18.155, FALL 2021, PROBLEM SET 7

Review / helpful information:

- Schwartz space: a function $f \in C^\infty(\mathbb{R}^n)$ lies in $\mathscr{S}(\mathbb{R}^n)$ if
$$\sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta f(x)| < \infty \quad \text{for all} \quad \alpha, \beta.$$
- Fourier transform on Schwartz functions:
$$\mathcal{F}\varphi(\xi) = \hat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)\,dx$$
- Fourier inversion formula: the inverse of the Fourier transform $\mathcal{F} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is given by
$$\mathcal{F}^{-1}\psi(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\psi(\xi)\,d\xi.$$
- Tempered distributions: $\mathscr{S}'(\mathbb{R}^n)$ consists of continuous linear forms on $\mathscr{S}(\mathbb{R}^n)$.
- Fourier transform on $\mathscr{S}'(\mathbb{R}^n)$: $(\hat{u}, \varphi) = (u, \hat{\varphi})$ for all $u \in \mathscr{S}'(\mathbb{R}^n)$, $\varphi \in \mathscr{S}(\mathbb{R}^n)$.

1. Assume that $a \in C^\infty(\mathbb{R}^n)$ has each derivative polynomially bounded, i.e. for each α there exist C, N such that $|\partial^\alpha a(x)| \leq C(1 + |x|)^N$ for all x. Explain how to define the operation of multiplication by a on $\mathscr{S}'(\mathbb{R}^n)$ (by duality, similarly to what we did for $\mathcal{D}'$).

2. (a) Show that $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathscr{S}(\mathbb{R}^n)$. (Hint: use multiplication by $\psi(\varepsilon x)$ for some cutoff function ψ. An important corollary of this is that an element of $\mathscr{S}'(\mathbb{R}^n)$ is determined by its pairing with functions in $C_c^\infty(\mathbb{R}^n)$, that is the map $\mathscr{S}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ is injective.)

(b) (Optional) Show that $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathscr{S}'(\mathbb{R}^n)$. (Hint: show that for an appropriate choice of $\psi, \chi \in C_c^\infty(\mathbb{R}^n)$ and each $u \in \mathscr{S}'(\mathbb{R}^n)$, we have $(\psi_\varepsilon u) * \chi_\varepsilon \to u$ in $\mathscr{S}'(\mathbb{R}^n)$ as $\varepsilon \to 0+$ where $\psi_\varepsilon(x) := \psi(\varepsilon x)$, $\chi_\varepsilon(x) := \varepsilon^{-n}\chi(x/\varepsilon)$. To do that, show first that for each $\varphi \in \mathscr{S}(\mathbb{R}^n)$ we have $((\psi_\varepsilon u) * \chi_\varepsilon, \varphi) = (u, \varphi_\varepsilon)$ where $\varphi_\varepsilon(x) = \psi_\varepsilon(x) \int_{\mathbb{R}^n} \chi_\varepsilon(y)\varphi(x + y)\,dy$; for that statement it helps to use the definition of convolution in §8.1 in lecture notes.)

3. For $w \in \mathbb{R}^n$, define the following operators on $C^\infty(\mathbb{R}^n)$:
$$\tau_w f(x) = f(x - w), \quad \sigma_w f(x) = e^{ix\cdot w} f(x).$$

(a) Show that τ_w, σ_w define continuous operators on $\mathscr{S}(\mathbb{R}^n)$. Use this to extend τ_w, σ_w to sequentially continuous operators on $\mathscr{S}'(\mathbb{R}^n)$.

(b) Show that for each $u \in \mathscr{S}'(\mathbb{R}^n)$

$$\widehat{\tau_w u} = \sigma_{-w}\hat{u}, \quad \widehat{\sigma_w u} = \tau_w \hat{u}.$$

4. (a) Show that for $\varphi \in \mathscr{S}(\mathbb{R}^n)$, $\psi \in \mathscr{S}(\mathbb{R}^m)$, we have $\varphi \otimes \psi \in \mathscr{S}(\mathbb{R}^{n+m})$ and the Fourier transform of $\varphi \otimes \psi$ is given by $\hat{\varphi} \otimes \hat{\psi}$.

(b) (Optional) Let $u \in \mathscr{S}'(\mathbb{R}^n)$, $v \in \mathscr{S}'(\mathbb{R}^m)$, and let $u \otimes v \in \mathcal{D}'(\mathbb{R}^{n+m})$ be their distributional tensor product. Show that $u \otimes v \in \mathscr{S}'(\mathbb{R}^{n+m})$ and the Fourier transform of $u \otimes v$ is given by $\hat{u} \otimes \hat{v}$.

5. Let $\varphi, \psi \in \mathscr{S}(\mathbb{R}^n)$. Show that the convolution $\varphi * \psi$ and the product $\varphi\psi$ both lie in $\mathscr{S}(\mathbb{R}^n)$.

6. (a) Assume that $A : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear map. Show that the pullback $A^* : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ restricts to a map $\mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ and for each $u \in \mathscr{S}'(\mathbb{R}^n)$ we have

$$\widehat{A^* u} = |\det A|^{-1}(A^{-T})^* \hat{u}$$

where A^{-T} is the inverse of the transpose of A.

(b) Assume that $u \in \mathscr{S}'(\mathbb{R}^n)$ is homogeneous of degree $a \in \mathbb{C}$. Show that $\hat{u}$ is homogeneous and compute its degree of homogeneity.

7. Denote by $\langle \varphi, \psi \rangle_{L^2} := \int_{\mathbb{R}^n} \varphi(x)\overline{\psi(x)}\,dx$ the inner product on $L^2(\mathbb{R}^n)$. Show that for all $\varphi, \psi \in \mathscr{S}(\mathbb{R}^n)$ we have $\langle \mathcal{F}\varphi, \psi \rangle_{L^2} = (2\pi)^n \langle \varphi, \mathcal{F}^{-1}\psi \rangle_{L^2}$ and $\|\hat{\varphi}\|_{L^2} = (2\pi)^{n/2}\|\varphi\|_{L^2}$.

8. (Optional) This exercise gives a method to compute Fourier transforms of certain distributions using analytic continuation.

(a) Assume that $u \in \mathscr{S}'(\mathbb{R})$ and $\operatorname{supp} u \subset [a, \infty)$ for some $a \in \mathbb{R}$. Take a cutoff $\chi \in C^\infty(\mathbb{R})$ such that $\chi = 1$ near $[a, \infty)$ and $\operatorname{supp} \chi \subset [a-1, \infty)$, and define the function

$$F(\eta) := (u(x), \chi(x)e^{-ix\eta}), \quad \eta \in \mathbb{C}, \quad \operatorname{Im} \eta < 0.$$

Explain why $F(\eta)$ is well-defined and independent of χ and show that it is holomorphic in $\{\operatorname{Im} \eta < 0\}$.

(b) Show that $F(\xi - i\varepsilon) \to \hat{u}(\xi)$ in $\mathscr{S}'(\mathbb{R})$ as $\varepsilon \to 0+$. (Hint: $F(\xi - i\varepsilon)$ is the Fourier transform of $e^{-\varepsilon x}u(x)$ but you should justify your arguments carefully.)

(c) Assume that $a \in \mathbb{C}$ and $\operatorname{Re} a > -1$. Show that the Fourier transform of x_+^a is given by $e^{-i\pi(a+1)/2}\Gamma(a+1)(\xi - i0)^{-a-1}$ where Γ is the Euler Gamma function and $(\xi - i0)^{-a-1}$ was defined in Problemset 3, Exercise 4. In particular, compute the Fourier transform of the Heaviside function. (Hint: use parts (a)–(b), computing $F(\eta)$ for $\eta = -is$, $s > 0$ and then arguing by analytic continuation in η. The result actually holds for all $a \in \mathbb{C}$ by analytic continuation in a.)