18.155, FALL 2021, PROBLEM SET 4

Review / helpful information:

- Sumset: if $X, Y \subset \mathbb{R}^n$ then $X + Y := \{x + y \mid x \in X,\ y \in Y\} \subset \mathbb{R}^n$.
- Convolution with smooth function: if $u \in \mathcal{D}'(\mathbb{R}^n)$, $\varphi \in C_c^\infty(\mathbb{R}^n)$, then $u * \varphi(x) = (u, \varphi(x - \bullet))$ and $u * \varphi \in C^\infty(\mathbb{R}^n)$.
- Tensor product: if $u \in \mathcal{D}'(U)$, $v \in \mathcal{D}'(V)$, then $u \otimes v \in \mathcal{D}'(U \times V)$ is uniquely determined by
$$(u \otimes v, \varphi \otimes \psi) = (u, \varphi)(v, \psi) \quad \text{for all} \quad \varphi \in C_c^\infty(U),\ \psi \in C_c^\infty(V)$$
where $(\varphi \otimes \psi)(x, y) = \varphi(x)\psi(y)$.
- Schwartz kernels: a sequentially continuous operator $A : C_c^\infty(V) \to \mathcal{D}'(U)$ has Schwartz kernel $Q \in \mathcal{D}'(U \times V)$ when
$$(A\psi, \varphi) = (Q, \varphi \otimes \psi) \quad \text{for all} \quad \varphi \in C_c^\infty(U),\ \psi \in C_c^\infty(V).$$

1. (a) Assume that $X \subset \mathbb{R}^n$ is closed and $Y \subset \mathbb{R}^n$ is compact. Show that $X + Y$ is closed.

(b) (Optional) Give an example when $X, Y \subset \mathbb{R}$ are both closed but $X + Y$ is not closed.

(c) Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$. Show that
$$\operatorname{supp}(u * \varphi) \subset \operatorname{supp} u + \operatorname{supp} \varphi.$$

(d) (Optional) Give an example of when the above inclusion is not an equality.

2. Let $u \in \mathcal{D}'(\mathbb{R}^n)$, $\varphi, \psi \in C_c^\infty(\mathbb{R}^n)$. Show that
$$(u * \varphi) * \psi = u * (\varphi * \psi).$$
(Hint: one way is to use density of C_c^∞ in $\mathcal{D}'$.)

3. Let $u \in \mathcal{D}'(U)$, $v \in \mathcal{D}'(V)$ where $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ are open and write elements of $\mathbb{R}^{n+m}$ as $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$. Show that:

(a) $\operatorname{supp}(u \otimes v) = \operatorname{supp} u \times \operatorname{supp} v$;

(b) $\partial_{x_j}(u \otimes v) = (\partial_{x_j} u) \otimes v$ and $\partial_{y_j}(u \otimes v) = u \otimes (\partial_{y_j} v)$.

4. Assume that $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ are open, $0 \in U$, and write elements of $\mathbb{R}^{n+m}$ as $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$. Show that the space of solutions $w \in \mathcal{D}'(U \times V)$ to the equations
$$x_1 w = \ldots = x_n w = 0$$

is given by distributions of the form $\delta_0 \otimes v$ where $\delta_0 \in \mathcal{D}'(U)$ is the delta distribution and $v \in \mathcal{D}'(V)$ is arbitrary.

5. Find the Schwartz kernels of the differentiation operators $\partial_{x_j} : C^\infty_{\mathrm{c}}(U) \to C^\infty_{\mathrm{c}}(U)$ and the multiplication operators $u \mapsto au$, where $a \in C^\infty(U)$.

6. Let $A : C^\infty_{\mathrm{c}}(V) \to \mathcal{D}'(U)$ be a sequentially continuous operator with Schwartz kernel $Q \in \mathcal{D}'(U \times V)$. Here $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ are open and we write elements of $\mathbb{R}^{n+m}$ as $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$. Denote by $\partial_{x_j} : \mathcal{D}'(U) \to \mathcal{D}'(U)$, $\partial_{y_\ell} : C^\infty_{\mathrm{c}}(V) \to C^\infty_{\mathrm{c}}(V)$ the differentiation operators. Show that the composition $\partial_{x_j} A$ has Schwartz kernel $\partial_{x_j} Q$ and $A\partial_{y_\ell}$ has Schwartz kernel $-\partial_{y_\ell} Q$.

7. (Optional) Let $A : C^\infty_{\mathrm{c}}(V) \to \mathcal{D}'(U)$ be a sequentially continuous operator with Schwartz kernel $Q \in \mathcal{D}'(U \times V)$. Show that Q is compactly supported (i.e. $Q \in \mathcal{E}'(U \times V)$) if and only if A extends to a sequentially continuous operator $\widetilde{A} : C^\infty(V) \to \mathcal{E}'(U)$. (Here the convergence of sequences on $\mathcal{E}'$ is defined as follows: $u_k \to u$ in $\mathcal{E}'(U)$ iff $(u_k, \varphi) \to (u, \varphi)$ for all $\varphi \in C^\infty(U)$. Equivalently, $u_k \to u$ in $\mathcal{D}'(U)$ and there exists $K \subset U$ compact such that $\operatorname{supp} u_k \subset K$ for all K.)