Scattering theory. What is it about?

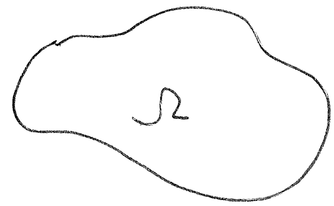
Scattering theory in particular studies

asymptotic behavior of waves in open systems

Example of a closed system:

$\mathbb{R}^3$

$\Omega \subset \mathbb{R}^3$ bdd domain



Wave equation: $(\partial_t^2 - \Delta_x)w = 0$

$$w|_{\mathbb{R}_t \times \partial\Omega} = 0 \qquad w = w(t,x)$$

$$t \in \mathbb{R},\ x \in \Omega$$

$$w|_{t=0} = f_0, \quad w_t|_{t=0} = f_1$$

Fourier method: $w(t,x) = \sum_j e^{-it\lambda_j} w_j(x)$

where $\lambda_j = \pm\sqrt{\text{eigenvalues of } -\Delta \text{ on } \Omega} \in \mathbb{R}$

Example of an open system:

take same Ω but solve the wave equation on the exterior $\mathbb{R}^3 \setminus \Omega$. There are no eigenvalues, instead there are resonances

$\lambda_j \in \mathbb{C}$ & we have (sometimes – this is trickier than the closed case)

$$w(t,x) \sim \sum_j e^{-it\lambda_j} v_j(x), \text{ as } t \to \infty,$$

in cpct sets in x, assuming supp f_0, f_1 cpct.

resonance expansion

$\cancel{Re}\ |e^{-it\lambda_j}| = e^{t\,|\mathrm{Im}\,\lambda_j|}$.

$\mathrm{Re}\ \lambda_j$ = rate of oscillation

$-\mathrm{Im}\ \lambda_j$ = rate of exponential decay.

Why can we decay (as $t \to \infty$) for x in a compact set?

Because most of the energy <u>escapes to infinity</u>.

[Semyon hits things]

(?) : How to define resonances?

(?) : Is the resonance expansion valid?

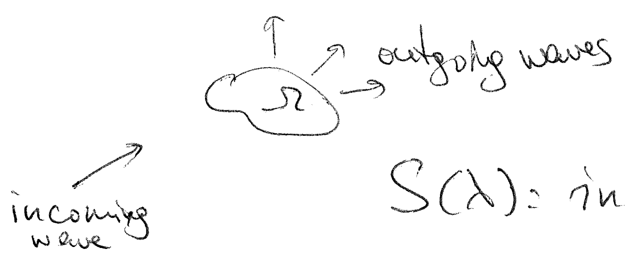
(?) : How close can λ_j come to the real line as $\mathrm{Re}\ \lambda_j \to \infty$?

(Semiclassical asymptotics)

Another object : scattering operator $S(\lambda)$ ← frequency

Write $w(t,x) = e^{-i\lambda t} u(x)$

$u(x) =$ incoming + outgoing



$S(\lambda)$: incoming ↔ outgoing.

Let's now do some math...

1D potential scattering (aka ODEs galore)

(presented more primitively than in the book. Will use the book's methods later.)

Book: §2.1

Consider the wave equation

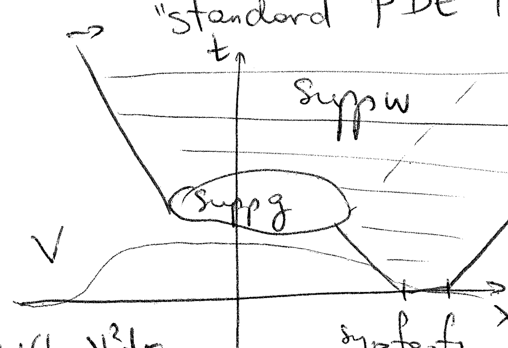
$$\begin{cases} (\partial_t^2 - \partial_x^2 + V(x))\,w(t,x) = g \\ w|_{t=0} = f_0(x) \\ w_t|_{t=0} = f_1(x) \end{cases} \qquad \text{(WE)}$$

$\boxed{V \in C_c^\infty(\mathbb{R};\mathbb{R})}$

We assume for now that $g \in C_0^\infty(\mathbb{R}^2)$, $\operatorname{supp} g \subset \{t>0\}$, $f_0, f_1 \in C_0^\infty(\mathbb{R})$.

There exists unique solution w ~~to~~ $\in C^\infty(\mathbb{R}^2_{t,x})$ to (WE) → "standard PDE theory"

Note the support of w: [Exercise 3]



Energy:

$$\mathcal{E}(t) = \frac{1}{2}\int_{\mathbb{R}} |w_t(t,x)|^2 + |w_x(t,x)|^2 + V|w(t,x)|^2\,dx$$

is constant in t once we pass $\operatorname{supp} g$. [Exercise 2]

It follows: • for $V \geq 0$, $w(t,x)$ grows at most polynomially in t

• for general V, $w(t,x)$ grows at most exponentially

will just do this case for simplicity of presentation

A useful fact: if supp f_0, supp f_1, supp g, supp $V \subset \{|x|<r_0\}$

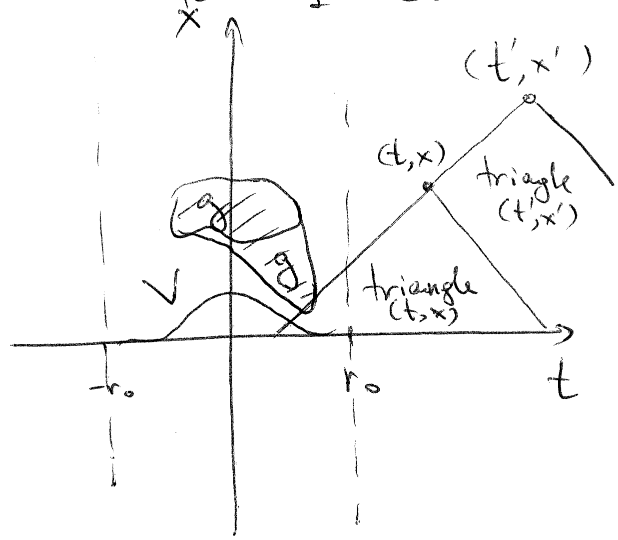
then $w(t,x) = w_\pm(x \mp t)$ for $\pm x \geq r_0$, $t \geq 0$.

[Exercise 1]

Why? d'Alembert's formula for

$(\partial_t^2 - \partial_x^2)w = g - Vw$. Assume for simplicity $f_0 \equiv f_1 \equiv 0$. Then

$$w(t,x) = \frac{1}{2}\int_{\text{triangle}(t,x)} g - Vw \, dt\,dx:$$



If $x-t = x'-t'$, $x, x' \geq r_0$, then the $\int$-s for $w(t,x)$, $w(t',x')$ are the same

---

Fourier transform in time: → converges exponentially fast

$$u(x;\lambda) := \hat{w} = \int_0^\infty e^{it\lambda} w(t,x)\,dt, \quad \operatorname{Im}\lambda > 0$$

Assume $f_0 \equiv f_1 \equiv 0$ (can reduce to this case).

Then Integrate By Parts (IBP) twice in t:

$$-\lambda^2 u(x;\lambda) = \int_0^\infty e^{it\lambda} w_{tt}(t,x)\,dt$$

$$(-\partial_x^2 + V)u(x;\lambda) = \int_0^\infty e^{it\lambda}(-w_{xx}(t,x) + V(x)w(t,x))\,dt$$

So, WE ~~for~~ w $\Rightarrow$ $\boxed{(P_V - \lambda^2)u(x;\lambda) = f(x;\lambda)}$

where $f(x;\lambda) = \hat{g} = \int e^{it\lambda} g(t,x)\,dt$.

Here $P_V = -\partial_x^2 + V = D_x^2 + V$.

$\boxed{\text{!!Important notation! } D = \frac{1}{i}\partial = -i\partial}$

$(P_V - \lambda^2)u(x) = f(x)$. This is an ODE.

If $\operatorname{supp} V \subset [-r_0, r_0]$, then
$\operatorname{supp} f$

$\cancel{u(x) = C_{1,\pm} e^{\pm i\lambda x}}$ $\quad$ u solves $(-\partial_x^2 - \lambda^2)u = 0$ on $\{|x| > r_0\}$.

So, $u(x) = C_{1,\pm} e^{i\lambda x} + C_{2,\pm} e^{-i\lambda x}$ for $\pm x \gg 1$

means $\pm x > r_0$ where r_0 depends on the situation!

Which solution is u?

Recall the useful fact:

$w(t,x) = w_\pm(x \mp t)$ for $\pm x \geq r_0$

for $\pm x \geq r_0$, $u(x;\lambda) = \int e^{-it\lambda} w_\pm(\underbrace{x \mp t}_{\tau})\, dt$

$= \int e^{\pm i(\tau - x)\lambda} w_\pm(\tau)\, d\tau = e^{\pm i\lambda x} \hat{w}_\pm(\mp\lambda)$

So, $u(x)$ is "outgoing":

$$u(x) = C_\pm e^{\pm i\lambda x} \quad \text{for } \pm x \gg 1. \quad \text{(out)}$$

This agrees with the fact that $u \in L^2$ when $\operatorname{Im} \lambda > 0$.

Meromorphic extension

Thm [IOU] There exists a meromorphic family of operators $R_V(\lambda) : C_c^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, called scattering resolvent, s.t. when λ not a pole of R_V, $u := R_V(\lambda) f$, $f \in C_c^\infty(\mathbb{R})$, is the unique solution to $(P_V - \lambda^2)u = f$ satisfying (out).

The poles of $R_V(\lambda)$, called resonances, correspond to λ for which there exists nontrivial u, $(P_V - \lambda^2)u = 0$, satisfying (out).

Contour deformation argument

Coming back to (WE), we write $u = \hat{w}$, $f = \hat{g}$, then $u(\lambda) = R_V(\lambda) f(\lambda)$.

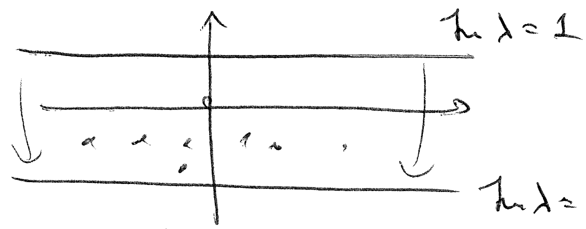
Now $f(\lambda)$ is entire in λ as g is compactly supported.

So $u(\lambda)$ has a meromorphic continuation to $\lambda \in \mathbb{C}$.

Fourier inversion formula: (applied to $e^{-t}w(t,x)$)

$$w(t,x) = \frac{1}{2\pi}\int_{\operatorname{Im}\lambda = 1} e^{-it\lambda}\,\hat{w}(\lambda,x) = \frac{1}{2\pi}\int_{\operatorname{Im}\lambda = 1} e^{-it\lambda} R_V(\lambda) f(\lambda)\,d\lambda$$

Fix $\nu > 0$



$$= \frac{1}{2\pi}\int_{\operatorname{Im}\lambda = -\nu} e^{-it\lambda} R_V(\lambda) f(\lambda)\,d\lambda$$

$$+ \sum_{\substack{\lambda_j \text{ resonance} \\ \operatorname{Im}\lambda_j > -\nu}} 2\pi i \operatorname{Res}_{\lambda=\lambda_j}\left(e^{-it\lambda} R_V(\lambda) f(\lambda)\right)$$

If R_V ~~for~~ has simple poles:

$$w(t,x) = \sum_{\substack{\lambda_j \text{ resonance} \\ \operatorname{Im}\lambda_j > -\nu}} e^{-it\lambda_j} v_j(x) + O(e^{-\nu t}).$$

Resonance expansion

But the contours were infinite, so we need more work!

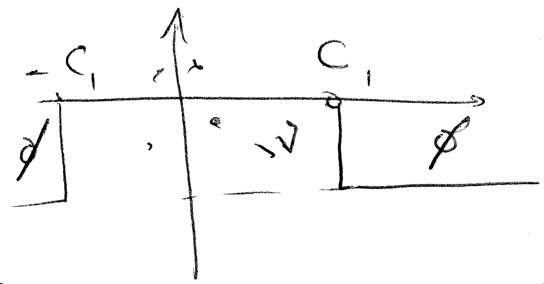
Spectral gap:

For each $\nu > 0$ there exists $C > 0$, $C_1 > 0$ s.t. for $|\operatorname{Im}\lambda| \le \nu$, $|\operatorname{Re}\lambda| \ge C_1$, λ is not a resonance and

$$\forall \chi \in C_c^\infty(\mathbb{R}), \quad \|\chi R_V(\lambda)\chi\|_{L^1 \to L^\infty} \le \frac{C}{|\lambda|}.$$

This gives the resonance expansion; note that since $f \in C_c^\infty$, we have

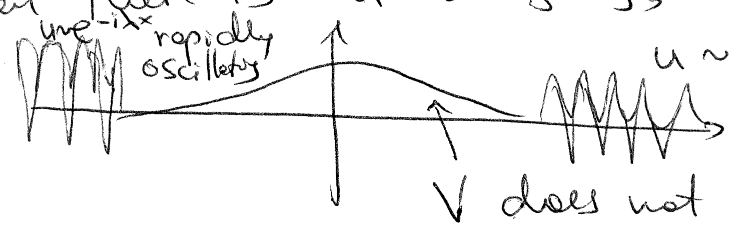
$$\|f(x;\lambda)\|_{L^1_x} \le C_N |\lambda|^{-N} \quad \text{when } |\operatorname{Im}\lambda| \le \nu.$$



Why spectral gap holds?

Exercise 6]. But basically, imagine we had a resonance with $|\operatorname{Im}\lambda| \le \nu$, $|\operatorname{Re}\lambda|$ large.

Then there is u outgoing, $(P_V - \lambda^2)u = 0$.



$u \sim e^{i\lambda x}$ rapidly oscillating

V does not oscillate

A slowly oscillating potential does not change much a rapidly oscillating solution.

Basic case: $V \equiv 0$, $R_V(\lambda) f(x) = \frac{i}{2\lambda} \int_{\mathbb{R}} e^{i\lambda|x-y|} f(y)\,dy$

One resonance at $\lambda = 0$ corresponds to d'Alembert's f-la: $w(t,x) = \frac{1}{2} \int_{\mathbb{R}^2} g(t,x)\,dt\,dx$ for $|x| \le r$, $t \gg 1$.