$P = -\Delta_g + V$, (M,g) manifold with Euclidean ends (potentially $\partial M \neq \emptyset$)
$V \in C_c^\infty(M;\mathbb{R})$.

Upper half-plane: ~~(proving for $\partial M = \emptyset$ only...)~~

- For $\operatorname{Im}\lambda > 0$, $P - \lambda^2 : H^2(M) \cap H^1_0(M) \to L^2(M)$ is Fredholm (proved last time when $\partial M = \emptyset$)
- In fact for $\operatorname{Im}\lambda > 0$, $\lambda \notin i\mathbb{R}$ $P-\lambda^2$ is invertible (& thus Fredholm of index 0 on $i\mathbb{R}$ as well):

(a) $P-\lambda^2$ has no kernel:
assume $u \in H^2 \cap H^1_0$, $(P-\lambda^2)u = 0$.
Integrating by parts (approximating u by compactly supported fns) we get
$$0 = \operatorname{Im}\langle (P-\lambda^2)u, u\rangle = -\operatorname{Im}(\lambda^2)\|u\|^2 \Rightarrow u \equiv 0.$$
Since $\langle Pu, u\rangle = \int_M |Du|^2 \, d\mathrm{vol}_g$

(b) $P-\lambda^2$ has no cokernel: will only do $\partial M = \emptyset$ case
assume $v \in L^2$ and $\forall u \in H^2$, ~~$\cap H^1_0$~~, $\langle (P-\lambda^2)u, v\rangle = 0$;
~~$\langle Pu, v\rangle = 0$; need to~~
need to show $v = 0$. This is true for all $u \in C_c^\infty(M) \Rightarrow$ integrating by parts, get $(P-\bar\lambda^2)v = 0$ in $\mathcal{D}'(M)$ (distributionally)

Elliptic regularity $\Rightarrow$ $v \in C^\infty(M)$

Enough to show $v \in H^2(M)$ since then integration by parts as in part ⓐ shows that $v \equiv 0$.

Since $v \in \cancel{H^2}\, C^\infty$ it's a question of the behavior of v near infinity, i.e. in the infinite ends. Assume for simplicity only 1 infinite end, $\{r \geq r_0\}$

take $\chi \in C_c^\infty(M)$, $\chi = 1$ near $\{r \leq r_0\}$.

Enough to prove $(1-\chi)v \in H^2(\mathbb{R}^n)$.

But $(-\Delta - \bar{\lambda}^2)(1-\chi)v = (P - \bar{\lambda}^2)(1-\chi)v$

$= [\Delta, \chi]v$ in distributions.

~~So:~~ So: $(1-\chi)v \in L^2(\mathbb{R}^n)$, $(-\Delta - \bar{\lambda}^2)(1-\chi)v = [\Delta, \chi]v \in C_c^\infty$.

Using Fourier transform, get $(1-\chi)v \in H^2(\mathbb{R}^n)$ as needed.

<u>Meromorphic continuation</u>

From now on, assume <u>n is odd</u>.

<u>Thm (see [DyZw, Thm 4.4])</u> The operator

$R(\lambda) := (P - \lambda^2)^{-1} : L^2 \to \overbrace{H^2(M) \cap H_0^1(M)}^{\mathcal{D}_P}$, $\operatorname{Im} \lambda > 0$

admits a meromorphic continuation w/poles of finite rank to

$R(\lambda) : L^2_{comp} \to H^2_{loc} \cap \{u : u|_{\partial M} = 0\}$, $\lambda \in \mathbb{C}$.

<u>Proof</u> Denote $\mathcal{D}_{P,loc} = \{u \in H^2_{loc}(M) : u|_{\partial M} = 0\}$

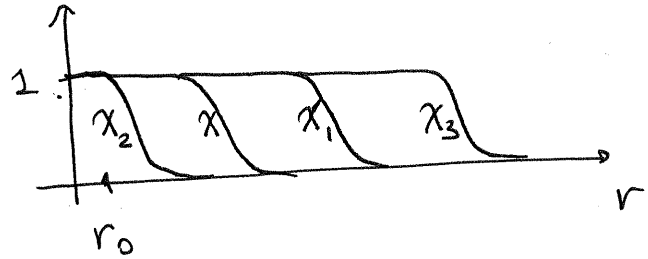
1. Pick $\chi, \chi_1, \chi_2, \chi_3 \in C_c^\infty(M)$:

$\chi_2 = 1$ near $\{r \le r_0\}$ (M is Euclidean for $r \ge r_0$)

$\chi = 1$ near supp χ_2

$\chi_1 = 1$ near supp χ

$\chi_3 = 1$ near supp χ_1



Fix $\lambda_0 \in \mathbb{C}$, Im $\lambda_0 > 0$.

We take a modified version of Q from previous lecture:

$$\boxed{Q := \chi_1 R(\lambda_0)\chi + (1-\chi_2) R_0(\lambda)(1-\chi).}$$

Here $R(\lambda_0) = (P-\lambda_0^2)^{-1} : L^2 \to \mathcal{D}_P$

$R_0(\lambda) = (-\Delta-\lambda^2)^{-1} : L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n)$, Im $\lambda > 0$

We have $Q : L^2 \to \mathcal{D}_P$ for Im $\lambda > 0$.

We compute (using $(1-\chi_2)P = (1-\chi_2)(-\Delta)$, $\chi_1\chi + (1-\chi_2)(1-\chi) = \chi + 1 - \chi = 1$)

$$\begin{aligned}(P-\lambda^2)Q &= \chi_1(P-\lambda^2)R(\lambda_0)\chi + [P,\chi_1]R(\lambda_0)\chi \\ &\quad + (1-\chi_2)(P-\lambda^2)R_0(\lambda)(1-\chi) \\ &\quad - [P,\chi_2]R_0(\lambda)(1-\chi) \\ &= \chi_1(I + (\lambda_0^2-\lambda^2)R(\lambda_0))\chi + [P,\chi_1]R(\lambda_0)\chi \\ &\quad + (1-\chi_2)(1-\chi) - [P,\chi_2]R_0(\lambda)(1-\chi) \\ &= I + Z(\lambda) \text{ where}\end{aligned}$$

$$\cancel{Z(\lambda) = (\lambda_0^2-\lambda^2)\chi_1 R(\lambda_0)\chi + [P,\chi_1]R(\lambda_0)\chi - [P,\chi_2]R_0(\lambda)(1-\chi)}$$

$$\boxed{Z(\lambda) = (\lambda_0^2-\lambda^2)\chi_1 R(\lambda_0)\chi + [P,\chi_1]R(\lambda_0)\chi - [P,\chi_2]R_0(\lambda)(1-\chi).}$$

2. Assume $I+Z(\lambda)$ is invertible for some λ.
We have $(P-\lambda^2)Q = I + Z(\lambda)$, $\operatorname{Im}\lambda > 0$

~~And If both $I+Z(\lambda)$ and $P-\lambda^2$ are invertible, then~~ If $I+Z(\lambda)$ is invertible, then

$(P-\lambda^2)Q(I+Z(\lambda))^{-1} = I$. Since $P-\lambda^2 : \mathcal{D}_P \to L^2$ is Fredholm of index 0, it is invertible and we have $R(\lambda) = (P-\lambda^2)^{-1} = Q(I+Z(\lambda))^{-1}$.

Next, $Z(\lambda) = \chi_3 Z(\lambda)$ so (assuming $I+Z(\lambda)\chi_3$ invertible ...)

$$(I+Z(\lambda))^{-1} = (I+Z(\lambda)\chi_3)^{-1}(I - Z(\lambda)(1-\chi_3))$$

(to check, multiply both sides by $I+Z(\lambda)$ on the right)

So then

~~$(*)\ R(\lambda) = Q(I+Z(\lambda)\chi_3)^{-1}$~~

$$\boxed{(*)\ R(\lambda) = Q(I+Z(\lambda)\chi_3)^{-1}(I - Z(\lambda)(1-\chi_3)).}$$

3. For general $\lambda \in \mathbb{C}$ we study the mapping properties:

- $R_0(\lambda) : L^2_{comp} \to H^2_{loc}$ is holomorphic (since n is odd)
- $Q(\lambda) : L^2_{comp} \to \mathcal{D}_{P,loc}$ is holomorphic
- $Z(\lambda) : L^2_{comp} \to H^1_{comp}$ is holomorphic

  thus $Z(\lambda)\chi_3 : L^2 \to L^2$ is compact.

  If $\exists \lambda : (I+Z(\lambda)\chi_3)$ is invertible (do it later), then $(I+Z(\lambda)\chi_3)^{-1} : L^2 \to L^2$ is meromorphic by Analytic Fredholm Theory.

- We have $(I+Z(\lambda)\chi_3)^{-1} = I - Z(\lambda)\chi_3(I+Z(\lambda)\chi_3)^{-1}$ actually maps $L^2_{comp} \to L^2_{comp}$ since $Z(\lambda) = \chi_3 Z(\lambda)$.

- Finally $I - Z(\lambda)(1-\chi_3): L^2_{comp} \to L^2_{comp}$

So by $(*)$ we get the meromorphic extension

$R(\lambda): L^2_{comp} \to \mathcal{D}_{P,loc}$, $\lambda \in \mathbb{C}$.

(4) It remains to show that $\exists \lambda$: $I + Z(\lambda)$, $I + Z(\lambda)\chi_3$ are invertible $L^2 \to L^2$

We put $\lambda := \lambda_0$ so that

$Z(\lambda) = [P, \chi_1] R(\lambda_0) \chi - [P, \chi_2] R_0(\lambda)(1-\chi)$.

We put $\lambda_0 := e^{i\pi/4} \cdot \alpha$, $\alpha \gg 1$, $\alpha \in \mathbb{R}$.

Then by spectral theory

$$\|R(\lambda_0)\|_{L^2 \to L^2} = \|(P - i\alpha^2)^{-1}\|_{L^2 \to L^2} \leq \frac{1}{\alpha^2}$$

And $\|PR(\lambda_0)\|_{L^2 \to L^2} = \|I + \lambda_0^2 R(\lambda_0)\|_{L^2 \to L^2} \leq C$

So $\|R(\lambda_0)\|_{L^2 \to H^2} \leq C$.

Then by interpolation $\|R(\lambda_0)\|_{L^2 \to H^1} \leq \frac{C}{\alpha}$

So $\|Z(\lambda)\|_{L^2 \to L^2} \leq \frac{C}{\alpha} \Rightarrow$ for $\alpha \gg 1$,

$\|Z(\lambda)\|$, $\|Z(\lambda)\chi_3\| \leq \frac{1}{2} \ldots$ $\square$

As before, we call the poles of $R(\lambda)$ <u>resonances</u>.

Arguing as in the case of $P = -\Delta + V$ we get:

- if $\lambda \neq 0$ <u>is not</u> a resonance, then $\forall f \in L^2_{comp}$ $u := R(\lambda) f$ is the unique outgoing solution to $(P - \lambda^2) u = f$
- if $\lambda \neq 0$ <u>is</u> a resonance, then $\exists$ nontrivial resonant state: $u \not\equiv 0$, $(P - \lambda^2) u = 0$, u outgoing.

What does it mean for u to be outgoing?

Def. $u \in \mathcal{D}_{P,loc}$ is outgoing at λ, if $\exists\, \varphi \in L^2_{comp}(\mathbb{R}^n)$ such that
$$u = R_0(\lambda)\varphi \quad \text{for large } |x|.$$

We have the following version of

Rellich's Uniqueness Theorem

If $\lambda \in \mathbb{R}\setminus\{0\}$ is a resonance and $(P-\lambda^2)u = 0$, u outgoing, then u is compactly supported.

The proof is similar to the case of $P = -\Delta + V$.

If M is connected (or at least it has no bounded connected components) then R.U.T. + unique continuation imply that there are no resonances on $\mathbb{R}\setminus\{0\}$.

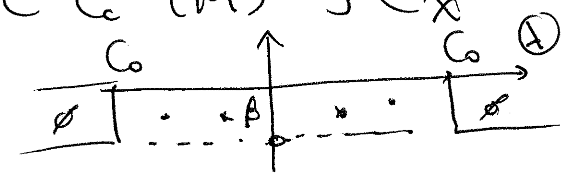
One can still define the scattering operator
$$S(\lambda) : \bigoplus_{\ell=1}^{L} L^2(\mathbb{S}^{n-1}) \circlearrowleft$$
where M has L infinite ends...

High frequency estimates & trapping

Imagine we want to show a spectral gap:

Def. P has an essential spectral gap of size $\beta > 0$ (with a polynomial resolvent bound) if $\exists\, C_0 > 0, N$ s.t. for $|\mathrm{Re}\,\lambda| \geq C_0$, $-\beta \leq \mathrm{Im}\,\lambda \leq 1$, λ is not a resonance and $\forall \chi \in C_c^\infty(M)\ \exists\, C_\chi$:
$$\|\chi R(\lambda) \chi\|_{L^2 \to L^2} \leq C_\chi |\lambda|^N.$$

1



<u>Note</u>: for any $f \in L^2_{comp}$,

$$R(-\bar\lambda) f = \overline{R(\lambda)\bar f}.$$

(indeed, ~~for~~ for $\operatorname{Im}\lambda > 0$ use that $\overline{(P-\lambda^2) f} = (P-\bar\lambda^2)\bar f$.)

So it's enough to ~~check~~ consider $\operatorname{Re}\lambda > 0$, in the context of essential spectral gap $\{\operatorname{Re}\lambda \geq C_0$.

<u>Note</u>: if P has a gap, then one can prove a resonance expansion with remainder $O(e^{-\beta t})$. If we also know that P has no resonances in $\{\operatorname{Im}\lambda \geq 0\}$, then we see that ~~for~~ <u>solutions to wave equation decay exponentially on compact sets.</u>

To handle large $\operatorname{Re}\lambda$, use <u>semiclassical rescaling</u>!

$$P_h := h^2 P = -h^2\Delta_g + h^2 V$$

We choose h so that $h^{-1} \approx \operatorname{Re}\lambda$ (e.g. can take $h := (\operatorname{Re}\lambda)^{-1}$),

then $h^2(P - \lambda^2) = P_h - \omega^2$ where

$\omega = h\lambda = 1 + O(h)$ if $|\operatorname{Im}\lambda| \leq C$

$\operatorname{Im}\lambda \geq -\beta$ corresponds to $\operatorname{Im}\omega \geq -\beta h \ldots$

So then $\underset{h^2(P-\lambda^2)}{P_h} = -h^2\Delta_g - 1 + h\Psi_h^0$ (when $\partial M = \emptyset$.)

We assume $\partial M = \emptyset$, then $P_h \in \Psi_h^2$ and

$\sigma_h(P_h) = p, \quad p(x,\xi) = \sum_{j,k=1}^{n} g^{jk}(x)\,\xi_j\,\xi_k.$

Thus $h^2(P_h - \lambda^2) \in \Psi_h^2$ and

$P_h - \omega^2$

$\sigma_h(h^2(p-\lambda^2)) = p - 1.$

It is thus reasonable to consider

- $S^*M = \{(x,\xi) \in T^*M : |\xi|_g = 1\}, \subset T^*M$

  the characteristic set of p: $S^*M = \{p = 1\}$

- $\varphi_t = \exp(tH_p) : T^*M \circlearrowleft$, the Hamiltonian flow

We know from pset 6 that φ_t is the geodesic flow on (M,g) rescaled by 2 in the time variable.

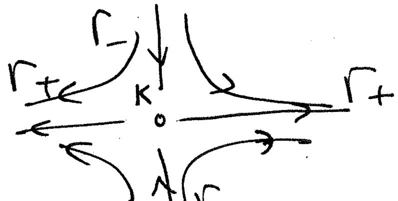
The question is <u>global dynamics</u> of the flow.

<u>Def.</u> We define the sets $\Gamma^\pm, K \subset T^*M \setminus 0 = \{(x,\xi) \in T^*M \mid \xi \neq 0\}$

as follows:

- $(x,\xi) \in \Gamma_\pm \iff \varphi_t(x,\xi)$ stays bounded as $t \to \mp\infty$

- $K := \Gamma_+ \cap \Gamma_-$



$\Gamma_\pm$ for $\varphi_t(x,\xi) = (e^t x, e^{-t}\xi)$

Call Γ_+ the outgoing tail

Γ_- the incoming tail

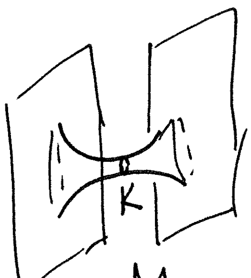
K the trapped set

<u>Note</u>: $\Gamma_\pm, K$ are homogeneous (preserved by $(x,\xi) \mapsto (x, \tau\xi)$, $\tau > 0$) $\cancel{(x,\lambda\xi) \mapsto (x,\lambda\xi)}$

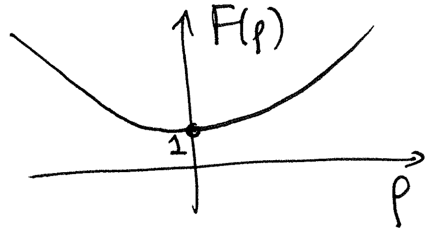
Example: stretched product

$$M = \mathbb{R}_\rho \times S^{n-1}_\theta, \quad g = d\rho^2 + F(\rho)^2 d\theta^2,$$

$$F(\rho): \begin{cases} F(\rho) = |\rho|, & |\rho| \geq r_0 \\ \rho F'(\rho) > 0 & \text{for } \rho \neq 0 \\ F''(\rho) > 0. \end{cases} \qquad F(0) = 1$$



M



Then $K = \{(\rho, \theta, \xi_\rho, \xi_\theta) : \rho = 0, \xi_\rho = 0, \xi_\theta \neq 0\}$ and what are $\Gamma_\pm$? Note that $|\xi_\theta|$ is conserved by the flow

$p = \xi_\rho^2 + F(\rho)^{-2} |\xi_\theta|^2$ is also conserved.

The set K has $p = |\xi_\theta|^2$. We will actually have

$$\Gamma_+ \cup \Gamma_- = \{p = |\xi_\theta|^2, \ \xi_\theta \neq 0\}$$

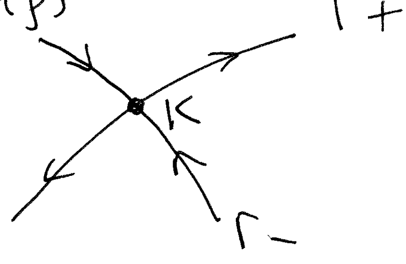
$$= \left\{ \xi_\rho^2 + \frac{|\xi_\theta|^2}{F(\rho)^2} = |\xi_\theta|^2, \ \xi_\theta \neq 0 \right\}$$

$$= \left\{ \xi_\rho^2 = \frac{|\xi_\theta|^2 (F(\rho)^2 - 1)}{F(\rho)^2}, \ \xi_\theta \neq 0 \right\}$$

Write $F(\rho)^2 - 1 = G(\rho)^2$ where $G \in C^\infty$, $\operatorname{sgn} G(\rho) = \operatorname{sgn} \rho$

Then $\Gamma_\pm = \left\{ \xi_\rho = \pm \frac{|\xi_\theta| G(\rho)}{F(\rho)}, \ \xi_\theta \neq 0 \right\}$

Intersect transversally at K:



This is an example of <u>normally hyperbolic trapping</u>.