Continuing the proof of Rellich's Uniqueness Theorem

Step 3: we already know that

u is compactly supported and $(P_V - \lambda^2)u = 0$.

Need to show $u \equiv 0$. This is a unique continuation result.
We assume $\operatorname{supp} u \subset B(0,R)$.

Lemma [Carleman estimate, simple version] Fix $R > 0$.

Then there exists a Carleman weight $\varphi \in C^\infty(\mathbb{R}^n; \mathbb{R})$ and $h_0 > 0$ such that for all $0 < h < h_0$ & $v \in H^2(\mathbb{R}^n)$, $\operatorname{supp} v \subset B(0,R)$, we have

$$(*) \quad \| h^2 e^{\varphi/h} \Delta e^{-\varphi/h} v \|_{L^2} \geq c h^{1/2} \|v\|_{L^2}.$$

How to use this? Put $v := e^{\varphi/h} u$.

We immediately get from $(*)$ that there are no nonzero compactly supported solutions to $\Delta u = 0$ (check it!)

But u solves instead $(-\Delta + V - \lambda^2)u = 0$.

So, $\Delta e^{-\varphi/h} v = \Delta u = (V - \lambda^2)u$. $\Rightarrow$

$\Rightarrow \| h^2 e^{\varphi/h} \Delta e^{-\varphi/h} v \|_{L^2} = \| h^2 (V - \lambda^2) e^{\varphi/h} u \|_{L^2}$

$\leq C h^2 \|v\|_{L^2}$. We get from $(*)$, for small enough h,

$$C h^2 \|v\|_{L^2} \geq c h^{1/2} \|v\|_{L^2} \Rightarrow$$

we obtain $v \equiv 0 \Rightarrow u \equiv 0$.

[End of proof of Rellich's Uniqueness Theorem]

$\square$

What's going on with the Carleman estimate?

General philosophy: Solutions to elliptic equations can get pretty small but they cannot be more than exponentially small... this is the quantum tunneling effect.

For more, see Zworski, §7.2 (and §7.1 is also useful).

Proof of Lemma Enough to prove in the case $v \in C_c^\infty(B(0,R))$, by approximation (C_c^∞ dense in H^2).

(1) Define $P_\varphi := -h^2 e^{\varphi/h} \Delta e^{-\varphi/h}$, so we need to show

$$\|P_\varphi v\|_{L^2} \geq C h^{1/2} \|v\|_{L^2}.$$

$\boxed{\text{Note: } P_\varphi \text{ is a semiclassical differential operator...}}$

We'll do a "positive commutator argument" (Hörmander):

$$\|P_\varphi v\|_{L^2}^2 = \langle P_\varphi v, P_\varphi v\rangle = \langle P_\varphi^* P_\varphi v, v\rangle$$
$$= \langle P_\varphi P_\varphi^* v, v\rangle + \langle [P_\varphi^*, P_\varphi] v, v\rangle$$
$$= \|P_\varphi^* v\|_{L^2}^2 + \langle [P_\varphi^*, P_\varphi] v, v\rangle$$

So it is enough to prove that

$$\langle [P_\varphi^*, P_\varphi] v, v\rangle_{L^2} \geq c h \|v\|_{L^2}. \quad (\star)$$

(2) We compute (using $h^2\Delta = \sum_j h^2\partial_{x_j}^2$ and $e^{\varphi/h} h\partial_{x_j} e^{-\varphi/h} = h\partial_{x_j} - \partial\varphi'_{x_j}$

$$P_\varphi = -h^2\Delta + 2h\langle \nabla\varphi, \nabla\rangle - |\nabla\varphi|^2 + h\Delta\varphi.$$

i.e. $P_\varphi v = -h^2\Delta v + 2h\langle\nabla\varphi, \nabla v\rangle - |\nabla\varphi|^2 v + (h\Delta\varphi)\cdot v.$

Next, compute P_φ^* by the identity

$$\langle P_\varphi v, w\rangle_{L^2} = \langle v, P_\varphi^* w\rangle_{L^2}, \quad v, w \in C_c^\infty(\mathbb{R}^n).$$

$$P_\varphi^* = -h^2\Delta - 2h\langle\nabla\varphi, \nabla\rangle - |\nabla\varphi|^2 - h\Delta\varphi.$$

Thus $P_\varphi + P_\varphi^* = -2h^2\Delta - 2|\nabla\varphi|^2$,

$$P_\varphi - P_\varphi^* = 4h\langle\nabla\varphi,\nabla\rangle + 2h(\Delta\varphi).$$

$$[P_\varphi^*, P_\varphi] = \frac{1}{2}[P_\varphi + P_\varphi^*, P_\varphi - P_\varphi^*]$$

$$= -2h[h^2\Delta + |\nabla\varphi|^2, 2\langle\nabla\varphi,\nabla\rangle + (\Delta\varphi)]$$

just a multiplication operator by $\Delta\varphi$

$= ?$

Compute $[\Delta, \langle\nabla\varphi,\nabla\rangle] = 2\sum_{j,k}\varphi''_{x_jx_k}\partial^2_{x_jx_k} + \langle\nabla(\Delta\varphi),\nabla\rangle.$

$$[|\nabla\varphi|^2, \langle\nabla\varphi,\nabla\rangle] = -\langle\nabla\varphi, \nabla(|\nabla\varphi|^2)\rangle$$

$$[\Delta, (\Delta\varphi)] = 2\langle\nabla(\Delta\varphi),\nabla\rangle + (\Delta^2\varphi).$$

$$[|\nabla\varphi|^2, (\Delta\varphi)] = 0.$$

So $[P_\varphi^*, P_\varphi] = 2h\Big[-4h^2\sum_{j,k}\varphi''_{x_jx_k}\partial^2_{x_jx_k} - 4h^2\langle\nabla(\Delta\varphi),\nabla\rangle$
$+ 2\langle\nabla\varphi, \nabla(|\nabla\varphi|^2)\rangle \mp h^2(\Delta^2\varphi)\Big].$

3

We choose $\varphi(x) := \frac{|x|^2}{2} + Mx_1$, M large constant

Then $\nabla(\Delta\varphi) = 0$, $\Delta^2\varphi = 0$.

~~$\nabla\varphi = x \Rightarrow |\nabla\varphi|^2 = |x|^2$, $\nabla(|\nabla\varphi|^2) = 2x$,~~

~~$2\langle\nabla\varphi, \nabla|\nabla\varphi|^2\rangle = 2\langle x, 2x\rangle =$~~

$\sum_{j,k}\varphi''_{x_jx_k}\partial^2_{x_jx_k} = \Delta$; $\nabla\varphi = x + Me_1$, $e_1 = (1,0,\dots,0)$

$|\nabla\varphi|^2 = |x+Me_1|^2$, $\nabla|\nabla\varphi|^2 = 2\nabla\varphi$.

$2\langle\nabla\varphi, \nabla|\nabla\varphi|^2\rangle = 4|\nabla\varphi|^2 = 4|x+Me_1|^2$. So

$$[P_\varphi^*, P_\varphi] = 8h[-h^2\Delta + 2|x+Me_1|^2].$$

(4) We now have

$$\langle [P_\varphi^*, P_\varphi] v, v \rangle_{L^2} = 8h \langle (-h^2\Delta + 2|x+Me_1|^2) v, v \rangle_{L^2}$$

$$= 8h \left(\|h\nabla v\|_{L^2}^2 + 2\int |x+Me_1|^2 \cdot |v(x)|^2 \, dx \right)$$

$\geq h\|v\|^2$ as soon as M is large enough

So that $|x+Me_1| \geq 1$ ~~on~~ when $x \in B(0,R)$.

This gives $(\star)$ and finishes the proof. $\square$

Scattering operator (matrix)

First introduce plane waves: take $\lambda \in \mathbb{R}\setminus\{0\}$, $\omega \in \mathbb{S}^{n-1}$.

$$w(x,\lambda,\omega) = e^{-i\lambda\langle x,\omega\rangle} + u(x,\lambda,\omega)$$

s.t. $\begin{cases} (P_V - \lambda^2) w = 0 \\ u \text{ is } \lambda\text{-outgoing.} \end{cases}$

How to construct? u has to solve the equation

$$(P_V - \lambda^2) u(x) = -(P_V - \lambda^2) e^{-i\lambda\langle x,\omega\rangle}$$

$$= -Ve^{-i\lambda\langle x,\omega\rangle} \quad \text{since } (-\Delta - \lambda^2) e^{-i\lambda\langle x,\omega\rangle} = 0.$$

$\lambda \in \mathbb{R}\setminus\{0\} \Rightarrow \lambda$ not a resonance

$\Downarrow$

$$\boxed{u = -R_V(\lambda) V e^{-i\lambda\langle \cdot,\omega\rangle}}$$

Note: plane waves can be used to make a "Fourier transform" which is adapted to P_V rather than to $-\Delta$.
(won't pursue it here though)

Our goal is now to construct solutions v to $(P_V - \lambda^2)v = 0$ such that $(\lambda \in \mathbb{R}\setminus\{0\})$

$v =$ (incoming) + (outgoing). (incoming at λ) = (outgoing at $-\lambda$)

Specifically we have

Thm (about scattering solutions) Let $\lambda \in \mathbb{R}\setminus\{0\}$. Then for each $g \in C^\infty(\mathbb{S}^{n-1})$ there exists unique $f \in C^\infty(\mathbb{S}^{n-1})$ and $v \in H^2_{loc}(\mathbb{R}^n)$ such that

$$(P_V - \lambda^2)v = 0,$$

$$v(r\theta) = r^{-\frac{n-1}{2}}\Big(\underbrace{e^{i\lambda r} f(\theta)}_{\text{outgoing}} + \underbrace{e^{-i\lambda r} g(\theta)}_{\text{incoming}}\Big) + O(r^{-\frac{n+1}{2}})$$

+ r-derivative is $O(r^{-\frac{n+1}{2}})$

Proof ⓪ Uniqueness: assume $g \equiv 0$. Then ~~$v(r\theta)$~~ v satisfies the Sommerfeld Radiation Condition (SRC). By the stronger version of Rellich's Uniqueness Theorem (that we did not prove; see Thm 3.32 in the Book) we get $v \equiv 0$.

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① Let us first understand the free case, that is P_V replaced by $-\Delta$.

Write $u_0(x) := \int_{\mathbb{S}^{n-1}} g(\omega) e^{i\lambda\langle x, \omega\rangle} dS(\omega)$. Note: $(-\Delta - \lambda^2)u_0 = 0$.

This is a superposition of plane waves; can we write it in incoming/outgoing terms?

Lemma (Thm 3.35). Let $g \in C^\infty(\mathbb{S}^{n-1})$ & put

$$u_0(x) := \int_{\mathbb{S}^{n-1}} g(\omega) e^{i\lambda\langle x, \omega\rangle} \, dS(\omega),$$ here $\lambda \in \mathbb{R}\setminus\{0\}$.

Then ~~$u_0(x) =$~~ for $r \to \infty$, $\theta \in \mathbb{S}^{n-1}$ we have

~~$u_0(r\theta) = \frac{1}{(\lambda r)^{\frac{n-1}{2}}}$~~

$$u_0(r\theta) = (\lambda r)^{\frac{1-n}{2}} \left[c_n^+ e^{-i\lambda r} g(-\theta) + c_n^- e^{i\lambda r} g(\theta) \right] + O(r^{-\frac{n+1}{2}})$$

where $c_n^\pm = (2\pi)^{\frac{n-1}{2}} e^{\pm \frac{\pi}{4}(n-1)i}$

Proof. We have

$$u_0(r\theta) = \int_{\mathbb{S}^{n-1}} g(\omega) e^{i\lambda r\langle \theta, \omega\rangle} \, dS(\omega)$$

$$= \int_{\mathbb{S}^{n-1}} e^{ir\Phi(\omega)} g(\omega) \, dS(\omega)$$ where

$\Phi(\omega) = \lambda \cdot \langle \theta, \omega \rangle$, $\Phi : \mathbb{S}^{n-1} \to \mathbb{R}$ depends on λ, θ as parameters

Zworski, §3.5 LOOK AT IT!

Time to apply **Method of Stationary phase**: assuming critical pts of Φ are nondegenerate (i.e. Φ a Morse fn)

$$\int_{\mathbb{S}^{n-1}} e^{ir\Phi(\omega)} g(\omega) \, dS(\omega) = \sum_{\omega_j : \nabla\Phi(\omega_j) = 0} e^{ir\Phi(\omega_j)} \cdot \Big[(2\pi)^{\frac{n-1}{2}} r^{\frac{1-n}{2}} \cdot \cancel{|\det}$$

$$\cdot |\det \nabla^2 \Phi(\omega_j)|^{-1/2} \cdot e^{\frac{i\pi}{4} \operatorname{sgn} \nabla^2 \Phi(\omega_j)} g(\omega_j) + O(r^{\frac{-1-n}{2}}) \Big]$$

as $r \to \infty$.

What are the stationary points?

Can rotate everything to make $\theta = e_1 = (1, 0, \dots, 0)$

So then $\Phi(\omega) = \lambda \cdot \omega_1$.

2 critical points: $\omega = \pm e_1$.

Can take local coordinates $\omega' = (\omega_2, \dots, \omega_n) \in \mathbb{R}^{n-1}$ near each of these, so then

$\Phi(\omega) = \pm \lambda \cdot \sqrt{1 - |\omega'|^2}$.

Can compute $\nabla^2 \Phi$ at $\omega' = 0$ & plug into stationary phase to finish the proof.

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2 Coming back to Thm about scattering solutions: we write

$$\cancel{v} \quad v(x) := \frac{\lambda^{\frac{n-1}{2}}}{c_n} \cdot \int_{S^{n-1}} g(\omega) \cdot w(x, \lambda, \omega)\, dS(\omega)$$

where $w(x, \lambda, \omega)$ is the plane wave. Then $(P_V - \lambda^2) v = 0$ & we can write

$$w = e^{-i\lambda \langle x, \omega \rangle} + \underbrace{u(x, \lambda, \omega)}_{\text{outgoing}}$$

$$v(x) = \frac{\lambda^{\frac{n-1}{2}}}{c_n} \int_{S^{n-1}} u_0(x) + \int (\text{outgoing things})$$

$$= r^{\frac{1-n}{2}} e^{i\lambda |x|} g\left(\frac{x}{|x|}\right) + (\text{outgoing}) \dots \quad \square$$