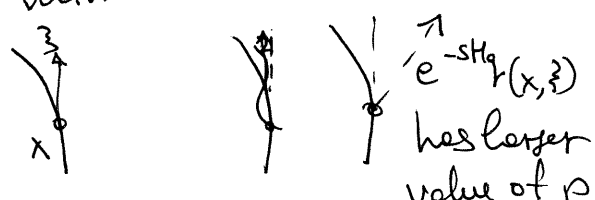
we'll get $q>0$:

So, $H_p^2 q>0$, i.e.

$$\{p,\{p,q\}\}>0 \quad (2)$$

What if we instead propagated along H_q?

We'd see that p has a local minimum there, i.e.



So, $H_q^2 p>0$, i.e.

$$\{q,\{q,p\}\}>0 \quad (3)$$

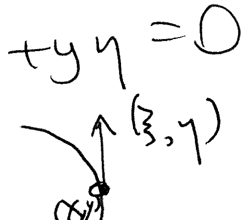
Note also: dp, dq lin. independent (4)

Example: the disk in $\mathbb{R}^2$. $p=\xi^2+\eta^2$, $q=x^2+y^2-1$

~~Sketch~~ $\{p,q\}=2(x\xi+y\eta)=2\langle(x,y),(\xi,\eta)\rangle$

Glancing: $\xi^2+\eta^2=1,\ x^2+y^2=1,\ x\xi+y\eta=0$

e.g. $x=1,\ y=0,\ \xi=0,\ \eta=1$:



Compute $\{p,\{p,q\}\} = \{\xi^2+\eta^2, x\xi + y\eta\}$

$$= 2(\xi^2+\eta^2)$$

$$\{q,\{q,p\}\} = -\{x^2+y^2-1, x\xi+y\eta\}$$

$$= 2(x^2+y^2)\ldots$$

Thm [Equivalence of glancing hypersurfaces;
Melrose 1976; Hörmander, Vol. III, Thm 21.4.8]

Assume $p, q \in C^\infty(T^*M_x)$ satisfy at (1) – (4) at some pt (x_0,ξ_0)
& so do $\tilde p, \tilde q \in C^\infty(T^*\tilde M)$, $\dim M = \dim \tilde M$.
near $(\tilde x_0, \tilde \xi_0)$

Then $\exists$ local symplectomorphism $\varphi : T^*M \to T^*\tilde M$ $(\ldots)$ are nonvanishing fns
$\varphi : (x_0,\xi_0) \mapsto (\tilde x_0, \tilde\xi_0)$, $\tilde p \circ \varphi = (\ldots) p$, $\tilde q \circ \varphi = (\ldots) q$.

So microlocally speaking (using theory of Fourier integral operators to quantize $\varphi\ldots$
all glancing situations are the same! (Note: false in the analytic category)
Can reduce to the basic Friedlander model: (in 2D…)

$q = x$, $p - 1 = \xi^2 - x - y$

Note: at $\{p,q\} = 2\xi$
Glancing set: $x=0$, $\xi=0$, $y=0$
$\{p,\{p,q\}\} = 2$, $\{q,\{q,p\}\} = 2$

Quantize: $P - 1 = (hD_x)^2 - x - y$
Need to solve $(P-1)u = 0$
which for given y becomes the Airy equation. Thus the solutions behave like Airy functions…